# Revision exercises

1. Prove that if $G$ is a locally compact unimodular group, then $L^2(G, dg)$ is a continuous representation of $G$ (acting by right translation).

2. Prove that the natural representation of $\mathbf{R}$ on $L^2(\mathbf{R})$ has no irreducible subrepresentation.

3. Prove that there is a natural homeomorphism
$$\mathbb{SL}_2(\mathbb{R})/\mathbb{SO}_2(\mathbf{R}) \simeq \mathscr{H} := \{z \in \mathbb{C} | \operatorname{Im}(z) > 0\}.$$

4. Prove that the measure $\frac{dxdy}{y^2}$ on $\mathscr{H}$ is $\mathbb{SL}_2(\mathbb{R})$-invariant.

5. Prove that $\Delta = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ is a self-adjoint operator on $C^\infty(\mathscr{H})$, commuting with the action of $\mathbb{SL}_2(\mathbb{R})$.

6. Describe the finite dimensional unitary representations of $\mathbb{SL}_2(\mathbb{R})$.

7. Give an example of a locally compact unimodular group $G$ and of an irreducible representation $V \in \operatorname{Rep}(G)$ for which the space $V^\infty$ of smooth vectors is not a simple $\mathfrak{g}$-module.

8. Give an example of a locally compact unimodular group $G$ and of a representation $V \in \operatorname{Rep}(G)$ with the property: there is a $\mathfrak{g}$-stable subspace $W \subset V^\infty$ whose closure is not stable under $G$.

9. Let $k \geq 0$ be even and let $d = \dim M_k(\mathbb{SL}_2(\mathbb{Z}))$. Prove that there are unique $f_0, ..., f_{d-1} \in M_k(\mathbb{SL}_2(\mathbb{Z}))$ such that for any $0 \leq i, j \leq d-1$ the coefficient of $q^i$ in the $q$-expansion of $f_j$ is 1 if $i = j$ and 0 otherwise. Prove that the $q$-expansion of $f_j$ has integral coefficients and that any form whose $q$-expansion has integral coefficients is an integral linear combination of the $f_i$.

10. Prove that if $p$ is a prime, then $\Gamma_0(p)$ has two inequivalent cusps.

11. a) Prove that for any prime $p$ and any $k \geq 1$ we have
$$|\mathbb{SL}_2(\mathbb{Z}/p^k\mathbb{Z})| = p^{3k}(1 - p^{-2}).$$

    b) Prove that for any $n \geq 1$ the map $\mathbb{SL}_2(\mathbb{Z}) \to \mathbb{SL}_2(\mathbb{Z}/n\mathbb{Z})$ is surjective and that
$$|\mathbb{SL}_2(\mathbb{Z}/n\mathbb{Z})| = N^3 \prod_{p|N}(1 - p^{-2}).$$

c) Prove that
$$[\mathbb{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p|N} (1+p^{-1}).$$
Hint: introduce an analogue $\Gamma_1(N)$ of $\Gamma_0(N)$ such that $\Gamma_1(N)/\Gamma(N) \simeq \mathbb{Z}/N\mathbb{Z}$ and $\Gamma_0(N)/\Gamma_1(N) \simeq (\mathbb{Z}/N\mathbb{Z})^*$, where $\Gamma(N) = \ker(\mathbb{SL}_2(\mathbb{Z}) \to \mathbb{SL}_2(\mathbb{Z}/n\mathbb{Z}))$.

12. Let $\Gamma = \mathbb{SL}_2(\mathbb{Z})$. Prove that (for $k$ even)
$$\dim S_k(\Gamma) = \max(0, \dim M_k(\Gamma) - 1)$$
and that $f \to f\Delta$ induces an isomorphism $M_k(\Gamma) \simeq S_{k+12}(\Gamma)$. Conclude that $(E_4^a E_6^b)_{4a+6b=k,a,b\geq 0}$ is a basis of $M_k(\Gamma)$ and that $\dim M_k(\Gamma)$ is $[k/12]$ for $k \equiv 2 \pmod{12}$ and $[k/12]+1$ otherwise.

13. Let $G \subset \mathbb{GL}_n(\mathbf{R})$ be a closed subgroup and $V \in \mathrm{Rep}(G)$.

a) Prove that $f.v \in V^\infty$ for $v \in V$ and $f \in C_c^\infty(G)$ and deduce that $V^\infty$ is dense in $V$.

b) Prove that $V^\infty$ is stable under $G$ and that it is a representation of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$.

14. For a modular form $f = \sum_{n\geq 0} a_n q^n \in M_k(\mathbb{SL}_2(\mathbb{Z}))$ define
$$L(f,s) = \sum_{n\geq 1} \frac{a_n}{n^s}.$$
Compute $L(E_k, s)$ in terms of the Riemann zeta function.

15. Prove by hand that $\frac{1}{|\det(x_{ij})|^n} \prod_{i,j=1}^n dx_{ij}$ is a left and right invariant measure on $\mathbb{GL}_n(\mathbf{R})$.

16. Prove that
$$\Delta = \frac{E_4^3 - E_6^2}{1728}$$
and
$$E_{12} - E_6^2 = \frac{2^6 \cdot 3^5 \cdot 7^2}{691}\Delta.$$
Deduce that if $\Delta = \sum_{n\geq 1} \tau(n) q^n$, then
$$\tau(n) \equiv \sigma_{11}(n) \pmod{691}.$$

17. Prove that $E_4^2 = E_8, E_4E_6 = E_{10}, E_6E_8 = E_{14}$. Deduce that
$$\sigma_{13}(n) - 21\sigma_5(n) + 20\sigma_7(n) = 10080 \sum_{k=1}^{n-1} \sigma_5(k)\sigma_7(n-k).$$

18. With the usual notations, prove that the product map $A \times N \times K \to \mathbb{SL}_2(\mathbf{R})$ is a homeomorphism.

19. Let $k \geq 4$ be an even integer and let $G_k$ be the corresponding Eisenstein series for $\mathbb{SL}_2(\mathbb{Z})$. Given a prime $p$, express in terms of $G_k$ the modular form $T_p(G_k)$.

20. Consider the operator

$$D = D_k : \mathscr{O}(\mathscr{H}) \to \mathscr{O}(\mathscr{H}), f \to \frac{1}{2i\pi}\frac{df}{dz} - \frac{k}{12}E_2 f.$$

a) Prove that $D$ induces an operator $D : M_k(\mathbb{SL}_2(\mathbb{Z})) \to M_{k+2}(\mathbb{SL}_2(\mathbb{Z}))$, and that $Df \in S_{k+2}(\mathbb{SL}_2(\mathbb{Z}))$ if and only if $f \in S_k(\mathbb{SL}_2(\mathbb{Z}))$.

b) Compute $D(E_4), D(E_6)$ and show that $D(\Delta) = 0$.

In the next problems $G$ is a unimodular locally compact group.

21. Let $H, H'$ be unitary representations of $G$, with $H$ irreducible. Prove that any $T \in \mathrm{Hom}_G(H, H')$ has closed image and induces an isomorphism between $H$ and a sub-representation of $H'$. Hint: use Schur's lemma.

22. Let $H, H'$ be unitary representations of $G$ such that $H \simeq H'$ in $\mathrm{Rep}(G)$. Prove that there is an isomorphism $U \in \mathrm{Hom}_G(H, H')$ such that $||U(h)|| = ||h||$ for all $h \in H$.

23. Let $K$ be a compact group. Prove that the characters $\phi_\pi$ of elements $\pi \in \hat{K}$ form an ON-basis of $L^2(K)$. Also, a finite dimensional representation $V$ of $K$ is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$.

In the next exercises $H$ is a separable Hilbert space and we use the notations $B(H), HS(H), TC(H)$, etc as in the lecture.

24. Let $T \in HS(H)$ and let $(e_n)$ and $(f_n)_n$ be an ON-bases of $H$. Using the Plancherel formula twice, prove that $\sum_n ||T(e_n)||^2 = \sum_n ||T^*(f_n)||^2$. Deduce that $T^* \in HS(H)$ and that $\sum_n ||T(e_n)||^2$ is independent of the ON-basis $(e_n)_n$.

25. Prove that any $T \in HS(H)$ is compact. Hint: pick an ON-basis $(e_n)$ and consider the operators $T_n(v) = \sum_{k \leq n} \langle v, e_k \rangle T(e_k)$.

26. Let $T \in B(H)$ and $S \in HS(H)$.

a) Prove that $TS, ST \in HS(H)$.

b) If $T \in HS(H)$, prove that $TS, ST \in TC(H)$.

27. In this exercise we will prove that any $T \in TC(H)$ can be written $T = AB$ with $A, B \in HS(H)$.

a) Explain why $T$ is compact and why $\ker(T^*T) = \ker(T)$. Deduce that $\ker(T)^\perp$ has an ON-basis $(v_n)_n$ such that $T^*Tv_n = \lambda_n v_n$ for some $\lambda_n > 0$ tending to 0.

b) Define operators $S, U$ by setting them equal to 0 on $\ker(T)$ and asking that $Sv_n = \sqrt[4]{\lambda_n} v_n$ and $Uv_n = \frac{1}{\sqrt{\lambda_n}} v_n$. Prove that $T = US^2$ and that $||Uv|| = ||v||$ for $v \in \ker(T)^\perp$.

c) Let $(e_n)$ be an ON-basis of $H$ such that $\sum ||Te_n|| < \infty$. Prove that $||Te_n|| \geq ||Se_n||^2$ (use Cauchy-Schwarz) and deduce that $S, U \in HS(H)$. Conclude.

d) Deduce that $\sum ||Tf_n|| < \infty$ for any ON-basis $(f_n)_n$ of $H$.

28. Let $T \in TC(H)$ and let $(e_n)_n$ and $(f_n)_n$ be two ON-bases of $H$.

a) Prove that

$$\sum_k |\langle Te_n, f_k \rangle \langle f_k, e_n \rangle| \leq ||Te_n||$$

and deduce that $\sum_{n,k} |\langle Te_n, f_k \rangle \langle f_k, e_n \rangle| < \infty$.

b) By computing $\sum_{n,k} \langle Te_n, f_k \rangle \langle f_k, e_n \rangle$ in two different ways, prove that

$$\sum_n \langle Te_n, e_n \rangle = \sum_n \langle Tf_n, f_n \rangle.$$

In the problems below $\Gamma$ is a finite index subgroup of $\Gamma(1) := \mathbb{SL}_2(\mathbb{Z})$.

29. Prove that $M(\Gamma) := \sum_{k \in \mathbb{Z}} M_k(\Gamma)$ is a sub-ring of $\mathscr{O}(\mathscr{H})$ and that $S(\Gamma) := \sum_{k \in \mathbb{Z}} S_k(\Gamma)$ is an ideal in $M(\Gamma)$.

30. Prove that the subspaces $M_k(\Gamma)$ (for various integers $k$, but fixed $\Gamma$) are in direct sum in $\mathscr{O}(\mathscr{H})$.

31. Prove that if $f \in M_k(\Gamma)$ and $\alpha \in \mathbb{SL}_2(\mathbb{Z})$, then $f|_k \alpha \in M_k(\alpha^{-1} \Gamma \alpha)$.

32. Prove that $\sum_{(c,d) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{(cz+d)^k}$ converges uniformly on compact subsets of $\mathscr{H}$ for $k \geq 3$, but that this fails for $k = 2$.

33. Let $k \geq 3$ and let $\varphi \in \mathscr{O}(\mathscr{H})$ be an $h$-periodic and bounded function, where $h > 0$ is such that $\Gamma_\infty := \Gamma \cap \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & h\mathbb{Z} \\ 0 & 1 \end{pmatrix}$. Prove that $p_\varphi := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \varphi|_k \gamma$ is well-defined and belongs to $M_k(\Gamma)$. What is $p_\varphi$ for $\varphi$ the constant function 1 and $\Gamma = \mathbb{SL}_2(\mathbb{Z})$?

34. The goal of this exercise is to prove that $G_2(-1/z) = z^2 G_2(z) - 2i\pi z$, where

$$G_2(z) = \sum_{c \in \mathbb{Z}} \left( \sum_{d \in \mathbb{Z}, (c,d) \neq (0,0)} \frac{1}{(cz+d)^2} \right).$$

a) Explain why $\sum_{d \in \mathbb{Z}} \frac{1}{(cz+d)(cz+d+1)} = 0$ for all $c$ and deduce that

$$G_2(z) = \sum_{d \neq 0} \frac{1}{d^2} + \sum_d \sum_{c \neq 0} \frac{1}{(cz+d)^2(cz+d+1)}.$$

b) Show that

$$z^{-2}G_2(-1/z) = \sum_{c\neq 0} \frac{1}{c^2} + \sum_d \sum_{c\neq 0} \frac{1}{(cz+d)^2}$$

c) Conclude that

$$z^{-2}G_2(-1/z) - G_2(z) = \sum_d \sum_{c\neq 0} \frac{1}{(cz+d)(cz+d+1)}.$$

d) Using Euler's identity, show that

$$\sum_d \sum_{c\neq 0} \frac{1}{(cz+d)(cz+d+1)} =$$

$$\lim_{N\to\infty} \sum_{c\neq 0} \left( \frac{1}{cz-N} - \frac{1}{cz+N} \right) = \frac{2i\pi}{z}.$$

35. Let $\varphi \in L^1(G)$ be left and right $K$-finite and $\mathscr{C}$-finite. Prove that the associated Poincaré series $p_\varphi$ is bounded.

36. Prove directly (i.e. without using theorems in the course) that if a discrete subgroup $\Gamma$ of $G$ contains a nontrivial unipotent matrix, then $\Gamma\backslash\mathscr{H}$ is not compact.