INTRODUCTION.

Consider a non-singular flow $f_t$ of class $C^2$ on a compact manifold $M$ and denote by $X$ the corresponding vector field. Recall that $f_t$ is an "Anosov flow" if there is a splitting of the tangent bundle $TM$ as a sum of the line field $\mathbb{R}X$ and two $df_t$-invariant sub-bundles $E^{ss}$ and $E^{uu}$ in such a way that vectors of $E^{ss}$ (resp. $E^{uu}$) are exponentially contracted (resp. expanded) by $df_t$ as $t$ goes to $+\infty$ (see for instance [1-3]). When $E^{uu}$ is one dimensional, one says that $f_t$ is a codimension one Anosov flow (by reversing the time, one might as well assume that $E^{ss}$ is one dimensional). These codimension one Anosov flows have been investigated by A. Verjovsky in [17] where he shows in particular that they are transitive as soon as the dimension of $M$ is bigger than 3 (this is no longer true in dimension 3, as shown in [7]). Moreover, A. Verjovsky conjectured that if the fundamental group of $M$ is solvable, then the flow $f_t$ must admit a global cross-section. Recall that such a global cross-section is a codimension one submanifold $\Sigma$ which cuts transversally every orbit of $f_t$. In such a situation, the flow $f_t$ can be reconstructed (by "suspension") from the first return map $f : \Sigma \to \Sigma$ and the first return time $t : \Sigma \to ]0, +\infty[$. Observe that, by a result of S. Newhouse [13] and J. Franks [6], codimension one Anosov diffeomorphisms are topologically conjugated to hyperbolic automorphisms of tori $T^n$. Therefore, Verjovsky's conjecture implies a classification of codimension one Anosov flows on manifolds with solvable fundamental groups.

This conjecture has been proven by P. Armandariz when $\dim M = 3$ and $f_t$ is transitive [2] and by J. Plante in the general case [15] (see also [16] for the end of the proof).
Obviously, the hypothesis on the fundamental group is necessary since the geodesic flow of a negatively curved compact surface provides an example of a codimension one Anosov flow with no global cross-section. However, this example is 3-dimensional and A. Verjovsky told us that he knew no higher dimensional example. The purpose of this paper is to prove two results that suggest that such an example might not exist.

**CONJECTURE.** Let \( f_t \) be a codimension one Anosov flow on a compact manifold of dimension bigger than 3. Then \( f_t \) admits a global cross-section.

We are going to prove this conjecture under some additional assumptions related to the smoothness of the sub-bundles \( E^{ss} \) and \( E^{uu} \). Recall first of all some facts (see [10]):

Usually, \( E^{ss} \) and \( E^{uu} \) are only (Hölder) continuous sub-bundles of \( TM \). However, they are uniquely integrable and define foliations \( F^{ss} \) and \( F^{uu} \) (called strongly stable and strongly unstable respectively). These foliations are absolutely continuous. In the same way, \( E^{ss} \oplus \mathbb{R}X \) and \( E^{uu} \oplus \mathbb{R}X \) also define foliations \( F^s \) and \( F^u \) (called, respectively, center stable and center unstable foliations).

In the codimension one case (\( \dim E^{uu} = 1 \)), the hyperplane field \( E^{ss} \oplus \mathbb{R}X \) turns out to be of class \( C^{1+\varepsilon} \) for some \( \varepsilon > 0 \). If, moreover, \( f_t \) is volume preserving and \( \dim M > 4 \), it is shown in [10] that the line field \( E^{uu} \) is of class \( C^{1+\varepsilon} \) (see also [14]). This last result is not true when \( \dim M = 3 \) by a theorem of J. Plante that we recall below. However, if \( f_t \) is of class \( C^\infty \), volume preserving, and \( \dim M = 3 \), the smoothness of \( F^s \) has been precisely studied in [11]; it is of class \( C^{1+\varepsilon} \) for every \( 0 < \varepsilon < 1 \). (Actually, the result in [11] is slightly better: the modulus of continuity of the first derivative can be chosen of the form \( \omega(s) = -ks\log s \).)

We can now state our first result.

**THEOREM 1.** Let \( f_t \) be a \( C^2 \) codimension one Anosov flow on a compact manifold of dimension bigger than 3. Assume that \( f_t \) is volume preserving and that the center stable foliation is of class \( C^2 \). Then \( f_t \) admits a global cross-section.
Observe that this theorem is invariant under reparametrization of the flow. Indeed, let $u : M \to \mathbb{R}$ be a smooth positive function. Then the flow $g_t$ associated to the vector field $uX$ is also Anosov and has the same center stable foliation. If $f_t$ preserves the volume form $\Omega$ then $g_t$ preserves $(1/u)\Omega$. However $f_t$ and $g_t$ do not have the same strong stable foliation unless $u$ is constant. The fact that $E^{ss}$ and $E^{uu}$ depend on the parametrization is illustrated by the following theorem of J. Plante [14]. Suppose $f_t$ is a $C^2$ Anosov flow on a compact 3-dimensional manifold and assume that $E^{ss} \oplus E^{uu}$ is of class $C^1$ and that $f_t$ admits a global cross-section. Then $f_t$ admits a global cross-section with constant return time. Note that this result is not true if we don’t assume that $f_t$ admits a global cross-section. Indeed, if $f_t$ is the geodesic flow of a $C^\infty$-negatively curved surface, it is easy to check that $E^{ss} \oplus E^{uu}$ is the orthogonal to $X$ in the natural metric of the unit tangent bundle and, therefore, $E^{ss} \oplus E^{uu}$ is of class $C^\infty$. Here, we prove a similar result in dimension bigger than 3 but we don’t assume the existence of a global section.

**THEOREM 2.** Let $f_t$ be a $C^2$-codimension one Anosov flow on a compact manifold of dimension bigger than 3. Assume that the hyperplane field $E^{ss} \oplus E^{uu}$ is of class $C^1$. Then $f_t$ admits a global cross-section with constant return time.

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1. PROOF OF THEOREM 1.

We begin by a homological characterization of flows with section.

**PROPOSITION 1.1.** Let $f_t$ be a transitive codimension one Anosov flow on a compact manifold $M$. Then $f_t$ admits a global cross-section if and only if no periodic orbit of $f_t$ is homologous to zero.

**PROOF.** The fact that the condition is necessary is clear and well known. Any section of $f_t$ determines a fibration $\pi$ of $M$ over the circle $S^1$ whose fibers are transverse to the flow. In particular, the inverse image by $\pi$ of the fundamental class...
of $S^1$ is non zero on periodic orbits of $f_t$. Hence, these periodic orbits can't be homologous to zero.

Consider the universal covering space $\tilde{M}$ of $M$ equipped with the lifted co-dimension one foliation $\tilde{F}^S$. It is shown in [17] that $\tilde{M}$ is diffeomorphic to a euclidean space $\mathbb{R}^n$ in such a way that the leaves of $\tilde{F}^S$ are the hyperplanes $\mathbb{R}^{n-1} \times \{s\}$. In particular, the leaf space of $\tilde{F}^S$ is $\mathbb{R}$ and the fundamental group $\Gamma$ of $M$ acts naturally on this leaf space $\mathbb{R}$. It is well known that a leaf of $F^S$ is diffeomorphic either to a cylinder $\mathbb{R}^{n-2} \times S^1$ if it contains a periodic orbit of $f_t$ or to a plane $\mathbb{R}^{n-1}$ if it does not contain such an orbit. More precisely, let $\gamma$ be an element of $\Gamma$. Then the action of $\gamma$ on $\mathbb{R}$ has a fixed point if and only if $f_t$ has a periodic orbit which is freely homotopic to $\gamma$. Under the assumption that no periodic orbit of $f_t$ is homologous to zero in $M$, we deduce that the first commutator group $[\Gamma, \Gamma]$ of $\Gamma$ acts on $\mathbb{R}$ without fixed point. Now, this implies that $[\Gamma, \Gamma]$, as any fixed point free group of homeomorphisms of $\mathbb{R}$, is Abelian (see for instance [9]). In particular, $\Gamma$ is solvable and the conclusion of the proposition follows from the already mentioned solution of Verjovsky's conjecture by J. Plante.

Before we can use this proposition, we recall some facts coming from foliation theory.

First of all, consider a foliation $F$ on a manifold $M$. We say that $F$ is of class $C^r$ if it can be defined by a $C^r$-foliated atlas. The tangent bundle to a $C^r$-foliation is usually a $C^{r-1}$ plane field (as a section of the Grassman bundle). However, it is shown in [8] (see also [5]) that, in any case, $F$ is $C^r$-conjugated to a $C^r$-foliation $F'$ for which the tangent bundle is of class $C^r$. For example, if $F$ is a (transversally orientable) codimension one foliation of class $C^2$, then we can always assume, (after conjugating, if necessary, by a $C^2$ diffeomorphism) that $F$ is defined by a non singular form of degree 1 and class $C^2$.

Now, let $\omega$ be such a $C^2$-form of degree 1 defining $F$. According to Frobenius' theorem, there exists a $C^1$-form $\eta$ of degree 1 such that $d\omega = \omega \wedge \eta$. It is well
known (and easy to prove [9]) that the restriction of $\eta$ to any leaf of $F$ is closed. Moreover, if $\gamma$ is a closed loop contained in a leaf of $F$, then the integral of $\eta$ along $\gamma$ is the logarithm of the absolute value of the linear part of the holonomy of $F$ along $\gamma$.

In case the foliation $F$ is not transversally orientable, it can still be defined by a non singular "odd differential form" $\omega$ of degree 1, i.e. by a differential form defined up to sign (see [4] for this notation). Note that, in this case, $d\omega$ has the same sign ambiguity and, therefore, there exists a usual (or "even") form $\eta$ for which $d\omega = \omega \wedge \eta$.

We now begin the proof of theorem i. We consider, as in the theorem, a codimension one Anosov flow $f_t$ on a compact manifold $M$ of dimension bigger than 3. We assume that $f_t$ is volume preserving and that the center stable foliation $F^s$ is of class $C^2$. As we have seen, we can assume that $F^s$ is defined by a non singular form $\omega$ of class $C^2$ (may be odd) and there exists a $C^1$-form $\eta$ such that $d\omega = \omega \wedge \eta$. Suppose, by contradiction, that there exists a periodic orbit $\gamma$ of $f_t$ which is homologous to zero. Then, it would be possible to find a compact oriented surface $S$ with one component in its boundary and a smooth map $i : S \to M$ such that $i(\partial S) = \gamma$. Note that $f_t \circ i$ also satisfies $f_t \circ i(\partial S) = \gamma$ and we can therefore use Stoke's theorem:

$$\int_{\gamma} \eta = \int_{\partial S} i^*(d\eta) = \int_{\partial S} i^*f_t^*(d\eta).$$

The left hand side of this equality is strictly positive since it is equal to the logarithm of the linear part of the holonomy of $F^s$ along $\gamma$ which is bigger than 1 since $d\eta \exp \omega$ expands $E^{uu}$.

We are going to find a contradiction in (1) by showing that the right hand side goes to zero as $t$ tends to $-\infty$.

**Lemma 1.2.** Let $E_1$ (resp. $E_2$) be a euclidean $(n-1)$-dimensional vector space, written as an orthogonal sum $U_1 \oplus S_1$ (resp. $U_2 \oplus S_2$) where $U_1$ (resp. $U_2$) is one dimensional. Let $f : E_1 \to E_2$ be a linear mapping satisfying the following properties:
1) \( f(S_1) = S_2; \ f(U_1) = U_2 \)

2) \( \det f = 1 \).

Let \( \mu \) be a constant such that \( \|f(v)\| \leq \mu\|v\| \) for every vector \( v \) of \( S_1 \). Then, for every vector \( v_u \) of \( U_2 \) and \( v_s \) of \( S_2 \), one has:

\[
\|f^{-1}(v_u \wedge v_s)\| \leq \mu^{n-3}\|v_u \wedge v_s\|
\]

where \( \| \\) denotes also the natural norm on exterior powers \( \Lambda^2(E_1) \) and \( \Lambda^2(E_2) \).

**PROOF.** Let \( w_u \in U_1 \) and \( w_s \in S_1 \) be two unit vectors. Choose vectors \( e_1, e_2, \ldots, e_{n-3} \) such that \( w_{u}, w_{s}, e_1, \ldots, e_{n-3} \) is an orthonormal basis of \( E_1 \).

Estimate the determinant of \( f \) by the Gram-Schmidt inequality:

\[
1 = \det f \leq \|f(w_u)\| \|f(w_s)\| \prod_{i=1}^{n-3} \|f(e_i)\|
\]

\[
\leq \|f(w_u)\| \|f(w_s)\| \mu^{n-3}.
\]

Now,

\[
\|f(w_u \wedge w_s)\| = \|f(w_u)\| \|f(w_s)\| \geq \mu^{-(n-3)}.
\]

In other words, \( f \) expands unit bivectors of the form \( w_u \wedge w_s \) by at least a factor of \( \mu^{-(n-3)} \). The lemma immediately follows.

\[\square\]

We can now finish the proof of theorem 1 by finding a contradiction in (1).

Choose a \( C^0 \)-Riemannian metric on \( M \) for which:

1) \( X \) has length 1

2) \( E^{ss} \oplus E^{uu} \oplus RX \) is an orthogonal splitting.

3) The given \( f_t \) invariant volume form is the Riemannian volume.

Consider the linear mapping \( df_t \) restricted to \( E^{ss} \oplus E^{uu} \) at a given point \( x \). By definition of an Anosov flow, this linear mapping satisfies the hypothesis of lemma 1.2 for a constant \( \mu \) of the form \( C\nu^\tau \) (\( C > 0 \) and \( 0 < \nu < 1 \)). Therefore,
one gets, for every $v_s \in E^{ss}$ and $v_u \in E^{uu}$

$$\|df_{-t}(v_s \wedge v_u)\| \leq C(n-3)v_s \wedge v_u$$

If, as we always assume, $n \geq 4$, this provides a uniform convergence to zero as $t$ goes to $+\infty$. Note that $\Lambda_2(TM)$ is the orthogonal sum of $\Lambda_2(E^{ss} \oplus \mathbb{R}X)$ and of the vector space of bivectors of the form $v_s \wedge v_u$. We know that the 2-form $\eta$ vanishes on $\Lambda_2(E^{ss} \oplus \mathbb{R}X)$. Therefore, the above inequality shows that for every bivector $w$ of $\Lambda_2(TM)$, one has:

$$|f_{-t}^* \eta(w)| = |\eta(df_{-t}(w))| \leq C(n-3)v_s \wedge v_u$$

This obviously implies the required contradiction in (1) since the right hand side converges to zero as $t$ goes to $-\infty$. This finishes the proof of theorem 1.

2. PROOF OF THEOREM 2.

We consider a codimension one Anosov flow $f_t$ on a compact manifold $M$ of dimension bigger than 3 and we assume, as in theorem 2, that the distribution $E^{ss} \oplus E^{uu}$ is of class $C^1$. Consider the differential form $\alpha$ of degree 1 which is equal to 0 on $E^{ss} \oplus E^{uu}$ and to 1 on the vector field $X$ associated to $f_t$. Obviously, $\alpha$ is invariant under $f_t^*$ and, therefore, $\beta = d\alpha$ is a 2-form of class $C^0$ which is also invariant under $f_t^*$. If we could show that $\beta$ is identically 0, that would mean that $E^{ss} \oplus E^{uu}$ is integrable and a theorem of J. Plante ([14], theorem 3.7) would imply that $f_t$ has a global cross-section with constant return time. Note that J. Plante proves the existence of such a section of class $C^1$; however it would be easy to deduce from [12] that this section is actually very smooth (of class $C^\infty$ if $f_t$ is $C^\infty$, and at least of class $C^{r-n/2}$ if $f_t$ is of class $C^2$ and dim $M = n$).

In order to prove the theorem, we are led to prove that the form $\beta$ vanishes. The following lemma shows that it suffices to study the restriction of $\beta$ to $E^{ss} \oplus E^{uu}$.

**LEMMA 2.1.** The vector field $X$ lies in the kernel of the 2-form $\beta$. 
PROOF. Consider first a vector \( v \) belonging to \( E^{ss} \). By invariance of \( \beta \), we have:

\[
\beta(X,v) = \beta(d\!f_t(X),d\!f_t(v)) = \beta(X,d\!f_t(v)) .
\]

The continuity of \( \beta \), the compactness of \( M \) and the fact that \( d\!f_t(v) \) goes to zero as \( t \) goes to \( +\infty \), show that:

\[
\beta(X,v) = 0 .
\]

Reversing the time, we see in the same way that \( \beta(X,v) = 0 \) if \( v \) belongs to \( E^{uu} \). In other words, \( X \) lies in the kernel of \( \beta \).

We shall analyse the situation at a periodic point using the following lemma which is analogous to lemma 1.1.

LEMMA 2.2. Let \( E \) be a \((n-1)\)-dimensional euclidean vector space \((n \geq 3)\) written as an orthogonal sum \( S \oplus U \) where \( U \) is one dimensional. Let \( f : E \to E \) be a linear mapping and \( \beta \) a non trivial skew symmetric bilinear form on \( E \). Suppose that:

1) \( S \) and \( U \) are invariant under \( f \).

2) \( \beta \) is invariant under \( f \).

3) There is a constant \( \mu \) such that \( 0 < \mu < 1 \) and for which one has \( \|f(v)\| \leq \mu \|v\| \) for \( v \in S \). Then,

\[
|\det f| \leq \mu^{n-3} .
\]

In particular, if \( n \geq 4 \), one has \( |\det f| < 1 \).

PROOF. Note first of all that, if \( v_1 \) and \( v_2 \) are vectors of \( S \), one has:

\[
\beta(v_1,v_2) = \beta(f^n(v_1),f^n(v_2)) \to 0 \quad \text{as} \quad n \to \infty
\]

Choose a unit vector \( v \) in \( U \) and let \( \lambda \) be such that \( f(v) = \lambda v \). Taking into account the fact that \( \beta \) is non trivial, one sees that the linear form:

\[
\ell : x \in S \to \beta(v,x) \in \mathbb{R}
\]
is non zero. Obviously, this linear form satisfies:

\[ \ell(f(x)) = \frac{1}{\lambda} \ell(x) \]  

(2)

Consider a unit cube \( C = [0,1]^{n-2} \) in \( S \) whose base \([0,1]^{n-3} \times \{0\}\) is contained in the kernel of \( \ell \). The image of \( C \) by \( f \) is a parallelepiped whose base \( f([0,1]^{n-3} \times \{0\}) \) has \((n-3)\)-volume bounded by \( \mu^{n-3} \) (by Gram-Schmidt inequality) and whose height is precisely \( 1/|\lambda| \) by (2). Therefore, the volume of \( f(C) \) is at most \( (1/|\lambda|)\mu^{n-3} \). Recalling that \( f(v) = \lambda v \), that \( S \) and \( U \) are orthogonal and invariant under \( f \), one gets:

\[ |\det f| \leq |\lambda| |1/\lambda| \mu^{n-3} = \mu^{n-3}. \]

In theorem 2, we do not assume that \( f_t \) is volume preserving. However, if we add this hypothesis, the proof is now easy:

**COROLLARY 2.3.** If \( f_t \) satisfies the hypothesis of theorem 2 and preserves some volume form, then \( f_t \) is a constant time suspension.

**PROOF.** If \( x \) is a periodic point of \( f_t \) of period \( T > 0 \), let \( E = E^{ss}_x \oplus E^{uu}_x \), \( S = E^{ss}_x \), \( U = E^{uu}_x \) and \( f = (df_t)_k : E \to E \). If \( k \) is sufficiently big, then the conditions 1), 2), 3) of the previous lemma are satisfied for some euclidean structure on \( E \). If the 2-form \( \beta \) were non zero at \( x \), we would get \( |\det f| < 1 \) contradicting the assumption that \( f_t \) is volume preserving. Using the continuity of \( \beta \) and the density of periodic points of \( f_t \) (dim \( M \geq 4 \)), we deduce that \( \beta \) vanishes everywhere. As we have already mentioned, this implies that \( f_t \) is a constant time suspension.

In the remaining part of this paragraph, we get rid of the condition that \( f_t \) is volume preserving.

We still assume, by contradiction, that \( \beta \) is not identically zero and we denote by \( U \) the non empty \( f_t \)-invariant open set consisting of points where \( \beta \) is non zero.
If $M$ is orientable, choose any volume form $\Omega$ on $M$ and denote by $u : M \to \mathbb{R}$ the divergence of $X$ with respect to $\Omega$, i.e., the function such that $\mathcal{L}_X \Omega = u \Omega$ where $\mathcal{L}_X$ denotes the Lie derivative. If $M$ is not orientable, we can choose an "odd" volume form $\tilde{\Omega}$ and remark that there is still a function $u$ such that $\mathcal{L}_X \tilde{\Omega} = u \tilde{\Omega}$. According to lemma 2.2, we know that if $x$ is a periodic point of $f_t$ of period $T > 0$ and if $x$ is in $U$, then:

$$\frac{1}{T} \int_0^T u(f_t(x)) \, dt < c < 0$$

for some constant $c$ independent of $x$ and $T$. Suppose for a moment that we prove the following two lemmas:

**Lemma 2.4.** Every $f_t$-invariant probability measure on $M$ can be approximated by convex combinations of invariant probabilities concentrated on periodic orbits contained in $U$.

**Lemma 2.5.** Let $u : M \to \mathbb{R}$ be a smooth function such that $\int u \, d\mu < 0$ for every $f_t$-invariant probability measure $\mu$ on $M$. Then there exists a smooth function $v : M \to \mathbb{R}$ such that $u + X(v) < 0$. Here, $X(v)$ denotes the derivative of $v$ in the direction of $X$.

Assuming these two lemmas, we can finish the proof of theorem 2 which reduces, as we have seen, to the fact that the open set $U$ cannot be non-empty. Indeed, assuming that $U \neq \emptyset$, lemma 2.4 and (3) would imply that $\int u \, d\mu < 0$ for every $f_t$-invariant probability measure. In turn, lemma 2.5 would imply that there is a smooth function $v$ such that $u + X(v) < 0$. Consider now the volume form $\tilde{\Omega}' = \exp(v) \tilde{\Omega}$. Then the divergence of $X$ with respect to $\tilde{\Omega}'$ is given by the formula:

$$\mathcal{L}_X \tilde{\Omega}' = \exp(v) X(v) \tilde{\Omega} + \exp(v) u \tilde{\Omega}$$

$$= (u + X(v)) \tilde{\Omega}'$$

This is the required contradiction since the negativity of $u + X(v)$ would imply that the measure associated to $\tilde{\Omega}'$ is contracted. But the total mass of $M$ has to be preserved by the flow $f_t$. 

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We are therefore left with the proof of lemmas 2.4 and 2.5.

PROOF OF LEMMA 2.4. This lemma is not typical to our situation but is valid for any transitive Anosov flow (we have seen that this condition is satisfied for codimension one Anosov flows in dimension bigger than 3).

Recall first of all how to approximate an ergodic invariant probability measure \( \mu \) by a probability measure concentrated on a periodic orbit. Consider a \( \mu \)-regular point \( x \), i.e., a point for which, for every continuous function \( h : M \to \mathbb{R} \), one has:

\[
\int h \, d\mu = \lim_{T \to \infty} \frac{1}{T} \int_0^T h(f_t(x)) \, dt
\]

Such a point is obviously recurrent and we can use Anosov's closing lemma (see [3]) to produce periodic orbits \( \gamma_k \) that approximate long pieces of orbits \( f_t(x) \), \( t \in [0,T_k] \). It is clear that the sequence of probability measures supported by \( \gamma_k \) converges to the given measure \( \mu \).

Now, the compact convex set of all invariant probability measures is the closed convex hull of ergodic invariant measures (Krein-Millman). In order to prove the lemma it suffices to show that the invariant probability measures concentrated on periodic orbits can be approximated by those contained in \( U \).

This can easily be done using symbolic dynamics. Under our assumption, the flow is transitive and therefore, there exists a Markov partition for \( f_t \) ([3]). We can even suppose that one of the boxes of the Markov partition is contained in \( U \). To this partition corresponds a finite connected graph \( G \) whose vertices are the boxes. A closed loop in this graph corresponds to a periodic orbit of \( f_t \). Let \( \gamma \) be such a closed loop originated at the vertex \( p \) of \( G \) and let \( \gamma^k \) denote the same loop iterated \( k \) times. Let \( p_0 \) be a vertex of \( G \) corresponding to a box contained in \( U \) and choose a path \( \delta \) (resp. \( \delta' \)) from \( p_0 \) to \( p \) (resp. \( p \) to \( p_0 \)). Then the sequence \( \delta \gamma^k \delta' \) represents a loop at \( p_0 \) and therefore represents a periodic orbit of \( f_t \) which is contained in \( U \). It is clear that the sequence of invariant probability measures \( \mu_k \) concentrated on these orbits converges to the invariant probabi-
ty measure concentrated on the periodic orbit corresponding to \( \gamma \).

\[ \square \]

**Proof of Lemma 2.5.** This lemma is true for any smooth flow \( f_t \) on any compact manifold \( M \), independently of the Anosov property.

We consider a function \( u : M \to \mathbb{R} \) such that \( \int u \, d\mu < 0 \) for every \( f_t \)-invariant measure \( \mu \). We claim that for \( T > 0 \) big enough, the function \( u_T \) defined by

\[
u_T(x) = \frac{1}{T} \int_0^T u(f_t(x)) \, dt
\]

is negative. Indeed suppose that there exists a sequence \( x_n \) in \( M \) and a sequence \( T_n \) going to \( +\infty \) such that \( u_T(x_n) \geq 0 \). Consider the probability measure \( \mu_n \) uniformly concentrated on the piece of orbit from \( x_n \) to \( f_T(x_n) \). Then any weak limit of the sequence \( \mu_n \) would be an invariant probability measure \( \mu \) for which \( \int u \, d\mu \geq 0 \) contrary to our assumption.

In order to prove the lemma, it is therefore sufficient to construct a function \( v_T \) such that:

\[
u + X(v_T) = u_T
\]

Consider the following two probability measures on \( \mathbb{R} \). The first one is \( \delta_0 \), the Dirac mass at the point 0 and the second one \( \gamma_T \) is uniformly distributed on \([0, T]\). Then, the difference \( \gamma_T - \delta_0 \) is derivative (in the sense of distributions) of the function

\[
t \in \mathbb{R} \rightarrow (\gamma_T - \delta_0)(] - \infty, t[) = \left(\frac{t}{T} - 1\right) \quad \text{if} \quad 0 \leq t \leq T
\]

\[= 0 \quad \text{otherwise.}\]

In other words, we get the following formula for every smooth function \( \phi : \mathbb{R} \to \mathbb{R} \):

\[
\int_0^T \left(1 - \frac{t}{T}\right) \phi'(t) \, dt = \frac{1}{T} \int_0^T \phi(t) \, dt - \phi(0).
\]

It is now clear that the following function \( v_T \) will satisfy (4):

\[
v_T(x) = \int_0^T \left(1 - \frac{t}{T}\right) u(f_t(x)) \, dt.
\]

\[ \square \]
REFERENCES.


