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# ON CODIMENSION ONE NILFOLIATIONS AND A THEOREM OF MALCEV

G. HECTOR, E. GHYS and Y. MORIYAMA

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A HOMOGENEOUS space of a (connected) Lie group $G$ is a manifold on which $G$ acts transitively; these manifolds have been studied for many years. A particularly interesting family of non transitive actions is that of actions whose orbits are the leaves of a foliation. If one tries to describe them, one naturally restricts to the codimension 1 case. Here we study foliations defined by locally free actions of nilpotent Lie groups on compact manifolds: the so-called *nilfoliations.*

Our aim is to generalize both

(i) Malcev's structure theorem for *nilmanifolds* (i.e. homogeneous spaces of connected nilpotent Lie groups) which states that any nilmanifold is diffeomorphic to the product of a compact nilmanifold with an Euclidean space (see Theorem B below);

(ii) results of Rosenberg–Roussarie–Weil–Châtelet stating that manifolds foliated by actions of $\mathbb{R}^n$ or Heisenberg groups are trivial cobordisms or fiber over $\mathbb{S}^1$ (see Theorem C).

Indeed a nilfoliation is almost without holonomy (see 4.1) and as for the study of these foliations in general, one can restrict to the simpler situation of *models of nilfoliations* (see 1.4 and 1.5). The most important models $(M, \mathscr{F})$ are those of type

(0) for which $\mathscr{F}$ is without holonomy;

(1) for which $\partial M \neq \emptyset$ and any compact leaf is a connected component of $\partial M$.

Consider $(M, \mathscr{F})$ a model of type 0 or 1, defined by a locally free action $\phi: G \times M \to M$. For any $x \in M$, the leaf $L_x$ through $x$ is isomorphic to $G/G_x$ (where $G_x$ is the isotropy subgroup of $G$ at $x$), and we denote with $\Gamma_x \subset G_x$ the kernel of the holonomy representation of $L_x$. The invariance of the Malcev completion $\hat{\Gamma}_x$ of $\Gamma_x$ (i.e. the unique closed connected subgroup of $G$ such that $\Gamma_x$ is uniform in $\hat{\Gamma}_x$) is a crucial point in the paper. Our results can be summarized as follows:

THEOREM A. *If* $(M, \mathscr{F})$ *is a model of type* 0 *or* 1, *the group* $\hat{\Gamma}_x$ *is a normal subgroup of* $G$ *which does not depend on* $x$ *and such that* $\hat{\Gamma}_x \supset [G, G]$.

THEOREM B. *Let* $(M, \mathscr{F})$ *be any nilfoliation. There exists a closed subgroup* $H$ *of* $G$ *containing* $[G, G]$ *whose action on* $M$ *defines a locally trivial fibration:*

$$(*): \Lambda \to M \xrightarrow{\pi} M' \text{ such that:}$$

(i) $\Lambda$ *is a compact nilmanifold;*

(ii) $\mathscr{F} = \pi^*(\mathscr{F}')$ *where* $\mathscr{F}'$ *is the foliation defined by the induced action* $\phi'$ *of the abelian group* $G' = G/H$.

THEOREM C. *Let $(M, \mathscr{F})$ be a nilfoliation,*

(i) *If $(M, \mathscr{F})$ is a type 1 model, $M$ is a trivial cobordism;*

(ii) *If $M$ is closed, $M$ fibers over $\mathbb{S}^1$, the fiber $F$ being diffeomorphic to the compact nilmanifold of $G$ if $\mathscr{F}$ has non-trivial holonomy.*

*In any case, $M$ has the homotopy type of a (non-compact) solvmanifold.*

In fact we will prove more precise versions of Theorems B and C for models of type 0 and 1 in sections 2 and 3 respectively. The general case will be discussed in §4. The proof of Theorem A for nilfoliations without holonomy requires some preliminary results on nilpotent Lie groups; they will be discussed in Appendix A. For the proof of the "cobordism theorem" (see 3.7.) which is an important step in the proof of Theorem C, we use the corresponding result for $\mathbb{R}^n$-actions due to Châtelet–Rosenberg; we give a new and short proof of the latter for great dimensions in Appendix B.

Results contained in an earlier version of this paper were announced in [6]. Later the paper could be deeply improved and simplified by proving the invariance of $\hat{\Gamma}_x$ in case of type 1 models. Finally the former weaker results concerning type 0 models were strengthened due to the contribution of the two last authors.

Although all results are valid in class $C^r$, $r \geqslant 2$, we will assume for the sake of simplicity that all structures considered in the paper are of class $C^\infty$.

## 1. PRELIMINARIES: MODELS OF NILFOLIATIONS

Let $\phi\colon G \times M \to M$ be a smooth action of a connected Lie group $G$ on a manifold $M$. As usual we write $g(x)$ instead of $\phi(g, x)$ for $(g, x) \in G \times M$ and we denote with $G(x)$ [resp. $G_x$] the *orbit* of $x$ under $\phi$ [resp. the *isotropy group* of $G$ at $x$]. If $\phi$ is *locally free* i.e. $\dim G_x = 0$ for any $x$, the orbits of $\phi$ are the leaves of a foliation $\mathscr{F}$ which is *defined* by $\phi$. The leaf $L_x$ of $\mathscr{F}$ through $x$ is diffeomorphic to $G/G_x$; without loss of generality, we can assume that $G$ is simply connected.

1.1. *Definition.* A foliation $\mathscr{F}$ defined by a locally free action of a connected nilpotent Lie group is a *nilfoliation* and all its leaves are *nilmanifolds.*

A Riemannian manifold $(L, \mathscr{R})$ has *polynomial growth of degree m* if given $x \in L$ there exists $\alpha > 0$ such that:

$$\operatorname{vol} B(x, r) \leqslant \alpha r^m$$

for any $r \in \mathbb{R}^+$ (where $\operatorname{vol} B(x, r)$ is the volume of the closed ball of radius $r$). One shows that this definition depends neither on $x \in L$ nor on the Riemannian metric if $L$ is a leaf in a compact foliated manifold $M$ endowed with the Riemannian metric induced by some metric on $M$ (see [7] t. $B$ for more details).

1.2. PROPOSITION. *Let $(M, \mathscr{F})$ be a codimension 1, transversely orientable nilfoliation on a compact manifold. Then any leaf of $\mathscr{F}$ has polynomial growth and*

*—either $\mathscr{F}$ is without holonomy.*

*—or the closure of any leaf of $\mathscr{F}$ contains a compact leaf.*

*Proof.* Suppose that $\mathscr{F}$ is defined by $\phi\colon G \times M \to M$. Using a right invariant Riemannian metric $\mathscr{R}_G$ on $G$, one constructs a (not unique) Riemannian metric $\mathscr{R}_\phi$ on $M$ which coincides on $L_x = G/G_x$ with the metric induced by $\mathscr{R}_G$. Thus the growth function of $L_x$ is dominated by that of $G$ (with respect to $\mathscr{R}_G$). As any nilpotent Lie group has polynomial growth with respect to any right-invariant metric (see [10]), the growth assertion follows.

Now recall that, $M$ being compact, the closure $\bar{L}$ of any leaf $L$ of $\mathscr{F}$ contains at least one minimal set $K$ and that any such set is of one of the following types:

(i) a compact leaf;
(ii) the manifold $M$ (and then all leaves are dense in $M$);
(iii) an exceptional minimal set.

But it is well known that if some leaf in $K$ has polynomial growth then either $\mathscr{F}$ is without holonomy and $K = M$ or $K$ is a compact leaf (see [7]). The proof is complete. ■

Not every nilpotent Lie group admits a discrete uniform subgroup (see [12]), therefore we get:

1.3. COROLLARY. *Let $(M, \mathscr{F})$ be a transversely orientable codimension 1 nilfoliation on a compact manifold defined by a locally free action of G. If the Lie algebra of G does not admit a rational basis, then $\mathscr{F}$ is without holonomy.*

Next we reduce the description of nilfoliations to that of a nice family of "models".

1.4. *Models of nilfoliations.* A codimension 1, transversely orientable *nilfoliation* $(M, \mathscr{F})$ tangent to the boundary, on a compact manifold is a *model (of nilfoliations) of type* (i) if the corresponding condition is fulfilled:

(0) $\mathscr{F}$ is without holonomy;

(1) $\partial M \neq \varnothing$ and all leaves in $\mathring{M}$ are non compact;

(2) $M$ is a trivial cobordism and $F$ being a component of $\partial M$, there exists a diffeomorphism $\psi: F \times I \to M$ such that $\psi_*(\partial/\partial t)$ is transverse to $\mathscr{F}$.

Examples of the three types of models already exist for $\mathbb{R}^n$-actions. A two-dimensional Reeb component on the annulus $A = \mathbb{S}^1 \times I$ is a model of type 1 which is not of type 2 (for $\mathbb{R}$-actions). The following result is analogous to Theorem 1 in [5]:

1.5. PROPOSITION. *For any codimension 1, transversely orientable nilfoliation $(M, \mathscr{F})$ on a compact manifold, there exist finitely many models $(M_i, \mathscr{F}_i)_{i\in\{1,\dots,q\}}$ such that $(M, \mathscr{F})$ is obtained by gluing these models along boundary components.*

Notice that if all models $(M_i, \mathscr{F}_i)$ are of type 2, then $M$ fibers over $\mathbb{S}^1$ with fiber a compact nilmanifold. One of our main results indeed is that this holds also in the case where some models are of type 1 (see Theorem C).

## 2. NILFOLIATIONS WITHOUT HOLONOMY

As for codimension 1 foliations in general, there are two kinds of nilfoliations without holonomy:

(i) either all leaves are compact and the foliation is a fibration over $\mathbb{S}^1$,
(ii) or all leaves are dense and the foliation is minimal.

We will call them *nilfibrations* and *minimal nilfoliations* respectively.

The following nice family provides examples of nilfoliations of both types:

2.1. *Homogeneous nilfoliations.* Take an extension of connected, simply connected Lie groups

$$0 \to G \to \tilde{G} \to \mathbb{R} \to 0$$

where $G$ is nilpotent, $\tilde{G}$ is solvable and admits a uniform discrete subgroup $\Delta$. Then $G$ acts on the left on the right-homogeneous compact solvmanifold $M = \tilde{G}/\Delta$ and the foliation $\mathscr{F}$ defined by this action is a *homogeneous nilfoliation*. It is a nilfibration or a minimal nilfoliation depending on the fact that $G \cap \Delta$ is uniform in $G$ or not.

In case of nilfoliations without holonomy, the kernel $\Gamma_x$ of the holonomy representation of the leaf $L_x$ coincides with $G_x$. Therefore, in order to prove Theorem A, we start describing the isotropy groups $G_x$ and their Malcev completions $\hat{G}_x$.

The first step will be used also in §3, therefore we consider any transversely orientable nilfoliation $\mathscr{F}$ defined by a locally free action $\phi: G \times M \to M$. Without loss of generality we may assume that $G$ is simply connected, thus $\phi$ lifts to a locally free action $\tilde{\phi}$ of $G$ on the universal covering $\tilde{M}$ of $M$ providing a commutative diagram:

$$\begin{array}{ccc} G \times \tilde{M} & \xrightarrow{\tilde{\phi}} & \tilde{M} \\ {\scriptstyle (id,q)}\downarrow & & \downarrow{\scriptstyle q} \\ G \times M & \xrightarrow{\phi} & M. \end{array}$$

The lifted action $\tilde{\phi}$ defines $\tilde{\mathscr{F}} = q^* \mathscr{F}$. Furthermore as any vanishing cycle of a foliation defined by a group action is trivial, it follows that $\tilde{\mathscr{F}}$ has no closed transversal, and all its leaves are planes; in particular $\tilde{\phi}$ is free (see [7] t. B. chap VII). Next let $\theta: \mathbb{R} \to M$ be an integral curve of some vectorfield $Y$ transverse to $\mathscr{F}$ and $\tilde{\theta}$ a lift of $\theta$ to $\tilde{M}$.

2.2. LEMMA. *The map $\tilde{\phi}_0: G \times \mathbb{R} \to \tilde{M}$ defined by $\tilde{\phi}_0(g,t) = \tilde{\phi}(g, \tilde{\theta}(t))$ is a diffeomorphism of $G \times \mathbb{R}$ onto an open saturated set $\tilde{W}$ of $(\tilde{M}, \tilde{\mathscr{F}})$ such that $\tilde{\phi}_0^* \tilde{\mathscr{F}}$ is the horizontal foliation of $G \times \mathbb{R}$.*

*Proof.* Because $\tilde{\mathscr{F}}$ has no closed transversal, $\tilde{\theta}$ is injective and any pair of leaves of $\tilde{\mathscr{F}}$ and $\tilde{Y} = q^* Y$ cut in at most one point. Thus if $(g_1, t_1)$ and $(g_2, t_2)$ are such that $\tilde{\phi}_0(g_1, t_1) = \tilde{\phi}_0(g_2, t_2)$ we get $t_1 = t_2$ and because $\tilde{\phi}$ is free $g_1 = g_2$; that is $\tilde{\phi}_0$ is injective. But $\tilde{\phi}_0$ is obviously a local diffeomorphism and its image is saturated under $\tilde{\phi}$, thus under $\tilde{\mathscr{F}}$. ∎

In the case where $\mathscr{F}$ is without holonomy we get a "continuous variation" for the groups $\hat{G}_x$.

2.3. *Isotropy groups in nilfoliations without holonomy.* Foliations without holonomy are studied in detail in ([7] t. B. chap. VIII).

(i) They are transversely orientable and the universal covering $\tilde{M} \xrightarrow{q} M$ is trivialized by the lifts $\tilde{\mathscr{F}}$ and $\tilde{Y}$ of $\mathscr{F}$ and $Y$; thus $\tilde{\theta}$ cuts any leaf of $\tilde{\mathscr{F}}$ in exactly one point and $\tilde{\phi}_0$ is a diffeomorphism of $G \times \mathbb{R}$ onto $\tilde{M}$.

(ii) The group of automorphisms $\mathrm{Aut}(q)$ preserves $\tilde{\mathscr{F}}$ and the action of $\mathrm{Aut}(q)$ on $\tilde{M}$ induces a homomorphism

$$\psi: \mathrm{Aut}(q) \to \mathrm{Diff}(\mathbb{R})$$

whose image is topologically conjugate to a finitely generated group of translations. All isotropy groups $G_x$ are isomorphic as well as their completions $\hat{G}_x$ and the corresponding subalgebras $\mathscr{G}_x$ of the Lie algebra $\mathscr{G}$ of $G$. Further all leaves of $\mathscr{F}$ are diffeomorphic to the same nilmanifold.

(iii) Identify $\tilde{M}$ with $G \times \mathbb{R}$ by means of $\phi_0$. For any $\gamma \in \operatorname{Ker}\psi$ and any $z = (g, t) \in G \times \mathbb{R}$, we write $\gamma(z) = (\gamma_1(g, t), t)$ and the map

$$\tilde{\gamma}\colon \tilde{M} \to G$$

defined by $\tilde{\gamma}(z) = \gamma_1(g, t)g^{-1}$ is smooth and has values in $G_{q(z)}$. Furthermore if $k = \dim \mathscr{G}_x$ and if $\{\gamma^{(1)}, \ldots, \gamma^{(k)}\}$ is a set of generators of $\operatorname{Ker}\psi$, the family $\{\exp^{-1}(\tilde{\gamma}^{(i)}(z)\}$ is a smooth family of linearly independent elements in $\mathscr{G}_{q(z)}$. It generates $\mathscr{G}_{q(z)}$ and the map

$$\tilde{\chi}\colon \tilde{M} \to Gr^k(\mathscr{G})$$
$$z \to \mathscr{G}_{q(z)}$$

(where $Gr^k(\mathscr{G})$ is the Grassmann manifold of $k$-planes in $\mathscr{G}$), factorizes through $q$ inducing a triangle of smooth maps:

$$\begin{tikzcd} \tilde{M} \arrow[rr, "\tilde{\chi}"] \arrow[dr, "q"'] & & Gr^k(\mathscr{G}) \\ & M \arrow[ur, "\chi"'] & \end{tikzcd}$$

The next lemma is trivial for nilfibrations.

2.4. LEMMA. *If $\mathscr{F}$ is a nilfoliation without holonomy, $\mathscr{G}_x$ is independent of $x$. In particular $\mathscr{G}_x$ is an ideal of $\mathscr{G}$ and $\hat{G}_x$ is a fixed normal subgroup of $G$.*

*Proof.* With the notations of Appendix A, we get a commutative diagram

$$\begin{tikzcd} G \times M \arrow[r, "\phi"] \arrow[d, "(\mathrm{id}, \chi)"'] & M \arrow[d, "\chi"] \\ G \times Gr^k(\mathscr{G}) \arrow[r, "\overline{Ad}"] & Gr^k(\mathscr{G}) \end{tikzcd}$$

and any minimal set of $\phi$ is mapped onto a minimal set of $\overline{Ad}$. By Theorem 1.1 of Appendix A, $\chi$ is constant for minimal nilfoliations and $\mathscr{G}_x$ as well as $\hat{G}_x$ does not depend on $x \in M$.

Next, notice that the inner conjugate of $\hat{G}_x$ by $g$ equals $\hat{G}_{g(x)}$ for any $g$; as $\hat{G}_x$ is independent of $x$, $\hat{G}_x$ is normal and $\mathscr{G}_x$ is an ideal of $\mathscr{G}$. ■

To derive the main theorems, we will have to deal with the restriction of the action $\phi$ to several normal subgroups $H$ of $G$ such that $G_x$ is uniform in $H$. Next we describe the situation defined by such an action.

2.5. *Induced actions.* Let $H$ be a connected closed subgroup of $G$. The restriction of $\phi$ to $H \times M$ is a locally free action with isotropy group $H_x = G_x \cap H$ in $x \in M$. If $G_x$ is uniform in $H$, it defines a locally trivial fibration

$$(*) \quad \Lambda \to M \xrightarrow{\pi} M'$$

where $\Lambda$ is the well defined compact nilmanifold $H/H_x$.

Furthermore, assuming $H$ is a normal subgroup of $G$, $\phi$ induces also a locally free action

$$\phi'\colon G' \times M' \to M'$$

of the nilpotent Lie group $G' = G/H$ which defines a nilfoliations $\mathscr{F}'$. It is easy to see that $\mathscr{F} = \pi^*(\mathscr{F}')$.

Theorems A and B for foliations without holonomy will be immediate consequences of the following:

2.6. Fibration Theorem. *If $(M, \mathscr{F})$ is a nilfoliation without holonomy, $\hat{G}_x \supset [G, G]$ for any $x \in M$. Moreover the action of $\hat{G}_x$ defines a locally trivial fibration*

$$(*): \quad \Lambda \to M \xrightarrow{\pi} M' \text{ such that:}$$

(i) $\Lambda$ is the compact nilmanifold $\hat{G}_x/G_x$;

(ii) $\mathscr{F} = \pi^* \mathscr{F}'$ where $\mathscr{F}'$ is the foliation by planes defined by the induced action $\phi'$ of the abelian group $G' = G/\hat{G}_x$.

*Proof.* Take $H = \hat{G}_x$ in 2.5. The action $\phi'$ of $G' = G/\hat{G}_x$ on $M'$ is free, $\mathscr{F}'$ is a foliation by planes and the manifold $M'$ is homeomorphic to $T^{s+1}$, $s = \dim G'$ (see [16] and [11]). By classical arguments any leaf of $\mathscr{F}'$ has exactly polynomial growth of degree $s$ (see [7] chap. IX, 2.1.9). This implies that $G'$ too has exactly polynomial growth of degree $s$. From Theorem 2.1. of Appendix A, we see that $G'$ is abelian, that is $\hat{G}_x \supset [G, G]$. The proof is complete. ∎

2.7. Theorem. *Any closed manifold $M$ which supports a nilfoliation without holonomy $\mathscr{F}$ has the homotopy type of a solvmanifold. Moreover $M$ fibers over $\mathbb{S}^1$.*

*Proof.* Because $\mathscr{F}$ is without holonomy, any integral curve of a transverse vector field $Y$ cuts any leaf of $\mathscr{F}$; therefore the map $\tilde{\phi}_0 \colon G \times \mathbb{R} \to \tilde{M}$ introduced in 2.2 is a diffeomorphism and $M$ is an Eilenberg–McLane space.

Now the exact homotopy sequence of the fibration $(*)$ (see 2.6):

$$0 \to G_x \to \pi_1(M, x) \to \mathbb{Z}^{s+1} \to 0$$

shows that $\pi_1(M, x)$ is strongly polycyclic and that its nilradical ${}^n\pi_1(M, x)$ is as follows:

$$[\pi_1(M, x), \pi_1(M, x)] \subset G_x \subset {}^n\pi_1(M, x).$$

Thus $\pi_1(M, x)$ fulfills condition $C$ of Ragunathan (see [15] p. 70) and $\pi_1(M, x)$ is a discrete subgroup of a connected, simply connected, solvable Lie group; that is $M$ is a solvmanifold.

Finally, if $\mathscr{F}$ is minimal, $M$ fibers over $\mathbb{S}^1$ by Tischler's theorem (see [19]). ∎

## 3. TYPE 1 MODELS OF NILFOLIATIONS

Let $(M, \mathscr{F})$ be a type 1 model of nilfoliations defined by a locally free action $\phi \colon G \times M \to M$ and transversely oriented by a vectorfield $Y$. Let $F$ be a connected component of $\partial M$ on which $Y$ points inwards; the fibration we are looking for in this section (see 3.6.), will be an extension of the "holonomy fibration" of $F$ which we describe first.

3.1. *The holonomy fibration of $F$.* By definition of type 1 models, $F$ is an isolated compact leaf thus has a non-trivial right-holonomy group $\mathrm{hol}^+(F)$. Moreover, if

$$0 \to \Gamma_x \to G_x \to \mathrm{hol}^+(F) \to 0$$

is the right-holonomy representation of $F$ at $x$ with kernel $\Gamma_x$, then $\mathrm{hol}^+(F)$ is abelian according to Lemma LI-A1 of [2] and topologically conjugate to a non-trivial linear representation by Lemma 1 of [13].

Let $s$ be the rank of $\mathrm{hol}^+(F)$. From the Malcev extension theorem, we get the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Gamma_x \arrow[r] \arrow[d] & G_x \arrow[r, "\mathrm{hol}^+(F)"] \arrow[d] & \mathbb{Z}^s \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \hat{\Gamma}_x \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & \mathbb{R}^s \arrow[r] \arrow[d] & 0 \\
 & \hat{\Gamma}_x/\Gamma_x \arrow[r] & G/G_x \arrow[r] & T^s &
\end{tikzcd}$$

where $\hat{\Gamma}_x$ is the Malcev completion of $\Gamma_x$. The bottom line is a locally trivial fibration of $F$ over $T^s$ with fiber $\Lambda = \hat{\Gamma}_x/\Gamma_x$. We call it the *holonomy fibration* of $F$.

Exactly as in 2.2, we construct a map $\tilde{\phi}_0 : G \times \mathbb{R}^+ \to \tilde{M}$ by taking $\theta : \mathbb{R}^+ \to M$ the positive orbit of $Y$ through the base point $x \in F$. It is a diffeomorphism of $G \times \mathbb{R}^+$ onto an open $\tilde{\mathscr{F}}$-saturated set $\tilde{W} \subset \tilde{M}$ whose image by $q : \tilde{M} \to M$ is an open saturated neighborhood $W$ of $F$ in $M$. The foliation $\tilde{\phi}_0^* \tilde{\mathscr{F}}$ is the horizontal foliation of $G \times \mathbb{R}^+$ and it follows easily that $\tilde{W}$ is a connected component of the saturated set $q^{-1}(W)$. Thus we have the diagram:

$$\begin{tikzcd}
G \times \mathbb{R}^+ \arrow[rr, "\tilde{\phi}_0"] \arrow[dr, "q \circ \tilde{\phi}_0"'] & & \tilde{W} \arrow[dl, "q"] \\
 & W &
\end{tikzcd}$$

and $q : \tilde{W} \to W$ is the universal covering of $W$.

3.2. *Holonomy maps for $F$.*

(i) Take $g \in G_x = \pi_1(F, x)$ represented by a loop $\sigma$ in $F$ at the point $x$. There exist $\varepsilon > 0$ and a well defined map

$$\hat{\sigma} : [0, 1] \times [0, \varepsilon] \to W \subset M$$

such that for any $u \in [0, 1]$ and any $t \in [0, \varepsilon]$, we get:

(a) $\hat{\sigma}(u, 0) = \sigma(u)$
(b) $\hat{\sigma}(0, t) = \theta(t)$
(c) the restriction $\hat{\sigma}_t$ of $\hat{\sigma}$ to $[0, 1] \times \{t\}$ is the unique vertical lift of $\sigma$ to the leaf $L_{\theta(t)}$ at the point $\theta(t)$.

Then the holonomy element $\mathrm{hol}^+(g)$ is the germ of the local "holonomy map"

$$\begin{aligned} \mathrm{hol}_g : \mathbb{R}^+ &\longrightarrow \mathbb{R}^+ \\ t &\longmapsto \theta^{-1} \hat{\sigma}(1, t). \end{aligned}$$

(ii) Let $\tilde{\sigma}$ be the lift of $\hat{\sigma}$ to $G \times \mathbb{R} \simeq \tilde{W}$ at the point $(0, 0)$. If $g_t$ is the horizontal component of $\tilde{\sigma}(1, t)$, we have the following properties:

(a) $g_0 = g$ and $\mathrm{hol}_g(t) = \theta^{-1}[g_t \cdot \theta(t)]$ for $t \in [0, \varepsilon]$;
(b) $g \in \Gamma_x$ if and only if $g_t \in G_{\theta(t)}$ for small $t$;
(c) if $k$ is any other element of $G_x$, then $(gk)_t = g_{\mathrm{hol}_k(t)} \cdot k_t$ for small $t$.

Now consider $G = G^0 \supset G^1 \supset \ldots \supset G^l \supset G^{l+1} = 0$ with $G^{j+1} = [G, G^j]$ the central series of the nilpotent group $G$ with *length* $l(G) = l$. The subgroup $G^l$ is the center $Z$ of $G$. The following lemma is crucial.

3.3. LEMMA. *If $g$ belongs to $\Gamma_x \cap Z$ then $g_t = g_0 = g$ for small $t$.*

*Proof.* As the holonomy group of $F$ is conjugate to a non-trivial group of linear transformations, there exists $k \in G_x = \pi_1(F, x)$ such that $\mathrm{hol}_k$ is a contraction on a neighborhood of 0. Further, for $g \in \Gamma_x \cap Z$, $gk = kg$ and $\mathrm{hol}_g(t) = t$ for small $t$. Thus applying (c) above we get:

$$g_{\mathrm{hol}_k(t)} \cdot k_t = (gk)_t = (kg)_t = k_{\mathrm{hol}_g(t)} \cdot g_t = k_t \cdot g_t;$$
$$g_{\mathrm{hol}_k(t)} = k_t \cdot g_t \cdot (k_t)^{-1},$$

where the right hand side is a product in the group $G$.

Let $m$ be the biggest integer for which there exists a neighborhood $V_m$ of 0 in $\mathbb{R}^+$ such that $g_t \in G^m$ for any $t \in V_m$. Then $G^m/G^{m+1}$ is the center of the nilpotent Lie group $G/G^{m+1}$, the classes $\bar{k}_t$ and $\bar{g}_t$ commute in $G/G^{m+1}$ and we get:

$$\bar{g}_t = \bar{g}_{\mathrm{hol}_k(t)} \quad \text{for any } t \text{ small enough.}$$

Because $\mathrm{hol}_k$ is a contraction, the continuous function $\bar{g}_t$ is locally constant and

$$\bar{g}_t = \bar{g}_0 = \bar{g} \quad \text{for small } t.$$

Our next claim is that $m = l$ and thus $g_t \in Z$ for small $t$. Indeed for $m < l$, $g$ would belong to $G^{m+1}$ and so would $g_t$ contradicting the definition of $m$.

Finally notice that if $m = l$, $G^{m+1} = \{0\}$ and the previous argument shows that $g_t = g$ for small $t$. ∎

We reach to global results:

3.4. PROPOSITION. *For any $y \in M$, the group $Z_y = G_y \cap Z$ is a discrete subgroup of $Z$ which does not depend on $y \in M$.*

Furthermore, if $G$ is not abelian, $Z_y$ is uniform in $Z$.

*Proof.* First notice that $Z_y$ is the same for all points $y$ of a fixed leaf $L \in \mathcal{F}$; we denote it with $Z_L$. Next suppose that $F$ is a compact leaf contained in the closure $\bar{L}$ of $L$; by continuity, we get $Z_L \subset Z_F$ and from 3.3, it follows that $Z_L = Z_F$. Now $Z_y$ is constant on the open neighborhood $V(F)$ of $F$ defined by the union of all leaves whose closure contains $F$.

Because the only minimal sets are the boundary leaves $F_1, F_2, \ldots, F_r$ it follows that the open sets $V(F_1), \ldots, V(F_r)$ cover $M$; by connectedness they intersect and $Z_y$ is constant on $M$. Finally as the holonomy group of any compact leaf $F$ is abelian, $Z_x$ is uniform in $Z$ for $x \in F$; this proves the proposition. ∎

As in section 2, we use "induced actions" in the sense of 2.5. for the proof of the main theorems:

3.5. THEOREM. *Let $(M, \mathcal{F})$ be a type* 1 *model of transversely orientable nilfoliations. Then*

(i) *$\mathcal{F}$ is almost without holonomy;*

(ii) *for any $y$, $\Gamma_y$ is uniform in $[G, G]$, and $\hat{\Gamma}_y$ is a normal subgroup of $G$ which does not depend on $y \in M$.*

*Proof.* The proof goes by induction on the length $l(G)$ of the central series of $G$.

If $l(G) = 0$, $G$ is abelian, (i) is proved in [3] and (ii) reduces to 3.4. So assume the assertions are true for any nilfoliation defined by a nilpotent Lie group whose central series has length less than or equal to $l - 1$ and consider $\mathscr{F}$ defined by an action $\phi$ of a group $G$ such that $l(G) = l$.

By 3.4., $\hat{Z}_y = Z$ for any $y$, so take $H = Z$ in 2.5. The group $G' = G/Z$ is such that $l(G') = l - 1$ and $\mathscr{F}'$ is almost without holonomy by the induction hypothesis. Furthermore $\pi$ is a principal torus bundle with group $T = Z/Z_y$; $\mathscr{F}$ is almost without holonomy and for any leaf $L_y$ we have a principal fibration:

$$T \to L_y \to L'_{\pi(y)}$$

with homotopy exact sequence:

$$0 \to Z_y \to G_y \to G'_{\pi(y)} \to 0.$$

The holonomy representation of $L_y$ factorizes through $\pi$ providing an exact sequence of kernels:

$$0 \to Z_y \to \Gamma_y \to \Gamma'_{\pi(y)} \to 0.$$

By 3.4 and the induction hypothesis, $Z_y$ and $\Gamma'_{\pi(y)}$ are uniform in $Z$ and $[G', G']$ respectively; thus $\Gamma_y$ is uniform in $[G, G]$ and $\hat{\Gamma}'_{\pi(y)}$ containing $[G, G]$ is a normal subgroup of $G$ which does not depend on $y$. The proof is complete. ■

Theorem B is again proved by restricting $\phi$ to $H = \hat{\Gamma}_y$. It provides a fibration of $M$ which extends the various holonomy fibrations of its boundary components:

3.6. Fibration Theorem. *Let $(M, \mathscr{F})$ be a type 1 model of transversely orientable nilfoliations. The action of $H = \hat{\Gamma}_y$ defines a locally trivial fibration:*

$$(*)\colon\ \Lambda \to M \xrightarrow{\pi} M' \text{ such that}$$

(i) *$\Lambda$ is a compact nilmanifold;*

(ii) *$\mathscr{F} = \pi^*(\mathscr{F}')$ where $\mathscr{F}'$ is the foliation defined by the induced action $\phi'$ of $\mathbb{R}^s = G/\hat{\Gamma}_y$, all of whose non compact leaves are planes.*

(iii) (*) *reduces to the holonomy fibration on each connected component of $\partial M$.*

Recall that the manifold $M'$ in 3.6., is a trivial cobordism $T^s \times [0, 1]$ according to Châtelet–Rosenberg (see [3]). Thus we get the following version of Theorem C:

3.7. Cobordism Theorem. *Let $(M, \mathscr{F})$ be a type 1 model of transversely orientable nilfoliations. Then $\partial M$ has exactly two components $F, F'$ and $(F, M, F')$ is a smooth trivial cobordism.*

*Proof.* Consider the fibration (*) of 3.6 and choose an adapted bundle metric on $M$. Let $X'$ be a vectorifield on $M'$ which trivializes $M'$; if $X$ is the unique lift of $X'$ orthogonal to the fibres of (*), then $X$ trivializes $M$. ■

In Appendix B we will give a new proof of 3.7 in the abelian case for great dimensions.

3.8. *Remark.* As noticed in §.1, it is not always possible to choose $X$ transverse to $\mathscr{F}$ (see for example the two-dimensional Reeb component).

## 4. GENERAL NILFOLIATIONS

For a general nilfoliation, it is no longer true that $\hat{\Gamma}_x$ is independent of $x$. Indeed for a compact isolated leaf $F$ it may happen that the two holonomy representations of $F$ (on the right and on the left) do not agree. Nevertheless the following weak version of Theorem A holds:

4.1. THEOREM. *Let $(M, \mathscr{F})$ be a transversely orientable nilfoliation on a compact manifold. Then*

(i) *$\mathscr{F}$ is almost without holonomy;*
(ii) *$\hat{\Gamma}_x \supset [G, G]$ for any $x \in M$ and $\Gamma_x$ is uniform in $[G, G]$ if $\mathscr{F}$ has non trivial holonomy.*

*Proof.* It is well known that the union $K$ of all compact leaves of $\mathscr{F}$ is closed (see [7] chap. V). Thus if the leaf $L_x$ of $\mathscr{F}$ through $x$ is non-compact, the connected component $C_x$ of $x$ in $M - K$ is an open saturated subset of $M$ which contains $L_x$. Its closure $\bar{C}_x$ is saturated; thus cutting along the compact leaves contained in the interior of $\bar{C}_x$, we obtain a nilfoliation $\hat{\mathscr{F}}$ on a compact manifold $\hat{M}$ all of whose compact leaves are contained in the boundary of $\hat{M}$. Then $(\hat{M}, \hat{\mathscr{F}})$ is a type 1 model (resp. a type 0 model if $K = \emptyset$), we apply 2.6 and 3.5. ■

As a consequence, we prove Theorem B in the remaining case of general nilfoliations with non trivial holonomy by considering the induced action of $H = [G, G]$ in the sense of 2.5. Next we get Theorem C in this same case.

4.2. *Proof of Theorem C.* According to 2.7 and 3.7, it remains only to consider the case where $\mathscr{F}$ is not a nilfibration but has at least one compact leaf $F$. Then by 1.5, there are finitely many models $(M_i, \mathscr{F}_i)_{i \in \{1, \dots, q\}}$ of type 1 or 2 such that $M$ is obtained by gluing the $M_i$'s along boundary components. Moreover each $M_i$ is a trivial cobordism; by definition for models of type 2 and by 3.7 for models of type 1. Thus all boundary components are isomorphic to $F$ and $M$ fibers over $\mathbb{S}^1$ with fiber isomorphic to $F$. We conclude as in 2.7. ■

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*Université Cl. Bernard*
*UA au CNRS* 746
69622 *Villeurbanne-Cedex*
*France*

*Université des Sciences et Techniques*
*de Lille I – U.E.R. de Mathématiques*
*Pures et Appliquées*
*U.A. au C.N.R.S.* 751
59655 - *Villeneuve d'Ascq Cedex, France*

*Department of Mathematics*
*Hokkaido University*
*Sapporo* 060, *Japan.*

## APPENDIX A: SOME PROPERTIES OF NILPOTENT LIE GROUPS.

(*1*) ***The adjoint action of a nilpotent Lie group* G *on the Grassmann manifold* $\mathrm{Gr}^k(\mathscr{G})$.**

Let $\mathscr{G}$ be the Lie algebra of a connected Lie group $G$ and let $Gr^k(\mathscr{G})$ be the Grassmann manifold of $k$-planes in $\mathscr{G}$. The adjoint representation Ad of $G$ extends to a smooth action

$$\overline{\mathrm{Ad}}: G \times Gr^k(\mathscr{G}) \to Gr^k(\mathscr{G})$$

by $\overline{\mathrm{Ad}}(g, W) = \mathrm{Ad}(g)(W)$ for any $k$-dimensional subspace $W$ of $\mathscr{G}$.

1.1. THEOREM. *Any minimal set $m$ of $\overline{\mathrm{Ad}}$ is a fixed point.*

First consider a nilpotent linear map $f: V \to V$ of a finite dimensional oriented vector space $V$ and define a flow

$$\alpha: \mathbb{R} \times Gr^k(V) \to Gr^k(V)$$

by $\alpha(t, W) = e^{tf}(W)$ on the Grassmann manifold of $V$. One has the following preliminary result:

1.2. LEMMA. *The $\omega$-limit set of any orbit of $\alpha$ is a fixed point.*

*Proof.* (a) First suppose that $k = 1$. For any non-zero element $v$ of $V$ there exists a non-negative integer $r$ such that $f^{r+1}(v) = 0$ and $f^r(v) \neq 0$. If $[v]$ denotes the equivalence class of $v$ in the projective space of $V$, we have:

$$e^{tf}(v) = \sum_{i=0}^{r} \frac{t^i}{i!} f^i(v),$$

thus

$$\alpha(t, [v]) = \left[ \sum_{i=0}^{r} \frac{r!}{i!\, t^{r-i}} f^i(v) \right],$$

and

$$\lim_{t \to +\infty} \alpha(t, [v]) = [f^r(v)]$$

which is fixed by $\alpha$.

(b) For $k > 1$, we consider the canonical embedding

$$\psi\colon Gr^k(V) \longrightarrow Gr^1(\bigwedge^k V)$$

$$W \longmapsto [e_1 \wedge e_2 \wedge \ldots \wedge e_k]$$

where $\bigwedge^k V$ is the $k$th-exterior power of $V$ and $(e_1, e_2, \ldots, e_k)$ is a positive frame in $V$. Then $\psi(Gr^k(V))$ is closed in $Gr^1(\bigwedge^k V)$; the map $\hat{f}\colon \wedge^k V \to \wedge^k V$ defined by

$$\hat{f}(v_1 \wedge \ldots \wedge v_k) = \sum_{i=1}^{k} v_1 \wedge \ldots \wedge f(v_i) \wedge \ldots \wedge v_k)$$

is nilpotent and $\psi$ conjugates $\alpha$ with the flow defined by $e^{t\hat{f}}$ on $\psi(Gr^k(V))$. The lemma follows applying (a). ∎

1.3. *Proof of the Theorem.* Let $G = G^0 \supset G^1 \supset \ldots \supset G^l \supset G^{l+1} = 0$ with $G^{j+1} = [G, G^j]$ be the central series of $G$; we will prove by induction that $G^j$ acts trivially on $\mathscr{m}$ for any $j$. This assertion being trivial for $G^{l+1}$, assume it holds for $G^{q+1}$ and let $g \in G^q$. If $X = \exp^{-1}(g)$, $\overline{\mathrm{Ad}}(g)$ is the time 1 map of the flow $\alpha$ on $Gr^k(\mathscr{G})$ generated by $f = \mathrm{ad}\ X$. Of course $\mathscr{m}$ is invariant by this flow and it follows from 1.2 that there is a point $u \in \mathscr{m}$ which is fixed by $\overline{\mathrm{Ad}}(g)$. Then $h$ being any element of $G$ the commutator $[g, h] \in G^q$ and from the induction hypothesis we get:

$$\overline{\mathrm{Ad}}\,(gh)(u) = \overline{\mathrm{Ad}}\,(hg)\,\overline{\mathrm{Ad}}\,([g, h])(u) = \overline{\mathrm{Ad}}\,(hg)(u)$$
$$\overline{\mathrm{Ad}}\,(gh)(u) = \overline{\mathrm{Ad}}\,(h)\,\overline{\mathrm{Ad}}\,(g)(u) = \overline{\mathrm{Ad}}\,(h)(u).$$

This means that $\overline{\mathrm{Ad}}(g)$ is the identity on the $G$-orbit of $u$ which is dense in $\mathscr{m}$. Therefore $g$ acts trivially on $\mathscr{m}$. ∎

### (2) *A remark on growth of nilpotent Lie groups.*

Let $G$ be a connected simply connected nilpotent Lie group of dimension $n$ and let $\mathscr{R}$ be a fixed right invariant Riemannian metric on $G$. It is well known that $G$ has polynomial growth with respect to $\mathscr{R}$ (see [10]). i.e. there exist $\alpha \in \mathbb{R}^+$ and $m \in \mathbb{N}$ such that the growth function of $G$ verifies $gr_G(r) = \mathrm{vol}\ B(0, r) \leqslant \alpha r^m$ for any $r \in \mathbb{R}^+$ (see 1.1).

In particular $gr_G(r) = \alpha r^n$ if $G$ is abelian (see [21]) and we get the following lower bound for the growth of non-abelian nilpotent groups:

2.1. THEOREM. *The nilpotent simply connected Lie group $G$ has polynomial growth of degree $n = \dim G$ if and only if $G$ is abelian.*

This result will be an immediate consequence of the following lemmas:

2.2 LEMMA. *Let $\mathscr{G}$ be a nilpotent Lie algebra of* dim $n \geqslant 4$. *If $\mathscr{G}$ is not abelian, it admits a codimension* 1, *non-abelian ideal $\mathscr{N}$.*

*Proof.* If $\mathscr{G}^1 = [\mathscr{G}, \mathscr{G}]$ is not abelian or if $\dim(\mathscr{G}/\mathscr{G}^1) \geqslant 3$, the kernel of any non trivial Lie algebra homomorphism $\psi\colon \mathscr{G} \to \mathbb{R}$ is a codimension 1 non-abelian ideal.

The result holds also if $\mathscr{G}^1$ is abelian and codim $\mathscr{G}^1 = 2$. Indeed if, in this case, any codimension 1 ideal would be abelian, any $X \in \mathscr{G}$ would act trivially on $\mathscr{G}^1$ and therefore $\mathscr{G}^1$ would be 1-dimensional. But this is impossible because $\dim \mathscr{G} \geqslant 4$.

Finally we claim that if $\mathscr{G}^1$ is abelian then codim $\mathscr{G}^1$ is indeed bigger than or equal to 2. For if codim $\mathscr{G}^1 = 1$, any $X \in \mathscr{G}$, $X \notin \mathscr{G}^1$, would act on $\mathscr{G}^1$ as an $(n-1)$-matrix both nilpotent and regular: this is impossible. ∎

2.3. LEMMA. *Let $G$ be a connected, simply connected nilpotent non-abelian Lie group of dimension $n$. There exists $\alpha > 0$ such that:*

$$gr_G(r) \geqslant \alpha r^{n+1} \quad \text{for any } r.$$

*Proof.* First recall that there exists exactly one such group of dimension $\leqslant 3$: the Heisenberg group $H$ which is of dimension 3 and admits a uniform discrete subgroup $H_0$. It is well known that $H$ and $H_0$ have the same growth type (see [7]) and according to theorem 3.2 of [21] the latter dominates a polynomial of degree 4.

Now we argue by induction on $n$. We assume that the proposition is true for $n = k$ and take $G$ of $\dim(k+1)$. We fix a subalgebra $\mathscr{N}$ of the Lie algebra of $G$ as in 2.2 and an element $X \in \mathscr{G}$ orthonormal to $\mathscr{N}$. A right invariant Riemannian metric $\mathscr{R}$ on $G$ is bundle-like with respect to the trivial fibration

$$N \to G \xrightarrow{\pi} \mathbb{R}$$

where $N$ is the Lie subgroup corresponding to $\mathscr{N}$. The flow $(\varphi_t)$ generated by $X$ trivializes $\pi$ and if $B_N(0, r)$ [resp. $B_G(0, r)$] is the ball of radius $r$ in $N$ [resp. $G$], we get:

$$C(r) = \bigcup_{t \in [-r, +r]} \varphi_t(B_N(0, r)) \subset B_G(0, 2r) \quad \text{for any } r \in \mathbb{R}^+.$$

Furthermore there is a decomposition $v = w \wedge \lambda$ of the Riemannian volume element on $G$, where $w$ is a volume element on $N$ and $\lambda \in \Lambda^1 \mathscr{G}$ is dual to $X$ with respect to $\mathscr{R}$. Then $w$, $\lambda$ and $v$ are invariant by $(\varphi_t)$ and from Fubini's theorem it follows easily that

$$gr_G(2r) \geqslant \int_{C(r)} v \wedge w = 2r \cdot \mathrm{vol}\, B_N(0, r) = 2r \cdot gr_N(r).$$

The result follows. ∎

## APPENDIX B: THE COBORDISM THEOREM FOR TYPE 1 MODELS OF $\mathbb{R}^n$-ACTIONS.

Here we give an alternative proof of the cobordism theorem for $\mathbb{R}^n$-actions which was used in the proof of 3.7. Our proof does not work for $n = 4$, but it extends to "foliations by planes" (not necessarily defined by actions).

(1) THEOREM. Let $M^{n+1}$ be a compact manifold with boundary and $\mathscr{F}$ a transversely orientable foliation on $M$, tangent to $\partial M$ and such that any leaf in $\mathring{M}$ is diffeomorphic to $\mathbb{R}^n$.

Then $\partial M$ has exactly two components $F$, $F'$ and $(F, M, F')$ is a $s$-cobordism.

We proceed in two steps:

(2) LEMMA. *Let $F$ be a connected component of $\partial M$, the natural embedding $j$ of $F$ in $M$ is a homotopy equivalence.*

*Proof.* Any connected component of $\partial M$ is a torus $T^n$ and it is not difficult to put it transverse to $\mathscr{F}$. Cutting along these transverse tori, we obtain on $M$ a foliation $\bar{\mathscr{F}}$ without holonomy, transverse to $\partial M$ with simply connected leaves. We can assume, up to topological conjugacy, that $\bar{\mathscr{F}}$ is defined by a closed form $\omega$ and therefore $\pi_1(M)$ is free abelian, isomorphic to the group $\mathrm{Per}(\omega)$ of periods of $\omega$(see [7]). Using a transverse flow preserving $\partial M$, one shows that $\bar{\mathscr{F}}$ lifts to the universal covering of $M$ as a trivial product foliation. This implies that $\mathrm{Per}(\omega)$ coincides with the group of periods of the restriction of $\omega$ to any boundary component. Then the homomorphism

$$j_*: \pi_1(F) \to \pi_1(M)$$

is-onto and because any vanishing cycle of $\mathscr{F}$ is trivial, $j_*$ is an isomorphism.

On the other hand, the universal covering of $\mathring{M}$ is homeomorphic to $\mathbb{R}^{n+1}$ therefore $\pi_i(M) = 0$ for $i > 0$ and $j$ is a homotopy equivalence. ∎

(3) *Proof of Theorem* 1. Consider the exact homology sequence with real coefficients of the pair $(M, \partial M)$:

$$\to H_1(\partial M) \xrightarrow{j_*} H_1(M) \to H_1(M, \partial M) \to H_0(\partial M) \to H_0(M) \to 0.$$

According to Lemma 2, $j_*$ is onto and we obtain a short exact sequence:

$$0 \to H_1(M, \partial M) \to H_0(\partial M) \to H_0(M) \to 0.$$

By Poincaré–Lefschetz duality and §2, $H_1(M, \partial M) \cong H^n(M) \cong H^n(F) \cong \mathbb{R}$, thus $H_0(\partial M) \cong \mathbb{R} \oplus \mathbb{R}$ and $\partial M$ has exactly two components $F, F'$. The Whitehead group of a free abelian group is zero according to [1] and $(F, M, F')$ is a $s$-cobordism. ■

Then by the s-cobordism theorem of Barden–Mazur–Stallings (see [14]), we get

(4) THEOREM. *Let $(M, \mathscr{F})$ be as in* §1. *For $n > 4$, the triple $(F, M, F')$ is a smooth trivial cobordism.*

Further using Stallings' fibration theorem and some results of Hatcher (see [4]), one could easily extend theorem 4 to the case $n \leqslant 3$.