CODIMENSION ONE ANOSOV FLOWS AND SUSPENSIONS

by

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INTRODUCTION.

Consider a non singular flow f_t of class C^2 on a compact manifold M and denote by X the corresponding vector field. Recall that f_t is an "Anosov flow" if there is a splitting of the tangent bundle TM as a sum of the line field $\mathbb{R}X$ and two df_t-invariant sub-bundles E^{ss} and E^{uu} in such a way that vectors of E^{ss} (resp. E^{uu}) are exponentially contracted (resp. expanded) by df_t as t goes to $+\infty$ (see for instance [1-3]). When E^{uu} is one dimensional, one says that f_t is a codimension one Anosov flow (by reversing the time, one might as well assume that E^{ss} is one dimensional). These codimension one Anosov flows have been investigated by A. Verjovsky in [17] where he shows in particular that they are transitive as soon as the dimension of M is bigger than 3 (this is no longer true in dimension 3, as shown in [7]). Moreover, A. Verjovsky conjectured that if the fundamental group of M is solvable, then the flow f_t must admit a global cross-section. Recall that such a global cross-section is a codimension one submanifold Σ which cuts transversally every orbit of f_t. In such a situation, the flow f_t can be reconstructed (by "suspension") from the first return map $f : \Sigma \to \Sigma$ and the first return time $t : \Sigma \to]0,\infty[$. Observe that, by a result of S. Newhouse [13] and J. Franks [6], codimension one Anosov diffeomorphisms are topologically conjugated to hyperbolic automorphisms of tori T^n . Therefore, Verjovsky's conjecture implies a classification of codimension one Anosov flows on manifolds with solvable fundamental groups.

This conjecture has been proven by P. Armandariz when $\dim M = 3$ and f_t is transitive [2] and by J. Plante in the general case [15] (see also [16] for the end of the proof).

Obviously, the hypothesis on the fundamental group is necessary since the geodesic flow of a negatively curved compact surface provides an example of a codimension one Anosov flow with no global cross-section. However, this example is 3-dimensional and A. Verjovsky told us that he knew no higher dimensional example. The purpose of this paper is to prove two results that suggest that such an example might not exist.

CONJECTURE. Let f_t be a codimension one Anosov flow on a compact manifold of dimension bigger than 3. Then f_t admits a global cross-section.

We are going to prove this conjecture under some additional assumptions related to the smoothness of the sub-bundles E^{ss} and E^{uu}. Recall first of all some facts (see [10]):

Usually, E^{ss} and E^{uu} are only (Hölder) continuous sub-bundles of TM. However, they are uniquely integrable and define foliations $\mathcal{F}^{ss}$ and $\mathcal{F}^{uu}$ (called strongly stable and strongly unstable respectively). These foliations are absolutely continuous. In the same way, $E^{ss} \oplus \mathbb{R}X$ and $E^{uu} \oplus \mathbb{R}X$ also define foliations $\mathcal{F}^{s}$ and $\mathcal{F}^{u}$ (called, respectively, center stable and center unstable foliations).

In the codimension one case $(\dim E^{uu} = 1)$, the hyperplane field $E^{ss} \oplus \mathbb{R}X$ turns out to be of class $C^{1+\varepsilon}$ for <u>some</u> $\varepsilon > 0$. If, moreover, f_t is volume preserving and $\dim M \geq 4$, it is shown in [10] that the line field E^{uu} is of class $C^{1+\varepsilon}$ (see also [14]). This last result is not true when $\dim M = 3$ by a theorem of J. Plante that we recall below. However, if f_t is of class C^{∞}, volume preserving, and $\dim M = 3$, the smoothness of $\mathcal{F}^{s}$ has been precisely studied in [11]; it is of class $C^{1+\varepsilon}$ for <u>every</u> $0 < \varepsilon < 1$. (Actually, the result in [11] is slightly better: the modulus of continuity of the first derivative can be chosen of the form $\omega(s) = -ks\mathrm{Log}\, s)$.

We can now state our first result.

THEOREM 1. Let f_t be a C^2 codimension one Anosov flow on a compact manifold of dimension bigger than 3. Assume that f_t is volume preserving and that the center stable foliation is of class C^2. Then f_t admits a global cross-section.

Observe that this theorem is invariant under reparametrization of the flow. Indeed, let $u : M \to \mathbb{R}$ be a smooth positive function. Then the flow g_t associated to the vector field uX is also Anosov and has the same center stable foliation. If f_t preserves the volume form Ω then g_t preserves $(1/u)\Omega$. However f_t and g_t do not have the same strong stable foliation unless u is constant. The fact that E^{ss} and E^{uu} depend on the parametrization is illustrated by the following theorem of J. Plante [14]. Suppose f_t is a C^2 Anosov flow on a compact 3-dimensional manifold and assume that $E^{ss} \oplus E^{uu}$ is of class C^1 and that f_t admits a global cross-section. Then f_t admits a global cross-section with <u>constant</u> return time. Note that this result is not true if we don't assume that f_t admits a global cross section. Indeed, if f_t is the geodesic flow of a C^∞-negatively curved surface, it is easy to check that $E^{ss} \oplus E^{uu}$ is the orthogonal to X in the natural metric of the unit tangent bundle and, therefore, $E^{ss} \oplus E^{uu}$ is of class C^∞. Here, we prove a similar result in dimension bigger than 3 but we don't assume the existence of a global section.

THEOREM 2. Let f_t be a C^2-codimension one Anosov flow on a compact manifold of dimension bigger than 3. Assume that the hyperplane field $E^{ss} \oplus E^{uu}$ is of class C^1. Then f_t admits a global cross-section with <u>constant</u> return time.

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1. PROOF OF THEOREM 1.

We begin by a homological characterization of flows with section.

PROPOSITION 1.1. Let f_t be a transitive codimension one Anosov flow on a compact manifold M. Then f_t admits a global cross-section if and only if no periodic orbit of f_t is homologous to zero.

PROOF. The fact that the condition is necessary is clear and well known. Any section of f_t determines a fibration π of M over the circle S^1 whose fibers are transverse to the flow. In particular, the inverse image by π of the fundamental class of S^1 is non zero on periodic orbits of f_t. Hence, these periodic orbits can't be homologous to zero.

Consider the universal covering space $\widetilde{M}$ of M equipped with the lifted codimension one foliation $\widetilde{F}^s$. It is shown in [17] that $\widetilde{M}$ is diffeomorphic to a euclidean space $\mathbb{R}^n$ in such a way that the leaves of $\widetilde{F}^s$ are the hyperplanes $\mathbb{R}^{n-1} \times \{*\}$. In particular, the leaf space of $\widetilde{F}^s$ is $\mathbb{R}$ and the fundamental group Γ of M acts naturally on this leaf space $\mathbb{R}$. It is well known that a leaf of F^s is diffeomorphic either to a cylinder $\mathbb{R}^{n-2} \times S^1$ if it contains a periodic orbit of f_t or to a plane $\mathbb{R}^{n-1}$ if it does not contain such an orbit. More precisely, let γ be an element of Γ. Then the action of γ on $\mathbb{R}$ has a fixed point if and only if f_t has a periodic orbit which is freely homotopic to γ. Under the assumption that no periodic orbit of f_t is homologous to zero in M, we deduce that the first commutator group $[\Gamma,\Gamma]$ of Γ acts on $\mathbb{R}$ without fixed point. Now, this implies that $[\Gamma,\Gamma]$, as any fixed point free group of homeomorphisms of $\mathbb{R}$, is Abelian (see for instance [9]). In particular, Γ is solvable and the conclusion of the proposition follows from the already mentioned solution of Verjovsky's conjecture by J. Plante.

□

Before we can use this proposition, we recall some facts coming from foliation theory.

First of all, consider a foliation F on a manifold M. We say that F is of class C^r if it can be defined by a C^r-foliated atlas. The tangent bundle to a C^r-foliation is usually a C^{r-1} plane field (as a section of the Grassman bundle). However, it is shown in [8] (see also [5]) that, in any case, F is C^r-conjugated to a C^r-foliation F' for which the tangent bundle is of class C^r. For example, if F is a (transversally orientable) codimension one foliation of class C^2, then we can always assume, (after conjugating, if necessary, by a C^2 diffeomorphism) that F is defined by a non singular form of degree 1 and class C^2.

Now, let ω be such a C^2-form of degree 1 defining F. According to Frobenius' theorem, there exists a C^1-form η of degree 1 such that $d\omega = \omega \wedge \eta$. It is well known (and easy to prove [9]) that the restriction of η to any leaf of $\mathcal{F}$ is closed. Moreover, if γ is a closed loop contained in a leaf of $\mathcal{F}$, then the integral of η along γ is the logarithm of the absolute value of the linear part of the holonomy of $\mathcal{F}$ along γ.

In case the foliation $\mathcal{F}$ is not transversally orientable, it can still be defined by a non singular "odd differential form" ω of degree 1, i.e. by a differential form defined up to sign (see [4] for this notation). Note that, in this case, $d\omega$ has the same sign ambiguity and, therefore, there exists a usual (or "even") form η for which $d\omega = \omega \wedge \eta$.

We now begin the proof of theorem 1. We consider, as in the theorem, a codimension one Anosov flow f_t on a compact manifold M of dimension bigger than 3. We assume that f_t is volume preserving and that the center stable foliation $\mathcal{F}^s$ is of class C^2. As we have seen, we can assume that $\mathcal{F}^s$ is defined by a non singular form ω of class C^2 (may be odd) and there exists a C^1-form η such that $d\omega = \omega \wedge \eta$. Suppose, by contradiction, that there exists a periodic orbit γ of f_t which is homologous to zero. Then, it would be possible to find a compact oriented surface S with one component in its boundary and a smooth map $i : S \to M$ such that $i(\delta S) = \gamma$. Note that $f_t \circ i$ also satisfies $f_t \circ i(\delta S) = \gamma$ and we can therefore use Stoke's theorem:

$$(1) \qquad \int_\gamma \eta = \int_S i^*(d\eta) = \int_S i^* f_t^*(d\eta) .$$

The left hand side of this equality is strictly positive since it is equal to the logarithm of the linear part of the holonomy of $\mathcal{F}^s$ along γ which is bigger than 1 since df_t expands E^{uu}.

We are going to find a contradiction in (1) by showing that the right hand side goes to zero as t tends to $-\infty$.

LEMMA 1.2. Let E_1 (resp. E_2) be a euclidean $(n-1)$-dimensional vector space, written as an orthogonal sum $U_1 \oplus S_1$ (resp. $U_2 \oplus S_2$) where U_1 (resp. U_2) is one dimensional. Let $f : E_1 \to E_2$ be a linear mapping satisfying the following properties:

1) $f(S_1) = S_2$; $f(U_1) = U_2$

2) $\det f = 1$.

Let μ be a constant such that $\|f(v)\| \le \mu\|v\|$ for every vector v of S_1 . Then, for every vector v_u of U_2 and v_s of S_2 , one has:

$$\|f^{-1}(v_u \wedge v_s)\| \le \mu^{n-3}\|v_u \wedge v_s\|$$

where $\| \ \|$ denotes also the natural norm on exterior powers $\Lambda_2(E_1)$ and $\Lambda_2(E_2)$.

PROOF. Let $w_u \in U_1$ and $w_s \in S_1$ be two unit vectors. Choose vectors $e_1, e_2, \ldots, e_{n-3}$ such that $w_u, w_s, e_1, \ldots, e_{n-3}$ is an orthonormal basis of E_1 . Estimate the determinant of f by the Gram-Schmidt inequality:

$$1 = \det f \le \|f(w_u)\| \|f(w_s)\| \prod_{i=1}^{n-3} \|f(e_i)\|$$

$$\le \|f(w_u)\| \|f(w_s)\| \ \mu^{n-3} .$$

Now,

$$\|f(w_u \wedge w_s)\| = \|f(w_u)\| \|f(w_s)\| \ge \mu^{-(n-3)} .$$

In other words, f expands unit bivectors of the form $w_u \wedge w_s$ by at least a factor of $\mu^{-(n-3)}$. The lemma immediately follows.

□

We can now finish the proof of theorem 1 by finding a contradiction in (1). Choose a C°-Riemannian metric on M for which:

1) X has length 1

2) $E^{ss} \oplus E^{uu} \oplus \mathbb{R}X$ is an orthogonal splitting.

3) The given f_t invariant volume form is the Riemannian volume.

Consider the linear mapping df_t restricted to $E^{ss} \oplus E^{uu}$ at a given point x . By definition of an Anosov flow, this linear mapping satisfies the hypothesis of lemma 1.2 for a constant μ of the form $C\nu^t$ $(C > 0$ and $0 < \nu < 1)$. Therefore, one gets, for every $v_s \in E^{ss}$ and $v_u \in E^{uu}$

$$\|df_{-t}(v_s \wedge v_u)\| \leq C^{(n-3)}\ \nu^{(n-3)t}\|v_s \wedge v_u\|$$

If, as we always assume, $n \geq 4$, this provides a uniform convergence to zero as t goes to $+\infty$. Note that $\Lambda_2(TM)$ is the orthogonal sum of $\Lambda_2(E^{ss} \oplus \mathbb{R}X)$ and of the vector space of bivectors of the form $v_s \wedge v_u$. We know that the 2-form $d\eta$ vanishes on $\Lambda_2(E^{ss} \oplus \mathbb{R}X)$. Therefore, the above inequality shows that for every bivector w of $\Lambda_2(TM)$, one has:

$$|f^*_{-t}\ d\eta(w)| = |d\eta(df_{-t}(w))| \leq C^{(n-3)}\ \nu^{(n-3)t}\|w\| \ .$$

This obviously implies the required contradiction in (1) since the right hand side converges to zero as t goes to $-\infty$. This finishes the proof of theorem 1.

2. PROOF OF THEOREM 2.

We consider a codimension one Anosov flow f_t on a compact manifold M of dimension bigger than 3 and we assume, as in theorem 2, that the distribution $E^{ss} \oplus E^{uu}$ is of class C^1 . Consider the differential form α of degree 1 which is equal to 0 on $E^{ss} \oplus E^{uu}$ and to 1 on the vector field X associated to f_t . Obviously, α is invariant under f^*_t and, therefore, $\beta = d\alpha$ is a 2-form of class C° which is also invariant under f^*_t . If we could show that β is identically 0, that would mean that $E^{ss} \oplus E^{uu}$ is integrable and a theorem of J. Plante ([14], theorem 3.7) would imply that f_t has a global cross-section with constant return time. Note that J. Plante proves the existence of such a section of class C^1 ; however it would be easy to deduce from [12] that this section is actually very smooth (of class C^∞ if f_t is C^∞ and at least of class $C^{r-n/2}$ if f_t is of class C^2 and $\dim M = n$) .

In order to prove the theorem, we are led to prove that the form β vanishes. The following lemma shows that it suffices to study the restriction of β to $E^{ss} \oplus E^{uu}$.

LEMMA 2.1. The vector field X lies in the kernel of the 2-form β .

PROOF. Consider first a vector v belonging to E^{ss}. By invariance of β, we have:

$$\beta(X,v) = \beta(df_t(X),df_t(v)) = \beta(X,df_t(v)) .$$

The continuity of β, the compactness of M and the fact that $df_t(v)$ goes to zero as t goes to $+\infty$, show that:

$$\beta(X,v) = 0 .$$

Reversing the time, we see in the same way that $\beta(X,v) = 0$ if v belongs to E^{uu}. In other words, X lies in the kernel of β.

□

We shall analyse the situation at a periodic point using the following lemma which is analogous to lemma 1.1.

LEMMA 2.2. Let E be a $(n-1)$-dimensional euclidean vector space $(n \geq 3)$ written as an orthogonal sum $S \oplus U$ where U is one dimensional. Let $f : E \to E$ be a linear mapping and β a non trivial skew symmetric bilinear form on E. Suppose that:

1) S and U are invariant under f.

2) β is invariant under f.

3) There is a constant μ such that $0 < \mu < 1$ amd for which one has $\|f(v)\| \leq \mu \cdot \|v\|$ for $v \in S$. Then,

$$|\det f| \leq \mu^{n-3} .$$

In particular, if $n \geq 4$, one has $|\det f| < 1$.

PROOF. Note first of all that, if v_1 and v_2 are vectors of S, one has:

$$\beta(v_1,v_2) = \beta(f^n(v_1),f^n(v_2)) \xrightarrow[n\to\infty]{} 0$$

Choose a unit vector v in U and let λ be such that $f(v) = \lambda v$. Taking into account the fact that β is non trivial, one sees that the linear form:

$$\ell : x \in S \to \beta(v,x) \in \mathbb{R}$$

is non zero. Obviously, this linear form satisfies:

$$\ell(f(x)) = 1/\lambda\ \ell(x) \tag{2}$$

Consider a unit cube $C = [0,1]^{n-2}$ in S whose base $[0,1]^{n-3} \times \{0\}$ is contained in the kernel of ℓ. The image of C by f is a parallelepiped whose base $f([0,1]^{n-3} \times \{0\})$ has (n-3)-volume bounded by μ^{n-3} (by Gram-Schmidt inequality) and whose height is precisely $1/|\lambda|$ by (2). Therefore, the volume of $f(C)$ is at most $(1/|\lambda|)\mu^{n-3}$. Recalling that $f(v) = \lambda v$, that S and U are orthogonal and invariant under f, one gets:

$$|\det f| \leq |\lambda|\ |1/\lambda|\ \mu^{n-3} = \mu^{n-3}.$$

□

In theorem 2, we do not assume that f_t is volume preserving. However, if we add this hypothesis, the proof is now easy:

COROLLARY 2.3. If f_t satisfies the hypothesis of theorem 2 and preserves some volume form, then f_t is a constant time suspension.

PROOF. If x is a periodic point of f_t of period $T > 0$, let $E = E_x^{ss} \oplus E_x^{uu}$, $S = E_x^{ss}$, $U = E_x^{uu}$ and $f = (df_T)^k : E \to E$. If k is sufficiently big, then the conditions 1), 2), 3) of the previous lemma are satisfied for some euclidean structure on E. If the 2-form β were non zero at x, we would get $|\det f| < 1$ contradicting the assumption that f_t is volume preserving. Using the continuity of β and the density of periodic points of f_t $(\dim M \geq 4)$, we deduce that β vanishes everywhere. As we have already mentioned, this implies that f_t is a constant time suspension.

□

In the remaining part of this paragraph, we get rid of the condition that f_t is volume preserving.

We still assume, by contradiction, that β is not identically zero and we denote by U the non empty f_t-invariant open set consisting of points where β is non zero.

If M is orientable, choose any volume form Ω on M and denote by $u : M \to \mathbb{R}$ the divergence of X with respect to Ω, i.e., the function such that $\mathcal{L}_X\Omega = u\Omega$ where $\mathcal{L}_X$ denotes the Lie derivative. If M is not orientable, we can choose an "odd" volume form Ω and remark that there is still a function u such that $\mathcal{L}_X\Omega = u.\Omega$. According to lemma 2.2, we know that if x is a periodic point of f_t of period $T > 0$ and if x is in U, then:

$$\frac{1}{T}\int_0^T u(f_t(x)\, dt < c < 0 \tag{3}$$

for some constant c independent of x and T. Suppose for a moment that we prove tha following two lemmas:

LEMMA 2.4. Every f_t-invariant probability measure on M can be approximated by convex combinations of invariant probabilities concentrated on periodic orbits contained in U.

LEMMA 2.5. Let $u : M \to \mathbb{R}$ be a smooth function such that $\int u\, d\mu < 0$ for every f_t-invariant probability measure μ on M. Then there exists a smooth function $v : M \to \mathbb{R}$ such that $u + X(v) < 0$. Here, $X(v)$ denotes the derivative of v in the direction of X.

Assuming these two lemmas, we can finish the proof of theorem 2 which reduces, as we have seen, to the fact that the open set U cannot be non empty. Indeed, assuming that $U \neq \phi$, lemma 2.4 and (3) would imply that $\int u\, d\mu < 0$ for every f_t-invariant probability measure. In turn, lemma 2.5 would imply that there is a smooth function v such that $u + X(v) < 0$. Consider now the volume form $\Omega' = \exp(v)\Omega$. Then the divergence of X with respect to Ω' is given by the formula:

$$\begin{aligned}\mathcal{L}_X\Omega' &= \exp(v)X(v)\Omega + \exp(v)u\Omega \\ &= (u + X(v))\Omega'\end{aligned}$$

This is the required contradiction since the negativity of $u + X(v)$ would imply that the measure associated to Ω' is contracted. But the total mass of M has to be preserved by the flow f_t.

We are therefore left with the proof of lemmas 2.4 and 2.5.

PROOF OF LEMMA 2.4. This lemma is not typical to our situation but is valid for any transitive Anosov flow (we have seen that this condition is satisfied for codimension one Anosov flows in dimension bigger than 3).

Recall first of all how to approximate an ergodic invariant probability measure μ by a probability measure concentrated on a periodic orbit. Consider a μ-regular point x, i.e. a point for which, for every continuous function $h : M \to \mathbb{R}$, one has:

$$\int h \, d\mu = \lim_{T\to\infty} \frac{1}{T} \int_0^T h(f_t(x))dt$$

Such a point is obviously recurrent and we can use Anosov's closing lemma (see [3]) to produce periodic orbits γ_k that approximate long pieces of orbits $f_t(x)$, $t \in [0,T_k]$. It is clear that the sequence of probability measures supported by γ_k converges to the given measure μ.

Now, the compact convex set of all invariant probability measures is the closed convex hull of ergodic invariant measures (Krein-Millman). In order to prove the lemma it suffices to show that the invariant probability measures concentrated on periodic orbits can be approximated by those contained in U.

This can easily be done using symbolic dynamics. Under our assumption, the flow is transitive and therefore, there exists a Markov partition for f_t ([3]). We can even suppose that one of the boxes of the Markov partition is contained in U. To this partition corresponds a finite connected graph G whose vertices are the boxes. A closed loop in this graph corresponds to a periodic orbit of f_t. Let γ be such a closed loop originated at the vertex p of G and let γ^k denote the same loop iterated k times. Let p_o be a vertex of G corresponding to a box contained in U and choose a path δ (resp. δ') from p_o to p (resp. p to p_o). Then the sequence $\delta\gamma^k\delta'$ represents a loop at p_o and therefore represents a periodic orbit of f_t which is contained in U. It is clear that the sequence of invariant probability measures μ_k concentrated on these orbits converges to the invariant probability measure concentrated on the periodic orbit corresponding to γ .

□

PROOF OF LEMMA 2.5. This lemma is true for any smooth flow f_t on any compact manifold M, independently of the Anosov property.

We consider a function $u : M \to \mathbb{R}$ such that $\int u \, d\mu < 0$ for every f_t-invariant measure μ . We claim that for $T > 0$ big enough, the function u_T defined by

$$u_T(x) = \frac{1}{T} \int_0^T u(f_t(x))dt$$

is negative. Indeed suppose that there exists a sequence x_n in M and a sequence T_n going to $+\infty$ such that $u_{T_n}(x_n) \geq 0$. Consider the probability measure μ_n uniformly concentrated on the piece of orbit from x_n to $f_{T_n}(x_n)$. Then any weak limit of the sequence μ_n would be an invariant probability measure μ for which $\int u \, d\mu \geq 0$ contrary to our assumption.

In order to prove the lemma, it is therefore sufficient to construct a function v_T such that:

$$u + X(v_T) = u_T \tag{4}$$

Consider the following two probability measures on $\mathbb{R}$. The first one is δ_o , the Dirac mass at the point 0 and the second one γ_T is uniformly distributed on $[0,T]$. Then, the difference $\gamma_T - \delta_o$ es derivative (in the sense of distributions) of the function

$$t \in \mathbb{R} \to (\gamma_T - \delta_o)(]-\infty,t[) = \left(\frac{t}{T} - 1\right) \quad \text{if} \quad 0 \leq t \leq T$$
$$= 0 \quad \text{otherwise.}$$

In other words, we get the following formula for every smooth function $\phi : \mathbb{R} \to \mathbb{R}$:

$$\int_0^T \left(1 - \frac{t}{T}\right) \phi'(t) \, dt = \frac{1}{T} \int_0^T \phi(t)dt - \phi(0) \ .$$

It is now cleat that the following function v_T will satisfy (4):

$$v_T(x) = \int_0^T \left(1 - \frac{t}{T}\right) u(f_t(x))dt \ .$$

□

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