The dynamics of vector fields in dimension 3

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July 8, 2013

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The year 2012 was the 125th anniversary of Srinivasa Ramanujan's birth. The Ramanuajan Mathematical Society has planned diverse mathematical activities to commemorate this anniversary. Among them are a series of 20 sets of lectures which has been named "Mathematical Panorama Lectures". These Lectures are envisaged to be a survey of a single topic starting at a level accessible to beginning graduate students and leading up to recent developments.

I was honoured to be invited to deliver such a series of lectures in March 2012, in the Tata Institute for Fundamenal Research in Mumbay. I would like to thank Prof. Raghunathan and Dani for this kind invitation and for the splendid organisation of the meeting. I would also like to thank all participants : they turned these lectures into a very stimulating experience for me.

My purpose was to introduce the students to two aspects of current research on the dynamics of vector fields in dimension 3. The first one is very venerable since it was inaugurated by Poincaré and concerns the quest for periodic trajectories. The notes contain in particular a description of the current state of Seifert conjecture about periodic trajectories for vector fields on the 3-sphere. The second aspect deals with the existence of Birkhoff sections for a wide class of vector fields that I call "left handed".

In practice, I gave nine lectures (and some exercises sessions) following some natural route : from surfaces to 3-manifolds; from Wilson and Schweitzer to the wonderful example of Kuperberg; and from the old and powerful ideas of Schwartman to recent constructions of left-handed vector fields.

These notes have been taken by Matthias Moreno and Siddhartha Bhattacharya. I would like to thank them sincerely for their work.

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Lecture I

Flows and vector fields on surfaces

The qualitative theory of dynamical systems originated in the work of Henri Poincaré. One of his main goals was to study the restricted 3-body problem, where the aim is to describe the motion of a planet A that is acted upon by the gravitational field of two other planets B and C. We assume that all three planets lie in the same plane, the planets B and C move in circular orbits about their center of mass, and the mass of the planet A is very small compared to the mass of the other two planets. In this case at any given time there are 4 parameters describing the motion of A : two position variables x and y, and two velocity variables v_x and v_y. In a rotating frame, in which B and C are fixed, one can show that there is some invariant quantity J, called the Jacobi invariant, and related to the invariance of the total energy. This gives rise to an equation of the form $J(x, y, v_x, v_y) = \mathit{constant}$. Hence the planet moves in a level surface of $\mathbb{R}^4$ which is (usually) a 3-manifold.

In order to study similar problems in a more abstract setting we recall a few basic notions from differential topology. Let M be a compact manifold. A C^r-flow ϕ on M is an action of $\mathbb{R}$ on M such that the induced map from $\mathbb{R} \times M$ to M is a C^r-map. For a flow ϕ on M, the map $m \mapsto t \cdot m$ will be denoted by ϕ^t. Any flow ϕ determines a vector field X_ϕ on M (of class C^{r-1}) defined by $X_\phi(m) = \frac{d}{dt}\phi^t(m)_{|t=0}$. Conversely, for any vector field X on M, there exists a flow ϕ such that X is the vector field associated to ϕ. A vector field X is said to be C^r if the associated map from M to TM, the tangent bundle of M, is a C^r-map.

Let X be a vector field on a compact manifold M, and let $\{\phi^t\}$ be the flow generated by X. For any $x \in M$ we define the *ω-limit set* of x by

$$\omega(x) = \{y \in M : \exists t_n \to \infty \text{ such that } \phi^{t_n}(x) \to y\}.$$

Similarly we define the *α-limit set* of x by

$$\alpha(x) = \{y \in M : \exists t_n \to -\infty \text{ such that } \phi^{t_n}(x) \to y\}.$$

It is easy to see that $\omega(x)$ and $\alpha(x)$ are always compact non-empty ϕ^t-invariant sets.

Theorem (Poincaré-Bendixson): *Let X be a vector field on the 2-sphere $\mathbb{S}^2$ vanishing in a finite number of points (which are fixed points of the associated flow). Then for any $x \in \mathbb{S}^2$, the set $\omega(x)$ is reduced to a fixed point, or to a periodic orbit, or is a connected set consisting of finitely many fixed points together with orbits joining them.*

We sketch a proof of this theorem in the special case when $\omega(x)$ does not contain a fixed point of the flow. In order to show that $\omega(x)$ is a periodic orbit, let us first recall the notion of a flow box. It is a subset B of M which is homeomorphic to some square $[-\epsilon, +\epsilon]^2$ in such a way that the induced local flow on the square is given by $\psi^t(x, y) = (x + t, y)$. Note that the subset $S \subset B$ corresponding to the line segment $x = 0$ in the square is transversal to the flow. We first claim that for any flow box B, the set $\omega(x)$ intersects the set S at at most one point. This follows from the observation that as we go along the orbit of X, the intersection with S forms a monotone sequence. Indeed, consider two consecutive points p, q of the intersection of the orbit with S. The piece of trajectory from p to q followed by the interval $[p, q]$ in S is a Jordan curve which therefore disconnects the sphere. Hence, any trajectory starting from an interior point r of $[p, q]$ enters one of the two connected components of the complement of this curve and cannot escape from this component. It follows that the trajectory of r only intersects $[p, q]$ in r and this proves our claim that the intersection with S of any trajectory forms a monotone sequence.

Now we choose a p in $\omega(x)$ and $q \in \omega(p)$. Since ω-limit sets are invariant it follows that $q \in \omega(x)$. By our assumption q is a regular point. We construct a flow box around q such that $q \in S$. Since $q \in \omega(p)$ we deduce that the orbit of p enters the flow box B and intersects S infinitely often. On the other hand, since $p \in \omega(x)$ and $\omega(x)$ is invariant, from the previous claim it follows that the orbit of p can intersect S at at most one point. Hence the orbit of p passes through q infinitely often, i.e., q lies in a periodic orbit. As $\omega(p) \subset \omega(x)$, this shows that $\omega(x)$ contains a non-degenerate periodic orbit. Now from an elementary topological argument we deduce that $\omega(x)$ is a periodic orbit. This concludes the proof in the special case.

The above theorem does not hold if $\mathbb{S}^2$ is replaced by $\mathbb{T}^2$. To see this, consider a linear flow on $\mathbb{T}^2$ with irrational slope. It is easy to see that $\omega(x) = \mathbb{T}^2$ for all x.

Hilbert's 16th problem : Consider the equation

$$\frac{dx}{dt} = P(x,y), \frac{dy}{dt} = Q(x,y),$$

where P and Q are polynomials of degree $d \geq 2$ in real variables (x, y). A periodic orbit is called a *limit cycle* if it is not in the interior of the set of periodic points. The problem is to find an upper bound for the number of limit cycles that depends only on d.

Ecalle and Illyaschenko have shown that the number of limit cycles is finite, but no explicit upper bound is known, even for $d = 2$.

Theorem (Peixoto, 1962): *Let M be a compact orientable surface, and let $X^\infty(M)$ be the space of all C^∞ vector fields on M. Let A denote the set of all $X \in X^\infty(M)$ with the property that for all $x \in M$, $\omega_X(x)$ is either a fixed point or a periodic orbit. Then A is a dense G_δ-subset of $X^\infty(M)$.*

Definition : Suppose X and Y are two vector fields on a manifold M. They are said to be *topologically equivalent* if there exists a homeomorphism

$h : M \to M$ that maps orbits of X to orbits of Y and preserves their natural orientations.

The following result classifies all non-singular C^2 vector fields on compact surfaces that do not admit periodic orbits.

Theorem (Poincaré, Denjoy): *Every non-singular C^2 vector field on a compact surface that has no periodic orbits is topologically equivalent to a linear flow on $\mathbb{T}^2$ with irrational slope.*

An example due to Denjoy shows that this result is not true for C^1 vector fields.

We end this section with a problem.

Question : There is no non-singular vector field on a compact manifold that admits exactly one non-closed orbit.

Exercises :

Exercise 1. Suppose X is a vector field on a compact manifold M. Show that every orbit of X which is closed as a subset of M is periodic.

Exercise 2. For $i = 1, 2$ suppose ϕ_i is a linear flow on $\mathbb{T}^2$ with slope $\alpha_i \in \mathbb{R} - \mathbb{Q}$. When are the flows ϕ_1 and ϕ_2 topologically equivalent ?

Exercise 3. Show that every non-singular flow on the Klein bottle has a periodic orbit.

Exercise 4. Suppose X is a non-singular flow on $\mathbb{T}^2$, and X^* is the corresponding map from $\mathbb{T}^2$ to $\mathbb{R}^2 - \{0\}$. If the induced map on fundamental groups $\pi_1(X^*) : \mathbb{Z}^2 \to \mathbb{Z}$ is non-trivial, show that X has a periodic orbit.

Lecture II

Suspensions

In the previous lecture, we concentrated on flows on $\mathbb{S}^2$ and other compact surfaces. In this lecture we present several constructions of flows on higher dimensional manifolds.

Suspensions : Let M be a compact manifold and let $\phi : M \to M$ be a diffeomorphism. We define an equivalence relation $\sim$ on $M \times \mathbb{R}$ by

$$(x, t) \sim (\phi^k(x), t + k) \quad \forall x \in M, k \in \mathbb{Z}.$$

Let M_ϕ denote the manifold $M \times \mathbb{R}/\sim$. We define a flow ϕ^t on M_ϕ by

$$\phi^t([x, s]) = [x, s + t].$$

The flow $\{\phi^t\}$ is called the *suspension* of the diffeomorphism ϕ. It is easy to see that this flow reflects many dynamical properties of the discrete system (M, ϕ). For example periodic points of ϕ correspond to periodic orbits of the suspension flow and this correspondence preserves the period.

As a special case we consider the example where $M = \mathbb{T}^2$ and ϕ is a hyperbolic automorphism of $\mathbb{T}^2$, induced by some 2 by 2 invertible and integral matrix. Then M_ϕ is a 3-manifold. Since periodic points of ϕ form a dense subset of $\mathbb{T}^2$ (exercise), it follows that the suspension flow on M_ϕ has a dense set of periodic orbits. Apart from having a dense set of periodic orbits, this flow has many other interesting properties. We state one such result without proof.

Theorem (Anosov): *Any vector field sufficiently close to X_ϕ in the C^1-topology is topologically equivalent to X_ϕ.*

Horseshoe map : Now we look at another special case of the above construction. Let $S \subset \mathbb{R}^2$ denote the subset consisting of a square capped by two semi-disks. The horseshoe map $f : S \to S$ is a diffeomorphism that is defined in two steps. In the first step S is contracted along the vertical direction by a factor $C < 1/2$. Next it is stretched in the horizontal direction by a factor of $1/C$, and the resulting strip is folded like a horseshoe and placed back to S.

Let Λ denote the set of all points x such that $f^i(x)$ lies in the square for all $i \in \mathbb{Z}$. Then Λ is a compact set and the restriction of f to Λ is a homeomorphism. It can be shown that Λ is homeomorphic with $K \times K$, where $K \subset [0, 1]$ is a Cantor set.

Now we look at the map $\Lambda \to \{L, R\}^{\mathbb{Z}}$ where the i-th co-ordinate depends on whether $f^i(x)$ lies in left or or right half of the square. It turns out that this map is a homeomorphism. Hence the restriction of the horseshoe map to Λ can be identified with the shift map on $\{L, R\}^{\mathbb{Z}}$. Now let M be a compact surface and let $f : M \to M$ is a diffeomorphism such that $f|_S$ is the horseshoe map for some region $S \subset M$. Let ϕ be the suspension flow on M_f. It is easy to see that for each n the shift map has 2^n periodic points with period n. Hence the suspension flow described above f has periodic orbits with arbitrarily large periods.

Geodesic flow : Let S be a compact surface with some Riemannian metric g, and let $M = T_1S$, the unit tangent bundle of S. For any $(x, v) \in M$ let $\alpha(x, v)$ be the unique geodesic starting at x in the direction v, and let (x_t, v_t) be the position and the direction after time t. We define a flow ϕ on M by $\phi^t(x, v) = (x_t, v_t)$. This flow on M is called the *geodesic flow* of S.

Proposition : *Let ϕ be a flow as described above. Then it is not a suspension flow.*

Proof : Suppose N is a compact n-manifold and $f : N \to N$ is a diffeomorphism such that ϕ can be identified with the suspension flow on N_f. We choose a periodic orbit γ in N_f. Now $[\gamma]$ defines an element of the first homology group $H_1(N_f)$ and $[N]$ an element of the n^{th} homology $H_n(N_f)$. We note that $[N][\gamma] > 0$, i.e., every periodic orbit intersects N positively. Clearly for any closed geodesic γ there exists another closed geodesic γ_-, which is the same geometric geodesic with the other orientation. On the other hand $[\gamma] = -[\gamma_-]$. Now, it is known that any closed Remannian manifold admits at least one closed geodesic. This contradicts the existence of N. $\square$

Horocycle flow : Let G be a Lie group and let Γ be a discrete subgroup of G. Then any one-parameter subgroup $\{g_t\}$ of G induces a flow on the manifold G/Γ defined by $\phi^t(x\Gamma) = g_t x\Gamma$. In the special case when $G = PSL(2,\mathbb{R})$ and Γ is a co-compact discrete subgroup of G, we have the following two well known examples :

$$\text{Geodesic flow : } g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \text{ Horocycle flow : } h_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Proposition *The Horocycle flow has no periodic orbits if Γ is co-compact.*

Proof : Suppose this is not the case. An elementary calculation shows that

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} = \begin{pmatrix} 1 & e^{2t}s \\ 0 & 1 \end{pmatrix}.$$

Let ϕ_g denote the geodesic flow and ϕ_h denote the Horocycle flow on $PSL(2,\mathbb{R})/\Gamma$. The above identity shows that

$$\phi_g(t) \circ \phi_h(s) \circ \phi_g(-t) = \phi_h(e^{2t}s).$$

Now suppose $\phi_h(s)(x) = x$ for some x. Then $\phi_g(-t)(x)$ lies in a periodic orbit of ϕ_h with period $e^{-2t}s$. By compactness, letting t go to infinity, we deduce that ϕ_h has a fixed point. This contradicts the discreteness of Γ and proves the above proposition. $\square$

Definition : A continuous action of a group G on a topological space X is said to be minimal if all orbits are dense.

Theorem (Hedlund): *Horocycle flows are minimal in the co-compact case.*

There is a conjecture due to Gottschalk that asserts that there are no minimal flows on the 3-sphere S^3. Katok, Fathi and Herman have shown that S^3 admits minimal C^∞-diffeomorphisms, disproving the discrete version of this conjecture.

Hopf flow on $\mathbb{S}^3$: We identify $\mathbb{S}^3 \subset \mathbb{C}^2$ with $\{(z_1, z_2) : |z_1|^2 + |z_2|^2 = 1\}$, and define a flow ϕ on $\mathbb{S}^3$ by

$$\phi^t(z_1, z_2) = (e^{2i\pi t} z_1, e^{2i\pi t} z_2).$$

It is easy to see that all orbits are periodic with period 1.

Later, we will prove the following result :

Theorem (Seifert): *Any perturbation of the Hopf vector field on $\mathbb{S}^3$ has at least one periodic orbit.*

Seifert also conjectured that every non-singular vector field on $\mathbb{S}^3$ has a periodic orbit. However, this stronger statement turned out to be false: Kuperberg found a C^∞ vector field on $\mathbb{S}^3$ that has no periodic orbit.

Exercises :

Exercise 1. Let ϕ be a hyperbolic automorphism of $\mathbb{T}^2$ and let X_ϕ be the vector field corresponding to the associated suspension flow. Show that there exists a 3-dimensional Lie group G and a discrete subgroup $\Gamma \subset G$ such that $M_\phi = G/\Gamma$, and X_ϕ corresponds to a right-invariant vector field on G.

Exercise 2 : Let S be the shift map on the topological space $X = \{L, R\}^{\mathbb{Z}}$. Show that there exists a point $x \in X$ such that $\omega(x)$ does not contain a periodic orbit.

Lecture III

Seifert's theorem

In this lecture we sketch a proof of Seifert's theorem. In order to illustrate the main idea, we first look at a simpler example.

Proposition : *Let M denote the manifold $\mathbb{S}^2 \times \mathbb{S}^1$, and let X denote the vector field $\frac{\partial}{\partial\theta}$ where θ is the co-ordinate on the second factor. Then there exists $\epsilon > 0$ such that any vector field Y with $||Y - X||_0 < \epsilon$ has a periodic orbit.*

Proof : We identify $\mathbb{S}^2$ with the set $\mathbb{S}^2 \times \{1\}$. It is easy to see that if ϵ is sufficiently small, then the Y-orbit of every $x \in \mathbb{S}^2$ comes back to another point in $\mathbb{S}^2$, close to x. This defines a map $h : \mathbb{S}^2 \to \mathbb{S}^2$ and a vector field V on $\mathbb{S}^2$ defined by $V(x) = (h(x) - x) - (x \cdot (h(x) - x))x$, (which is indeed a vector tangent to the 2-sphere). As $\mathbb{S}^2$ does not admit non-vanishing vector fields, we can find a point x_0 such that $V(x_0) = 0$. Then $h(x_0) = x_0$, and this implies that the orbit of x_0 is periodic. $\square$

Now we turn to the proof of Seifert's theorem. Recall its statement:

Theorem (Seifert): *Any perturbation of the Hopf vector field on $\mathbb{S}^3$ has at least one periodic orbit.*

Let ϕ be a flow on an oriented 3-manifold M, and let $x \in M$ be a point in a periodic orbit γ. We choose a small disk D around x and define $h : D \to D$ to be the first return map. It is easy to see that h is a well defined local diffeomorphism. Using a chart we can view h as a local diffeomorphism from

$\mathbb{R}^2$ to $\mathbb{R}^2$ at the origin, with $h(0) = 0$. We define a vector field X_h in a neighbourhood of the origin by $X_h(v) = h(v) - v$, and define $Index(\gamma)$ to be the index of X_h at 0. It can be verified that $Index(\gamma)$ does not depend on the choice of D or x in γ.

For a vector field X on $\mathbb{S}^3$ and $\delta > 0$, let $Per(X, \delta)$ denote the collection of periodic orbits of X with period $T \in (1-\delta, 1+\delta)$. We will deduce Seifert's theorem from the following stronger result about vector fields on $\mathbb{S}^3$.

Theorem (Seifert): *Let H denote the Hopf vector field on $\mathbb{S}^3$. Then there exist $\epsilon, \delta > 0$ such that*

$$\sum_{\gamma \in Per(X,\delta)} Index(\gamma) = 2,$$

whenever $||H - X|| < \epsilon$ and $Per(X, \delta)$ is finite.

Before turing to the proof of the above theorem we introduce a few notations. First, we identify S^2 with $\mathbb{C} \cup \{\infty\}$, and $\mathbb{S}^3$ with the set

$$\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}.$$

Let $\pi : \mathbb{S}^3 \to \mathbb{S}^2$ denote the map defined by $\pi(z_1, z_2) = z_1/z_2$. We define two subsets $C, D \subset \mathbb{S}^2$ by

$$C = \{z : |z| \geq \frac{1}{4}\}, \ D = \{z : |z| \leq \frac{1}{2}\}.$$

Let Σ denote the section from $\mathbb{C} \subset \mathbb{S}^2$ to $\mathbb{S}^3$ defined by

$$\Sigma(z) = (\frac{z}{\sqrt{1+|z|^2}}, \frac{1}{\sqrt{1+|z|^2}}).$$

We note that $\Sigma(D)$ is transversal to the Hopf fibration.

We will deduce the theorem stated above from the following three lemmas.

Lemma 1 : *Let H denote the Hopf vector field on $\mathbb{S}^3$, and let X_1 and X_2 be two vector fields on $\mathbb{S}^3$ that agree on $\pi^{-1}(D)$. Then there exist $\epsilon, \delta > 0$*

such that $||H - X_i|| < \epsilon$ *and* $|Per(X_i, \delta)| < \infty$ *implies that*

$$\sum_{\gamma \in Per(X_1,\delta)} Index(\gamma) = \sum_{\gamma \in Per(X_2,\delta)} Index(\gamma).$$

Proof : It is enough to prove the equality when the sum on both sides are taken over periodic orbits intersecting the set C. Let h_1 and h_2 be the first return maps of the vector fields X_1 and X_2. They are defined on a disk containing C if ϵ is sufficiently small. Define two vector fields Z_1 and Z_2 on C by $Z_i(x) = h_i(x) - x$. Since X_1 and X_2 agree on D, it follows that $Z_1 = Z_2$ on ∂C. Now the above lemma can be deduced from the identity

$$\sum_{Z_i(x)=0} index(x) = Index(Z_i, \partial C).$$

□

Lemma 2 : *Let* H *denote the Hopf vector field on* $\mathbb{S}^3$*. Then there exist* $\epsilon, \delta > 0$ *such that for any two vector fields* X_1 *and* X_2 *on* $\mathbb{S}^3$ *satisfying* $||H - X_i|| < \epsilon$ *and* $Per(X_i, \delta)$ *is finite, we have*

$$\sum_{\gamma \in Per(X_1,\delta)} Index(\gamma) = \sum_{\gamma \in Per(X_2,\delta)} Index(\gamma).$$

Proof : Using partitions of unity find a vector field X_3 on $\mathbb{S}^3$ such that X_3 agrees with X_1 on $\pi^{-1}(D)$ and agrees with X_2 on $\pi^{-1}(D')$ where

$$D' = \{i/z : z \in D\}.$$

From a symmetry argument it is easy to deduce that the previous lemma holds when D is replaced by D'. Now Lemma 2 follows from two applications of Lemma 1. □

Lemma 3 : *For every $\epsilon > 0$ there exists a vector field X on $\mathbb{S}^3$ such that $||H - X|| < \epsilon$, $Per(X, \delta)$ is finite for some $\delta > 0$ and*

$$\sum_{\gamma \in Per(X,\delta)} Index(\gamma) = 2.$$

Proof. We identify $\mathbb{S}^3$ with $\{(z_1, z_2) : |z_1|^2 + |z_2|^2 = 1\}$, and for every $s > 0$ we define a flow ϕ_s on $\mathbb{S}^3$ by

$$\phi_s^t(z_1, z_2) = (e^{2\pi i t} z_1, e^{2\pi i(1+s)t} z_2).$$

Let X_s denote the vector field associated to this flow. It is easy to see that for sufficiently small s, $||H - X_s|| < \epsilon$. We also note that for sufficiently small s, the set $\mathrm{Per}(X_s, \frac{1}{2})$ contains only two fixed points, $(1, 0)$ and $(0, 1)$. Furthermore, it is easy to verify that both these fixed points have index 1. Hence

$$\sum_{\gamma \in Per(X_s,\frac{1}{2})} Index(\gamma) = 2.$$

□

This concludes the proof of Seifert's theorem.

Exercises :

Exercise 1 : Let X be a vector field on a compact surface. Is it true that the union of closed orbits of X is always closed ?

Exercise 2 : Let ϕ be the Horseshoe map. Compute the number of periodic points of period n. Similarly, for any $A \in SL(2, \mathbb{Z})$, compute the number of A-periodic points in $\mathbb{T}^2$ with period n.

Exercise 3 : If $A, B \in SL(2, \mathbb{Z})$ are two automophisms of $\mathbb{T}^2$, then show that they are topologically conjugate if and only if they are conjugate in $GL(2, \mathbb{Z})$. Is the condition $Trace(A) = Trace(B)$ sufficient ?

Exercise 4 : Show that the horocycle flow on $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ is not a suspension.

Exercise 5 : Let M be a compact manifold. Show that the set of minimal homeomorphisms of M is a G_δ subset of $Homeo(M)$.

Lecture IV

Wilson and Scheitzer plugs

1966 **Wilson surgery: vector fields with only one periodic orbit**

The main idea introduced by Wilson in [9] is to start with an ordinary nonsingular vector field X on a manifold M and *operate* it by making a *surgery.* For this purpose, we define what we call a *plug.* This object can be "inserted" in a flow box and locally replace the dynamics, while controlling the changes outside the plug. We will give sufficiently precise conditions, which will guarantee that such an operation is possible.

Definition (Plug): *A C^r plug, $0 \leq r \leq \infty$, P is a nonsingular C^r vector field $\mathcal{W}$ on the product of a $(n-1)$-dimensional compact connected manifold Σ and the interval $[-1,+1]$, such that*

1. $\Sigma \times [-1,+1]$ *can be embedded in* $\mathbb{R}^n$ *so that all the fibers* $\{\star\} \times [-1,+1]$ *are parallel line segments.*

2. *In a neighborhood of the boundary* $\partial(\Sigma \times [-1,+1])$*, the vector field is tangent to the fibers* $\{\star\} \times [-1,+1]$*.*

   When we integrate the field $\mathcal{W}$, we get a local flow on P. The subset $\Sigma \times \{-1\}$ will be called the *entrance* of P and the subset $\Sigma \times \{+1\}$ will be called the *exit* of P. We require the following:

3. *If the orbit starting at an entrance point* $(x,-1)$ *reaches the exit at* $(y,+1)$*, then* $x = y$*.*

4. *At least one trajectory (or an open section* $\Omega \subset \Sigma \times \{-1\}$ *of trajectories) starting at the entrance never reaches the exit.*

For $n = 3$, we recall that any compact orientable surface with one point removed Σ can be immersed in $\mathbb{R}^2$, and can be embedded in $\mathbb{R}^3$ in such a way that the constant field $\partial/\partial z$ is transverse to any point of Σ. Thus, by pushing Σ along $\partial/\partial z$, we get an embedded manifold with the shape of a plug.
From now on, z will denote the second coordinate in the product $\Sigma \times [-1, +1]$ and the n^{th}-coordinate in $\mathbb{R}^n$.
The first two conditions guarantee that the embedding of P can be connected with the constant vector field $\partial/\partial z$ (remember that any non-singular vector field locally looks like $\partial/\partial z$) in a C^r way.

The following lemma, inspired by [6], shows that we can insert many copies of our plug and traps all the existing periodic orbits of X. It is important to note that this surgery doesn't change the topology of the manifold.

Lemma: *Let M be a n-manifold and X a smooth non-singular vector field on M. Let Σ be a $(n-1)$-dimensional compact connected manifold, containing an open subset Ω ("the trapping zone").*

Then there exists $k \in \mathbb{N}$ and embeddings $f_i : \Sigma \to M$ for $i = 1, \ldots, k$ such that:

- $\forall\ i \neq j\ f_i(\Sigma) \bigcap f_j(\Sigma) = \emptyset$
- *X is transverse to $\bigcup\limits_i f_i(\Sigma)$*
- *Every orbit of X meets one of the $f_i(\Omega)$*

For the proof, see [6].

We now show how to use surgery to create nonsingular vector fields with finitely many periodic orbits. Note however that in the case of $\mathbb{S}^3$, this step is useless since is not hard to give explicit examples of nonsingular vector fields with only two periodic orbits (or even only one: exercise for the reader!)

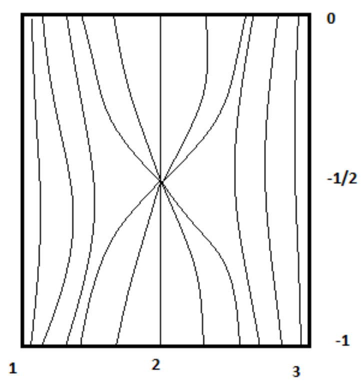
The next theorem gives a construction for general three manifolds. Actually one can prove that there exists non-singular vector fields with only two periodic orbits. We omit the proof of the stronger statement and leave it as an exercise.

Theorem (Wilson, 1966): *For every three dimensional compact manifold M there exists a non-singular vector field with a finite number of periodic orbits.*

Theorem (Wilson, 1966): *For every compact manifold M with $\chi(M) = 0$ and $dim(M) \geq 4$, there exists a non-singular vector field with no periodic orbit.*

Proof: Our entrance surface Σ will be the ring $\mathbb{T}^{n-2} \times [1, 3]$, and the "*trapped zone*" $\mathbb{T}^{n-2} \times [\frac{3}{2}, \frac{5}{2}]$. The final plug P will be $\mathbb{T}^{n-2} \times [1, 3] \times [-1, +1]$. All pictures correspond to the case $n = 3$.

Let's now construct the vector field on P. We begin with the following vector field on $R := [1, 3] \times [-1, 0]$:



Then, let $f : R \to [0, 1]$ be a positive function, equal to 1 in a neighborhood of $(2, -\frac{1}{2})$, and to 0 in the neighborhood of the boundary of R. We use f to twist a well chosen field Γ on $\mathbb{T}^{n-2}$. If $n = 3$ we take Γ to be the tangent vector field $\partial/\partial\theta$ on $\mathbb{S}^1$. If $n \geq 4$ then we choose a vector field with irrational slope on $\mathbb{T}^{n-2}$, that is $\Gamma := \alpha_1 \frac{\partial}{\partial x_1} + \cdots + \alpha_{n-2} \frac{\partial}{\partial x_{n-2}}$, with $dim_{\mathbb{Q}}(\alpha_1, \ldots, \alpha_{n-2}) = n-2$. In all the cases, we define a vector field on P by $W := f \cdot \Gamma + \mathcal{R}$.

This produces a vector field on half the plug. To complete it, we just take the *mirror-image* with respect to the plane $\{z = 0\}$, with reversed orientation. This insures that if an orbit enters at some point $(x, -1)$ and reaches the plane $\{z = 0\}$, then it will exit at the symmetric point $(x, +1)$. If it doesn't happen, then it is trapped.

In dimension 3, the obtained plug will have two (and only two) periodic orbits. In this way, every orbit entering the ring $\mathbb{S}^1 \times [\frac{3}{2}, \frac{5}{2}] \times \{-1\}$ is trapped, winds up, and converges to the circular periodic orbit contained in the plane $\{z = -\frac{1}{2}\}$. This and lemma complete the proof of the first Wilson theorem.

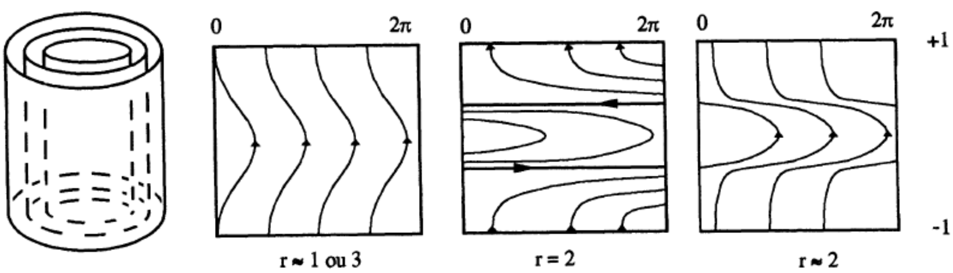


Figure 1: Wilson plug

In dimension ≥ 4, the ω-limit set of the trapped orbits is $\mathbb{T}^{n-2} \times \{-\frac{1}{2}\}$, which does not contain new periodic orbits. This proves the second theorem.

We therefore obtained a construction of vector fields with no periodic orbit in dimension ≥ 4. The solution described above is smooth, and could be made analytic (to do so, we can't ask the original field W to be equal to the constant field $\partial/\partial z$ on the boundary, but we can replace this condition by "being conjugated"). We would then use a difficult theorem by Morey-Grauert that

asserts that every smooth compact manifold has a unique compatible analytic structure, up to smooth diffeomorphisms.
We can now point out what is the main goal of our work: finding a convenient plug in dimension 3 containing no periodic orbit.

1974 **Schweitzer's example: a C^1 nonsingular vector field without periodic orbits**

The basic idea of using plugs to destroy periodic orbits of a nonsingular vector field in dimension 3 remains the same. But instead of trapping them on two circles, we will use a more complicated minimal set that was constructed by Denjoy in [1].

This time, the shape of our plug will be $(\mathbb{T}^2 \setminus \mathbb{D}^2) \times [-1, 1]$, which can be embedded in $\mathbb{R}^3$. The next paragraph explains how to construct a nonsingular field on $\mathbb{T}^2$ without periodic orbits, which has a non-trivial closed invariant subset with empty interior $\mathcal{D}$. We take the disk outside of $\mathcal{D}$ to construct our surface $\mathbb{T}^2 \setminus \mathbb{D}^2$.

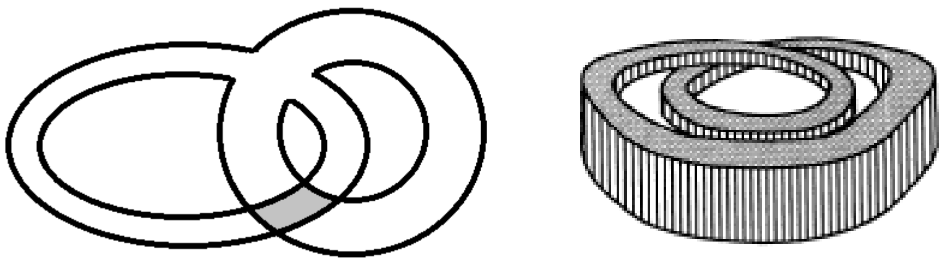


Figure 2: Shape of the immersed entrance $\mathbb{T}^2 \setminus \mathbb{D}^2$ and the embedded Schweitzer plug

Denjoy's example (1932) :

The desired flow on $\mathbb{T}^2$ will be obtained from a suspension of a diffeomorphism f on $\mathbb{S}^1$.

To build it, we start with an irrational rotation ρ, and we insert an infinite family of suitable intervals $(I_n)_{n\in\mathbb{Z}}$ in the circle, one interval for each point $x_n := \rho^n(x)$ of some orbit. A first assumption should be that the length of the intervals is summable, i.e.,

$$\sum_{n\in\mathbb{Z}} |I_n| < \infty$$

.

Thus, if d denotes the distance between two points in $\mathbb{S}^1$, the new distance between two points x_m and x_n (which is also the distance between two intervals I_m and I_n) is $d(x_m, x_n) + \sum\limits_{x_k\in[x_m,x_n]} |I_k|$ and we obtain a manifold that is homeomorphic to the circle $\mathbb{S}^1$. We could even put a C^1 differentiable structure on it.

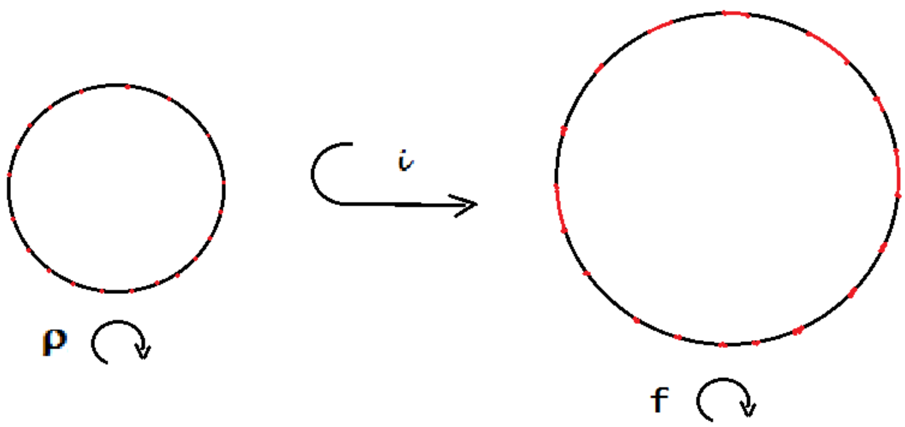


Figure 3: The construction of Denjoy diffeomorphism

Denjoy showed that, for a good choice of intervals lengths $|I_k|$, one can construct a C^1 diffeomorphism f on our manifold such that $f(I_n) = I_{n+1}$ and $f_{|\mathbb{S}^1\setminus \bigcup\limits_{n\in\mathbb{Z}} I_n}$ is semiconjugate to ρ. That is, for $x \in \mathbb{S}^1 \setminus \bigcup\limits_{n\in\mathbb{Z}} I_n$, we set

$f \circ i(x) = i \circ \rho(x)$, and if we choose for $f_{|I_n}$ suitable smooth functions. Denjoy showed that one can organize this construction in such a way that the resulting f is a C^1 diffeomorphism of the circle (in fact its first derivative has α-Holder derivative, for some $\alpha \in (0,1)$), with no periodic points, and with an invariant Cantor set $\mathbb{S}^1 \setminus \bigcup_{n \in \mathbb{Z}} I_n$. For details, see [3].

By taking a suspension of f, we get a C^1 flow Φ^t on the torus $\mathbb{T}^2$ such that

- There exists a Φ^t-invariant, nowhere dense, compact set $\mathcal{D}$ (the Denjoy set).
- every orbit of $x \in \mathcal{D}$ is dense in $\mathcal{D}$ (i.e., $\mathcal{D}$ is minimal).
- $\mathbb{T}^2 \setminus \mathcal{D}$ is homeomorphic to $]0,1[\times\mathbb{R}$ equipped with the linear flow $\partial/\partial t$.

Figure 4 below represents an approximation of this construction (we only draw a finite number of intervals. The "approximated" Denjoy set appears in white).

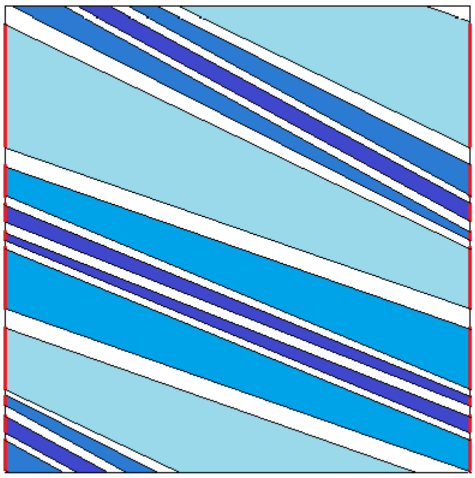


Figure 4: Suspension of f

It is important to note that neither the diffeomorphism f, nor the Denjoy vector field, can be of class C^2. This has been proved by Denjoy who showed, more generally, that

Theorem: *If a C^2 diffeomorphism of the circle does not have periodic points then all the orbits are dense.*

Corollary: *Any C^2 vector field on $\mathbb{T}^2$ without periodic orbits is topologically equivalent to a linear flow with irrational slope. In particular it does not have non-trivial closed invariant sets.*

Schweitzer's plug (1974) As we already said, we can embed $\Sigma \times [-1, +1]$, where $\Sigma = \mathbb{T}^2 \setminus \mathbb{D}^2$, in any open set of $\mathbb{R}^3$ in a C^∞ way. We can even choose this embedding in such a way that segments of the form $\{\star\} \times [-1, 1]$ are mapped to vertical segments in $\mathbb{R}^3$. We consider a positive function $g : \Sigma \times [-1, 0] \to [0, 1]$ such that

- $g = 0$ in the neighboorhood of the boundary of $\Sigma \times [-1, 0]$.
- $g = 1$ on $\mathcal{D} \times \{-\frac{1}{2}\}$.

Then, if we equip the embedded $\Sigma \times [-1, 0]$ with the vector field $g \cdot X + (1 - g)\frac{\partial}{\partial z}$, where X is induced on $\mathbb{T}^2 \setminus \mathbb{D}^2$ by a Denjoy vector field, and paste this half-plug with its mirror image, we get a C^1 plug without periodic orbits as in definition .

Schweitzer used this plug to prove:

Theorem: *Every nonsingular vector field on a three-dimensional manifold is homotopic (in the space of non-singular vector fields) to a C^1 non-singular vector field without periodic orbits.*

Exercises:

Exercise 1. Let X be a non-zero vector field on $\mathbb{S}^2$ such that the associated flow preserves the area. Prove that almost all orbits are closed.

Exercise 2. Let X be a vector field on a compact oriented surface such that the associated flow preserves the area. Prove that the union of periodic orbits is an open subset.

Exercise 3. Let ϕ be a non-singular area preserving flow on $\mathbb{T}^2$. Show that if the flow admits a closed orbit then all orbits are closed.

Exercise 4. Show that there exist volume preserving vector fields on $\mathbb{S}^3$ with only two periodic orbits that are arbitrarily close to the Hopf vector field.

Exercise 5. Show that the Wilson plug is not volume preserving.

Exercise 6. Let X be a vector field on M^3 with a finite number of periodic orbits, as constructed by the Wilson method. Describe all invariant measures of the associated flow.

Exercise 7. Let X be a non-singular vector field on a compact n-manifold M. Show that one can find a finite number of pairwise disjoint embedded balls $\{B_i^{n-1}\}$ such that each B_i is transversal to to X, and the union intersects all orbits.

Exercise 8. Show that in the previous exercise one ball B^{n-1} is enough.

Exercise 9. Show that on any compact 3-manifold there exists a vector field X with only two periodic orbits.

Lecture V

Kuperberg plug

1994 **Kuperberg's example: a smooth nonsingular vector field without periodic orbit**

Twenty years after Schweitzer's work, we finally had the smooth generalization of his theorem:

Theorem (Kuperberg 1993): *Every three-dimensional compact manifold M admits a non-singular* analytic *vector field without periodic orbits.*

We only present the smooth C^∞ version. This theorem will be proved using a plug, "*à la Wilson*", and building a more complicated attractor, as in Schweitzer's work.

The construction of Kuperberg's plug. We have to stay with two images in mind. The first one is a simplified version of the Wilson plug we drew above, in which all the orbits lay in cylinders $\{r = cst\}$ (left part of figure 5). This is our "abstract" plug. The second one is the three-dimensional manifold represented in the right part of figure 5. This is our "actual" plug, that is, in the way it will be embedded in $\mathbb{R}^3$.

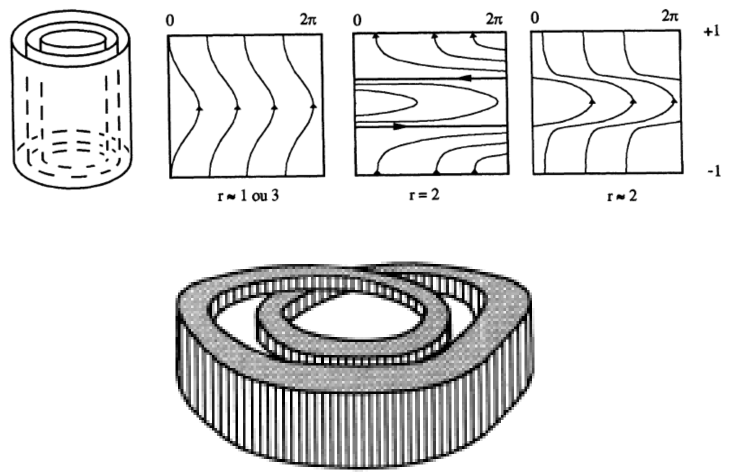


Figure 5: "Abstract" plug and its actual shape

The first image, which is topologically not so different from the second, is only here to help us visualize what's happening. For example, when we say "cylinders", it actually means surfaces $\{r = constant\}$ in the first image, and their straightforward cousins in the second.

As in the previous sections, we construct on our plug P a vector field $\mathcal{W}$ with the following properties:

- Close to the boundary, $\mathcal{W}$ is the constant vector field $\partial/\partial z$,
- $\mathcal{W}$ is tangent to the cylinders $\{r = cst\}$,
- $\mathcal{W}$ is antisymmetric with respect to the plane $\{z = 0\}$,
- $\mathcal{W}$ has a positive vertical component, except on two periodic orbits contained in the cylinder $\{r = 2\}$.

As showed in figure 5, any cylinder $\{r \neq 2\}$ is partitioned into orbits entering at some point $(r, \theta, -1)$ and exiting in the opposite point $(r, \theta, +1)$, after some time tending to infinity as r tends to 2.

On the cylinder $\{r = 2\}$, we have a so called "*Reeb component*".

The rest of the process is a construction which is called *self-insertion*. We will dig two "*mortises*", where we delete the existent vector field, and insert in it two twisted parts of the plug, called the "*tenons*", so as to create another manifold equipped with a smooth non-singular vector field. The final topological object is depicted in figure 6.

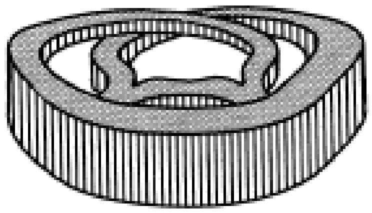


Figure 6: Shape of the plug after self-insertion

The details of the construction are as follows. All the formulas are given with an index i, which takes only two values 1 or 2, and corresponds to one tongue, tenon, or the corresponding mortise.

For a start, we define two "tongues", L_1 (respectively L_2), on the top $\{z = +1\}$ of the cylinder, which are two topological discs delimited by two smooth arcs α_1 and β_1 (respectively α_2 and β_2) as in the figure 7.

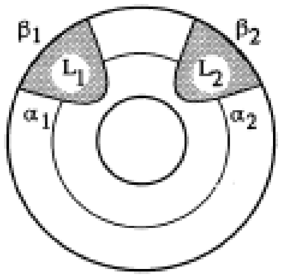
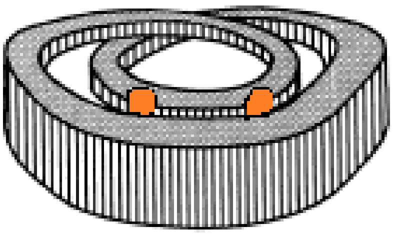


Figure 7: Tongues

The two "tenons" mentioned above are $T_1 := L_1 \times [-1, +1]$ and $T_2 := L_2 \times [-1, +1]$. We reproduce figures 6 with the tenons coloured in orange.



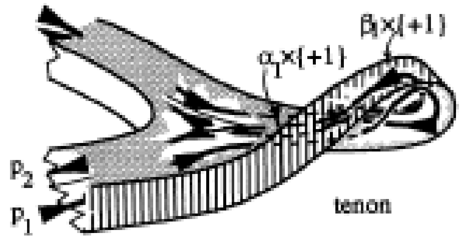


Figure 8: Tenons

The mortises are two zones M_1 and M_2 in front of the tenons, where we erase the vector field $\mathcal{W}$, and whose shapes will correctly be defined together with the insertions.

Insertions of the tongues We want to insert the tenons, that is, to define two embeddings Σ_i from the tenons to the mortises. We first insert the two tongues, transversally to $\mathcal{W}$, and then push the images along the orbits of $\mathcal{W}$. The fact of $\mathcal{W}$ being vertical close to the boundary and horizontal near the periodic orbits forces Σ_i to rotate by 90 degrees.

We sum up here all the conditions imposed to the insertions Σ_i, $i = 1, 2$. The reader can check that this construction is possible. Their usefulness will only appear clearly in next paragraph, where we prove theorem .

Condition 1

- Σ_i is C^∞
- The image of $\Sigma_i(\alpha_i)$ is of the form $\alpha_i' \times \{-1\}$, where α_i' is an arc of the circle $\{r = 1,\ z = -1\}$ of the boundary.
- The image of $\Sigma_i(L_i)$ is transverse to $\mathcal{W}$.

To destroy the periodic orbits, it is necessary to trap them. That's why we also ask for (figure 9):

Condition 2

- $\Sigma_i(L_i)$ meets p_i at a unique point.
- $\Sigma_i(L_i)$ does not meet $p_j,\ j \neq i$.

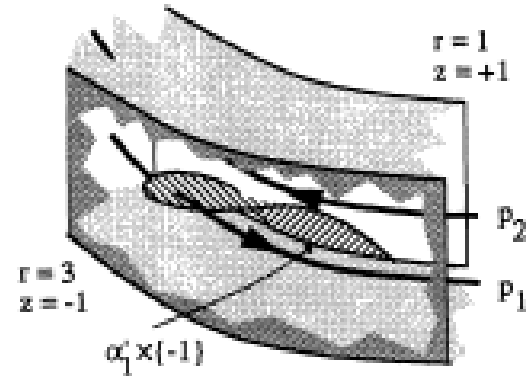


Figure 9: Insertion of a tongue

Insertion of the tenon We now extend the previous insertions to new ones, also called Σ_i, whose starting sets are the tenons T_i, and targets at the mortises M_i. We now require that:

Condition 3

- These insertions extend the previous ones.
- The vertical arcs of the form $\{*\} \times [0,1]$, with $*$ a point of L_i, match with arcs contained in orbits of $\mathcal{W}$.
- The images of $\alpha_i \times \{-1\}$ by Σ_i are $\alpha_i' \times \{-1\}$.
- The two mortises $\Sigma_i(L_i \times [-1,+1]),\ i = 1,2$ are disjoint.

Remark: By "*changing time*", ie multiplying $\mathcal{W}$ by a C^∞ strictly positive function, we do not change the orbits of $\mathcal{W}$, and this allows us to assume that Σ_i matches the vertical segments of the field $\partial/\partial z$ in the tenons to $\mathcal{W}$ in a C^∞ way.

The following condition is only a more precise version of condition 2:

Condition 4 The tongue L_i contains a point of the form $(2, \theta_i)$ such that the vertical segment $\{(2, \theta_i)\} \times [-1, +1]$ is inserted by Σ_i in an arc of the periodic orbit p_i of $\mathcal{W}$.

On the other hand, this one is new, and will play a central role in the proof of the theorem:

Radius Condition If a point $(\overline{r}, \overline{\theta})$ of $L_1 \cup L_2$ is sent (by Σ_1 or Σ_2) on a point (r, θ, z) of W, then $\overline{r} > r$ unless $(\overline{r}, \overline{\theta})$ is one of the two points $(2, \theta_1)$ or $(2, \theta_2)$.

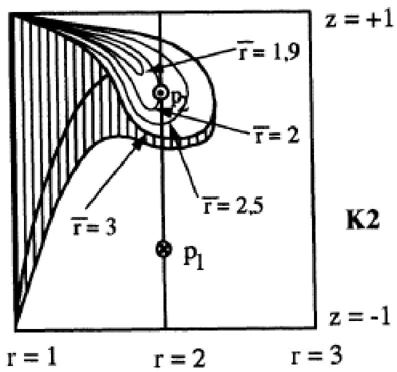


Figure 10: A mortise sideways on

We use our insertion to identify, for each x in $L_i \times \{-1, +1\} \cup \beta_i \times [-1, +1]$ the points x and $\Sigma_i(x)$. This gives the compact set K of figure 6. We denote by τ the natural projection on K.

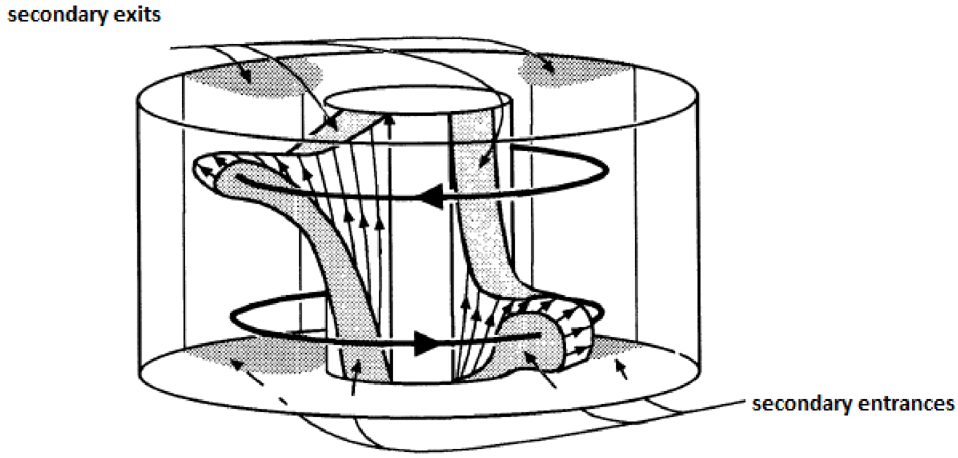


Figure 11: A better view of the quotient

The compact set K is a smooth 3-manifold with nonempty boundary. Since the insertions Σ_i match the vertical field on $\mathcal{W}$, and $\mathcal{W}$ is equal to $\partial/\partial z$ close to the boundary, we get a smooth vector field $\mathcal{K}$ in K.

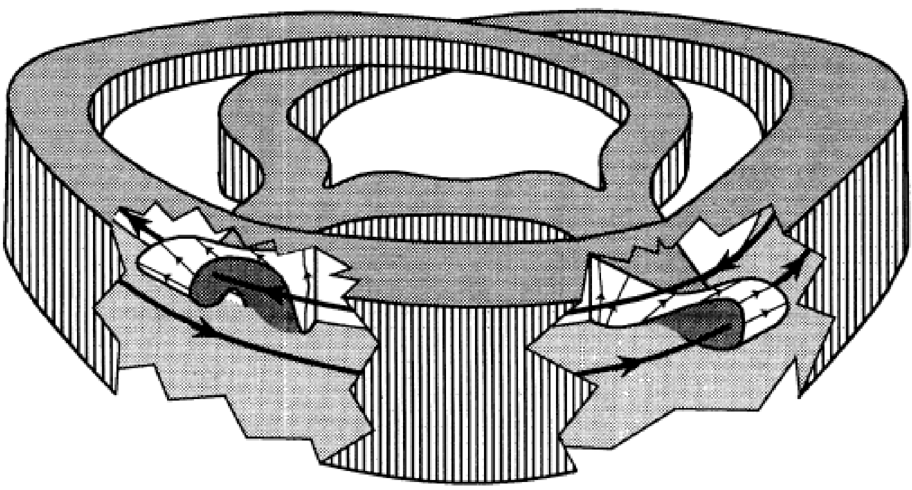


Figure 12: The plug

The boundary of K is composed of a lateral surface S, on which $\mathcal{K}$ is tangent, and corresponds to $\partial/\partial z$, and two components which are transverse to $\mathcal{K}$, which we call respectively "*entering*" and "*exiting*". Each of these component is homeomorphic to a torus $\mathbb{T}^2$ minus of two discs. It is important to remark that these components are canonically isomorphic: to each point $\tau(r, \theta, -1)$ of the entering component corresponds the point $\tau(r, \theta, +1)$ of the exiting one. We say that two such points are "*in front of each other*."

Picture 12 shows that K can be embedded in $\mathbb{R}^3$ in a smooth way so that:

- Two points in front of each other are sent in the same vertical.
- The field $\mathcal{K}$, extended by $\partial/\partial z$ outside of K is C^∞.

We now are going to use K so as to trap periodic orbits, without creating new ones. The next section explains this.

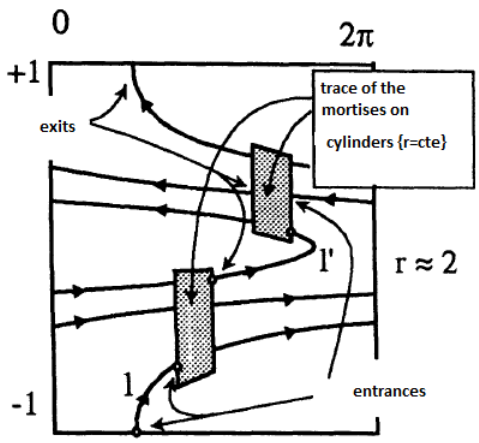
Lecture VI

Aperiodicity of Kuperberg's plug

Kuperberg's theorem

We are now going to prove that Kuperberg's plug K is indeed an aperiodic trap, ie that it satisfies all the four conditions of the plug definition , as well as aperiodicity. Of course, conditions 1 and 2 are already satisfied by our construction. To check the other properties, we need to handle a correct description of its orbits. Let's first see what happens to the Wilson orbits during the process of self-insertion.

Wilson arcs: A Wilson orbit for the vector field $\mathcal{W}$ now meets several times each mortise, and is split into compact arcs that we will call "*Wilson arcs*".



Each Wilson arc begins and ends with a primary or secondary entrance or exit point (respectively P.En, P.Ex, S.En and S.Ex). It is also interesting to note that Wilson arcs are made of points with the same radius (of course, since the Wilson orbits are contained in cylinders $\{r = cst\}$), so we can talk about "*the radius of a Wilson arc*". Every Wilson orbit is oriented by the vector field in a natural way, so that we can say that *a Wilson arc is before another* if they are on the same Wilson orbit and the extremity of the first one is met before the beginning of the second (when we follow the orbit in the sense given by the orientation).

For instance, each periodic orbit p_i meets exactly once the mortise M_i, and thus gives rise to exactly one Wilson arc $\overset{p'_i}{\overline{\text{S.Ex} \qquad \text{S.En}}}$.

An orbit beginning at $\{r = 2\}$ gives rise to an infinite sequence of Wilson arcs

$$\overset{\lambda_1}{\overline{\text{P.En or S.En} \quad \text{S.En}}}\,;\quad \overset{\lambda_2}{\overline{\text{S.Ex S.En}}}\,;\quad \overset{\lambda_3}{\overline{\text{S.Ex S.En}}}\,\dots\quad \overset{\lambda_k}{\overline{\text{S.Ex S.En}}}\,\dots$$

An orbit beginning at $\{r \simeq 2\}$, $r \neq 2$, gives rise to a finite sequence of Wilson arcs (we emphasise the change of mortise) :

$$\overset{\lambda_1}{\overline{P.En \text{ or } S.En_1 \; S.En_1}}\,;\quad \overset{\lambda_2}{\overline{S.Ex_1 S.En_1}}\,\dots\quad \overset{\lambda_{i-1}}{\overline{S.Ex_1 S.En_1}}\,;\quad \overset{\lambda_i}{\overline{S.Ex_1 S.En_2}}\,;\quad \overset{\lambda_{i+1}}{\overline{S.Ex_2 S.En_2}}\,\dots$$

$$\dots\overset{\lambda_{k-1}}{\overline{S.Ex_2 S.En_2}}\,;\quad \overset{\lambda_k}{\overline{S.Ex_2 \; P.Ex \text{ or } S.Ex_2}}\,.$$

An orbit beginning at $r \simeq 1$ or $r \simeq 3$ is either destroyed (only in the case $r = 1$) when we dig the mortises, or gives rise to a sequence of one or two Wilson arcs (exactly one of the four following cases: $\overline{P.En\,P.Ex}$ or $\overline{S.En\,S.Ex}$ or $\overset{\lambda_1}{\overline{P.En S.En_1}}$; $\overset{\lambda_2}{\overline{S.Ex_1 P.Ex}}$ or $\overset{\lambda_1}{\overline{P.En S.En_2}}$; $\overset{\lambda_2}{\overline{S.Ex_2 P.Ex}}$)

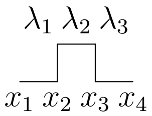
Kuperberg arcs We call a *Kuperberg arc* a compact arc γ of an orbit of the Kuperberg flow $\mathcal{K}$ which begins and ends with a primary or secondary entrance or exit point. Considering the properties of the insertions, it is clear that a Kuperberg arc is naturally divided in a finite sequence of Wilson arcs $\lambda_1, \dots, \lambda_k$. Note that the radiuses of the Wilson arc constituting γ are generally not the same, due to the numerous insertions, so we have a sequence of changing radiuses. The changes are ruled by the condition and will be crucial to prove aperiodicity.

Levels The last notion to be introduced (it was suggested by Matsumoto) is the concept of levels, which helps to describe all the transitions of the arc γ. The levels are a sequence of integers $lev(i),\ i = 1, \dots, k$ defined by $lev(1) = 0$ and for $i = 1, \dots, k-1$ by $lev(i+1) = lev(i) + 1$ if x_i is an entrance point or $lev(i+1) = lev(i) - 1$ if it is an exit point.

We are now able to state and prove the main lemma. It essentially says (see the immediate consequence after the proof) that "*two points that are connected after the insertion were connected before*".

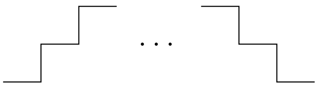
Lemma *Let $[\lambda_1, \dots, \lambda_k]$ be a Kuperberg arc such that $lev(1) = lev(k) = 0$ and $\forall\ i\ lev(i) \geq 0$. Then λ_1 and λ_k are on the same Wilson orbit, and λ_1 is before λ_k.*
In particular, if λ_1 begins by an entrance point x, and λ_k ends with an exit point y, then y is in front of x.

Proof : The proof is by induction on k. The condition $lev(1) = lev(k) = 0$ forces k to be odd.

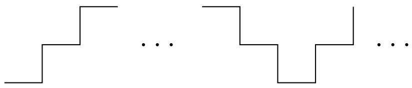
For $k = 3$ the only possibility is 

; x_2 and x_3 are respectively secondary entrance and exit points, so they belongs to a tenon T_i and are in front of each other. We know that, by construction, the image by Σ_i of the segment $\{x_2\} \times [-1, +1]$ is a portion of Wilson orbit, which connects λ_1 and λ_3. The desired conclusion immediately follows, and this proves the lemma for $k = 3$.

Let's take $k \geq 5$. We have to consider two situations:

Case 1: $\forall\ i\ \neq 1, k\ lev(i) > 0$. 

Then the lemma applied to $[\lambda_2, \ldots, \lambda_{k-1}]$ says that λ_2 is before λ_{k-1}. In particular, the secondary exit point x_k is in front of the secondary entrance point x_2. Then again we know that, by construction, the image by Σ_i of the segment $\{x_2\} \times [-1, +1]$ is a portion of Wilson orbit, which connects λ_1 and λ_k. This proves the lemma in this case.

Case 2: $\exists\ i\ \neq 1, k\ lev(i) = 0$. 

Then the lemma applied to $[\lambda_1, \ldots, \lambda_i]$ and to $[\lambda_i, \ldots, \lambda_k]$ says that λ_1 is before λ_i and λ_i is before λ_k so that λ_1 is before λ_k. This proves the lemma in this case, and ends the proof.$\Box$

Condition 3 now directly follows from the lemma. Let's take two points $x = (r, \theta, -1)$ and $y = (r', \theta', -1)$ connected by a Kuperberg arc $[\lambda_1, \ldots, \lambda_k]$. We suppose that x (respectively y) is a primary entrance (respectively exit) point. The first observation is that $lev(2) = 2$ and $lev(k) = lev(k-1) - 1$, because we cannot go from a primary entrance to a secondary exit in just one arc, and from a secondary entrance to a primary exit. This forbids the situation where $lev(i+1) = lev(i) + 1$ for all i or $lev(i+1) = lev(i) - 1$ for all i .

Consequently, we must show that the associated sequence of levels cannot reach negative values. Indeed, if it did, then by stopping just before the first arc with negative level (let's say λ_{i+1}), we would have a kuperberg arc $[\lambda_1, \ldots, \lambda_i]$ with nonnegative level sequence, beginning and ending by zero, whose first point is a primary entrance point, and last point an exit point. Thus, by applying the lemma, λ_i would be in the same orbit as λ_1, and then it would end with a primary exit point, which of course cannot be.

Let's now imagine that the last level is not zero. Then, by stepping backward, we must encounter a first Wilson arc (say λ_i) with the same level value than λ_k, and beginning with an entrance point. The lemma again, applied to the Kuperberg arc $[\lambda_i, \ldots, \lambda_k]$, gives the conclusion that λ_i is before λ_k in the same Wilson orbit, which would mean that λ_i begins with a primary entrance point, which of course cannot be.

We conclude that all the levels are nonnegative and that $lev(1) = lev(k) = 0$, so that, by the lemma, λ_1 is before λ_k, and $(r, \theta) = (r', \theta')$.

The previous proof also shows that condition 4 is satisfied: the orbit which enters the plug at point $(2, \theta, -1)$ never exits. Indeed, if it were the case, then it would do so through the point $(2, \theta, +1)$, and by applying the main lemma , there would exist a Wilson trajectory connecting these two points, which is obviously not the case.

We shall now check the most interesting and central assertion: the Kuperberg plug doesn't house any periodic orbit. This will end with the proof of Kuperberg theorem.

Aperiodicity We use *reductio ad absurdum* and suppose that there exists a Kuperberg arc $P := [\lambda_1, \ldots, \lambda_k]$ with $\lambda_1 = \lambda_k$. We can assume that $lev(i) \geq 0$ for all i, only by beginning P with the Wilson arc of lowest level; and also that for all i such that $1 \leq i < j \leq k-1$ $\lambda_i \neq \lambda_j$ by taking P minimal.

Case 1 We suppose that $lev(k+1) = lev(1) = 0$.

Then the lemma implies that λ_1 is before λ_{k+1}, and they are equal. But the only Wilson arcs that are before themselves are p'_1 and p'_2! Thus, we have $\lambda_1 = \lambda_{k+1} = p'_i$, with $i = 1$ or 2, beginning with the secondary exit point x and ending with the secondary entrance point y, which are matched with two points of radiuses 2 in the corresponding tongues. Those points are not connected by a Wilson arc, and then, they cannot be connected by a Kuperberg arc. This completes the proof in this case.

Case 2 We suppose that $lev(k+1) = n > 0$ (the negative case is analogous and is solved by reversing the orientation of $\mathcal{K}$).

We have to remember the way we built our insertions. Note that this is the first and only time where we use the radius condition . Every time our arc hits the mortise M_i in a secondary entrance, it is “deflected” to the corresponding point of tenon T_i (remember figure), with an increasing radius. That is, if we have two Wilson arcs λ_i and λ_{i+1} consecutive in a same Kuperberg orbit, with $lev(i+1) = lev(i) + 1$, then the radiuses satisfy $r_{i+1} \geq r_i$ with equality if and only if λ_i is one of the two arcs p'_1 or p'_2.
In particular, the sequence of radiuses cannot be increasing all the time, because as $r_1 = r_{k+1}$, we would have $r_i = r_1$ for all i, thus $\lambda_i = p'_1$ or p'_2 for all i, which is impossible because those arcs are not $\mathcal{K}$-consecutive.

In the general case, we can find a sequence of indices $1 = i_1 < i_2 < \cdots < i_{n+1} \leq k+1$ such that $lev(i_a) = a - 1$ and $lev(i) \geq a - 1$ for $i \geq i_a$. Then, lemma allows us to “fill in the blanks”: every time that $i_{a+1} > i_a + 1$, we apply the lemma to the sequence of $\mathcal{K}$-consecutive Wilson arcs $l_{i_a}, \ldots, l_{i_{a+1}-1}$, and we have that λ_{i_a} is $\mathcal{W}$-before $\lambda_{i_{a+1}-1}$, so that $r(i_a) = r(i_{a+1} - 1)$ for $a = 1, \ldots, n$. Then again, the fact that $r_1 = r_{k+1}$ implies that all the $r(i_a)$ are equal, and thus that each arc is equal to p'_1 or p'_2 for $a = 1, \ldots, n$.

If $i_{a+1} > i_a + 1$, we know that l_{i_a} is $\mathcal{W}$-before $l_{i_{a+1}-1}$, and thus that l_{i_a} is equal to p'_1 or p'_2, which is impossible because the $l_1, \ldots, l_k$ are all distinct.□

This ends the proof of theorem .

Kuperberg's plug is not volume-preserving. In fact, we can show that there exists some $\epsilon > 0$ such that all the orbits with entrance points (r, θ) with $2 - \epsilon < r < 2$ never exit. Thus, we conclude that for

$$\Omega := \{\ \Phi^t(r,\theta) \mid t \geq 0,\ 2 - \epsilon < r < 2,\ (r,\theta) \textit{ is a primary entrance point }\},$$

then $\Phi^1(\Omega)$ is stricly contained in Ω. Hence

$$\mu(\{\ \Phi^t(r,\theta)\mid t>1,\ 2-\epsilon<r<2,\ (r,\theta)\ \textit{is a primary entrance point}\ \})=0.$$

Since it is an open set, this proves the claim.

G. Kuperberg proved the following:

Theorem: *There exists a nonsingular vector field on any three-dimensional manifold which is C^1, volume-preserving, and without periodic orbit.*

However, it is still unknown if we can ask the vector field to be both C^2-smooth and volume-preserving.

Lecture VII

Schwartzman's cycles

In this lecture we give a characterization of suspension flows on compact manifolds. The result was obtained by several people, including Schwartzman, Sullivan and Fried.

Suppose ϕ is a flow on a compact manifold V. Let $\mathcal{M}$ denote the set of all ϕ-invariant probability measures on V. Then $\mathcal{M}$ is a non-empty compact convex set. We fix an element μ of $\mathcal{M}$, and for any closed 1-form ω on V we set

$$\theta(\mu)(w) = \int \omega(X) d\mu.$$

Now suppose $\omega = df$ for some function f. Since μ is ϕ-invariant, it follows that $\int f \circ \phi^t d\mu = \int f d\mu$. Hence

$$\frac{d}{dt} \int (f \circ \phi^t - f) d\mu = 0 = \int_V \omega(X) d\mu.$$

This shows that $\theta(\mu)$ induces a linear map from $H^1(V, \mathbb{R})$ to $\mathbb{R}$, i.e. θ induces a map from $\mathcal{M}$ to $H_1(V, \mathbb{R})$.

Theorem : *Let ϕ be a flow on a compact 3-manifold V, and let $\mathcal{M}$ and θ be as described above. Then the flow ϕ is a suspension flow if and only if $\theta(\mathcal{M}) \subset H_1(V, \mathbb{R})$ does not contain the origin.*

Proof. Since μ is invariant under ϕ, almost all $x \in V$ are recurrent, i.e., there exists a set of full μ-measure such that for any element x in that set we can find a sequence $\{t_n\}$ with the property that $\phi_{t_n}(x) \to x$ as $n \to \infty$. Now we fix a point x, and for each n join the points $\phi_{t_n}(x)$ and x by a short geodesic. Let γ_n denote the resulting loop. It is easy to see that for sufficiently large n, $[\gamma_n] \in H_1(V, \mathbb{R})$ is "almost" independent of the choices involved: these choices can only change $[\gamma_n]$ by some bounded amount. One can also show that

$$\int_V \lim_{n\to\infty} \frac{1}{t_n}[\gamma_n] d\mu(x) = \theta(\mu).$$

Now suppose ϕ is a suspension flow. Let $M \subset V$ and $h : M \to M$ be such that ϕ is the suspension of h. Note that there is a projection map from V to $\mathbb{S}^1$ with copies of M transversal to the flow as fibers. For any loop $\gamma \in H_1(V, \mathbb{R})$ let $l(\gamma)$ denote the number of times γ intersects M. It is easy to see that this induces a linear map from $H_1(V, \mathbb{R})$ to $\mathbb{R}$. Furthermore from the above remark it follows that $l(\theta(\mu)) \geq 1$ for all $\mu \in \mathcal{M}$. This proves the first half of the theorem.

For any 1-form ω and for any $t > 0$ we define

$$\omega_t = \frac{1}{t}\int_0^t \phi_s^*(\omega) ds.$$

From the given condition one can deduce that there exist ω and t_0 such that $\omega_t(X) > 0$ for all $t > t_0$. In particular, we can find ω such that $\omega(X) > 0$. We can now approximate ω by some other 1-form, still positive on X, and such that the periods of ω (integrals over loops) constitute a discrete rank one subgroup of $\mathbb{R}$. Finally, multiplying by some constant, we can even assume that all periods are integral. We choose a basepoint b. For any $x \in V$, we join x and b by a path and define

$$\pi(x) = \int_b^x \omega.$$

It is easy to see that $\pi(x)$ is well defined as an element of $\mathbb{R}/\mathbb{Z}$. One can use the fibration by π to show that ϕ is a suspension flow. □

Example : Suppose X has a periodic orbit γ. Let μ be the Lebesgue measure on γ. It is easy to check that $\theta(\mu) = [\gamma]/T$, where T is the period of γ.

Birkhoff sections : Let X be a non-singular flow on a compact 3-manifold V. Let S be a surface with boundary that is imbedded in V. We call it a Birkhoff section if the following three conditions are satisfied.

1) The flow is transversal to the interior of V.
2) The set ∂S consists of periodic orbits.
3) The surface S intersects all orbits.

Let Σ be a compact orientable surface and let m be a Riemannian metric on Σ with constant negative curvature. Let ϕ be the geodesic flow on $T_1\Sigma$, the unit tangent bundle of Σ. Since ϕ is invariant under the symmetry sending a vector to its opposite, the set $\theta(\mathcal{M})$ is symmetric with respect to the origin. Hence $0 \in \theta(\mathcal{M})$ and ϕ is not a suspension flow. However, we have the following :

Theorem : (Birkhoff) *There is a Birkhoff section S for this flow such that S can be identified with the 2-torus with the union of twelve disks removed.*

Proof. We consider the six geodesics beneath (in blue and green):

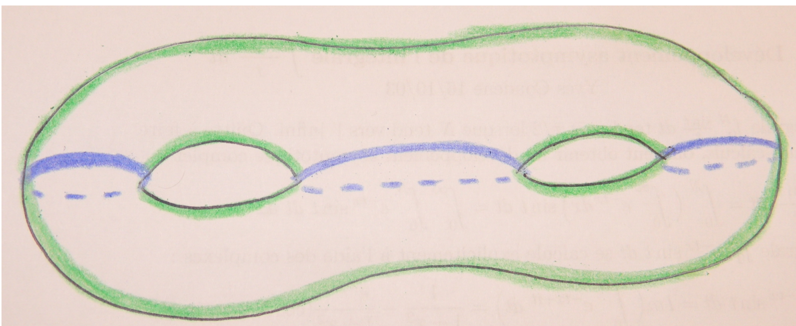


Figure 13: bitorus with 6 geodesics

They lift to twelve periodic orbits of the geodesic flow on $T^1\Sigma$. Now if we cut the bitorus along these geodesics, we get four hexagonal pieces with geodesic boundaries and corners. We take two opposite hexagons and we fill their interiors with one point and a family of concentric strictly convex curves (this is obviously possible if one look at an hexagon in the Klein model where geodesics are straight lines):

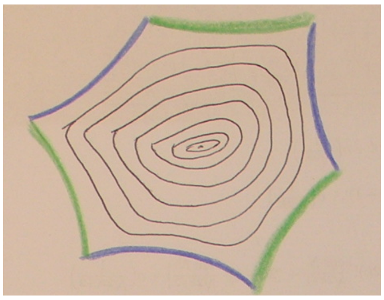


Figure 14: An hexagon

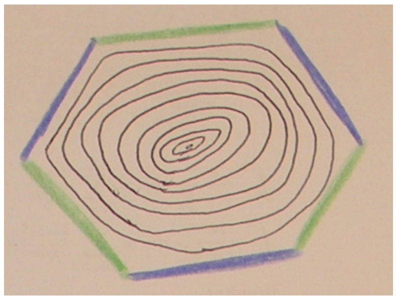


Figure 15: An hexagon in the Klein model

For each point x neither in the center, the boundaries, nor the corners, we exactly have two unit tangent vectors to the unique strictly convex curve that passes through x. This gives a section from almost all the hexagon to $T^1\Sigma$ with exactly two images for each point. The image of the hexagon deprived of its center and its boundaries is thus a couple of surfaces.

What happens on the center and the boundaries?

Let's first forget about the corners and imagine that we are lifting from our surface to the unit tangent bundle a disc bounded by closed curves, whose boundary would be smooth. Passing to limits as the convex curves get close to the center, we have a whole circle of unit tangent vectors above the center. This circle connects the two precedent surfaces in a self-closed double spiral staircase.

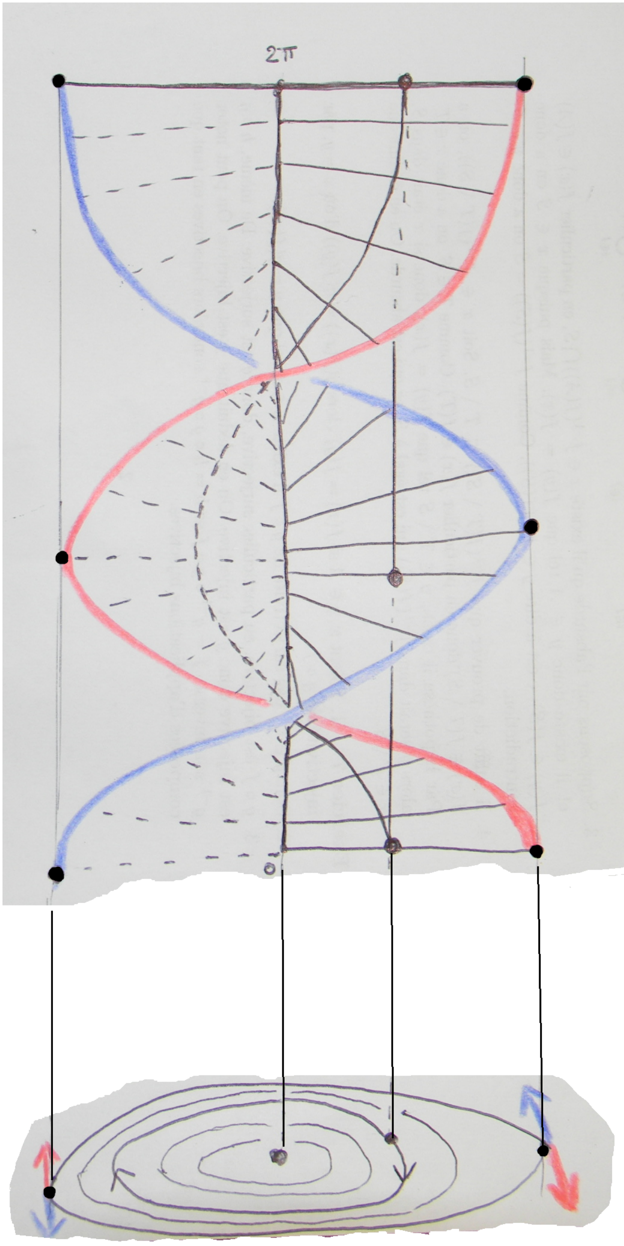


Figure 16: Double spiral staircase. The projection is a disc, with a circle above the center and two points above each point of the interior.

For each geodesic segment of the boundary of our hexagons we have two

possible orientations, which give two lifted curves (in red and blue in the picture).
Above each corner x, we have four directions tangent to the geodesics of the boundaries that join at x. They divide the circle of unit tangent vectors at x into four segments (fig. 17). Our surfaces lean on two of these segments as in figure 18:

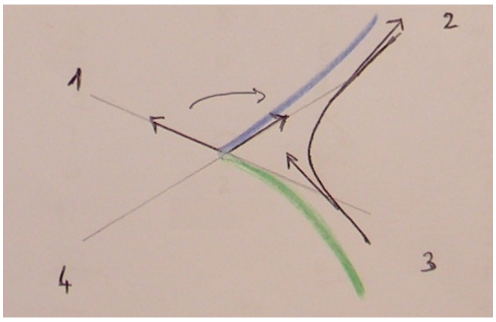


Figure 17: A corner

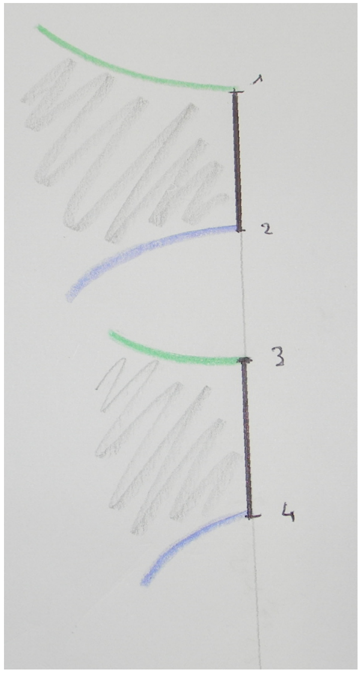


Figure 18: A lifted corner

The final result for one hexagon is a surface of $T^1\Sigma$ with geodesic boundaries (in blue and red in the figures) and non-geodesic segments (in black in the figures). Its Euler-Poincaré characteristics is 0, as one easily convince (an hexagon minus the center has the homotopy type of a circle with Euler characteristics 0, and two circles glued along one circle gives $0 + 0 - 0 = 0$).
But two hexagons are connected through corners, and this makes disappear the non-geodesic segments from the boundaries. The final topological result is a surface with twelve boundary components and its Euler characteristics is equal to $0+0-2\times 6 = -12$. If we fill all the boundaries with discs, we get a surface without boundary with Euler characteristics -12 + 12 = 0, hence is a torus.

The next step is to prove that our surface is a Birkhoff section.
First, the surface is transverse to the geodesic flow, by strict convexity of the convex curves (ie the curves have geodesic curvature ¿ 0 while the geodesics are of curvature 0).
Second, the surface intersects transversally all but the twelve chosen orbits. This is true because a geodesic is infinite so it cant stay in one hexagon. We leave it as an exercise to the reader to show that a geodesic couldnt pass from one hexagon to another only through corners. □

Remark. One can construct this section in such a way that the first return map is an element of $GL(2, \mathbb{Z})$.

Lecture VIII

Birkhoff sections

In this lecture we introduce the notions of linking number and left handed vector fields. Let $\gamma_1, \gamma_2 : \mathbb{S}^1 \to \mathbb{S}^3$ be two disjoint embedded loops in $\mathbb{S}^3$. We delete a point in $\mathbb{S}^3$ and view these loops in $\mathbb{R}^3$. We project them generically onto a plane, and at each point of intersection of these two projections, assign $+1$ or -1, depending on whether the corresponding tangent vectors form an oriented or unoriented basis. We add all these numbers and denote the resulting number by $\mathrm{lk}(\gamma_1, \gamma_2)$. It is called the linking number of γ_1 and γ_2.

The linking number $\mathrm{lk}(\gamma_1, \gamma_2)$ can be defined in various other ways. For example, since γ_1 is a an embedded loop, it follows that $H_1(S^3 - \gamma_1, \mathbb{Z})$ is isomorphic with $\mathbb{Z}$. It can be shown that $[\gamma_2]$, as an element of $H_1(S^3 - \gamma_1, \mathbb{Z})$ is equal to $\mathrm{lk}(\gamma_1, \gamma_2)$. The linking number can also be defined as the number of times γ_2 intersects S, where S is any oriented surface with γ_1 as it's boundary. Alternatively suppose γ_1 and γ_2 are two disjoint loops in $\mathbb{R}^3$. We define a map h from $\mathbb{S}^1 \times \mathbb{S}^1$ to $\mathbb{S}^2$ by

$$h(t_1, t_2) = \frac{\gamma_1(t_1) - \gamma_2(t_2)}{||\gamma_1(t_1) - \gamma_2(t_2)||}.$$

The linking number of γ_1 and γ_2 is the degree of this map. If γ_1 and γ_2 are two disjoint loops in $\mathbb{R}^3$ then $\mathrm{lk}(\gamma_1, \gamma_2)$ can be computed by the following explicit formula due to Gauss :

$$lk(\gamma_1, \gamma_2) = \frac{1}{4\pi} \int\int \frac{det(\gamma_1(t_1) - \gamma_2(t_2), \gamma_1'(t_1), \gamma_2'(t_2))}{||\gamma_1(t_1) - (\gamma_2(t_2)||^3} dt_1 dt_2.$$

Let ϕ be a non-singular flow on $\mathbb{S}^3$, and let $\mathcal{M}$ denote the compact convex set of all ϕ-invariant probability measures on $\mathbb{S}^3$. Note that each periodic orbit

γ gives rise to an element of μ of $\mathcal{M}$, where μ is the Lebesgue measure on γ. Hence elements of $\mathcal{M}$ can be viewed as generalized periodic orbits. We now generalize the notion of linking number in this context.

Let μ_1 and μ_2 be two elements of $\mathcal{M}$. It can be shown that set of non-recurrent points has measure zero with respect to any element of $\mathcal{M}$. We choose two recurrent points x and y, and choose $\{t_n\}$ and $\{s_n\}$ such that $\phi_{t_n}(x) \to x$ and $\phi_{s_n}(y) \to y$. For each n obtain loops γ_x^n and γ_y^n at x and y respectively by joining $\phi_{t_n}(x)$ to x and $\phi_{s_n}(y)$ to y. We define a function $h : \mathbb{S}^3 \times \mathbb{S}^3 \to \mathbb{R}$ by

$$h(x, y) = \lim_{n\to\infty} \frac{1}{s_n t_n} \mathrm{lk}(\gamma_x^n, \gamma_y^n).$$

It turns out that this limit exists for μ_1 almost every x and μ_2 almost every y and depends only on x and y. Now we define

$$\mathrm{lk}(\mu_1, \mu_2) = \int\int h(x, y) d\mu_1(x) d\mu_2(y).$$

Definition : A non-singular vector field on $\mathbb{S}^3$ is called *left handed* if $\mathrm{lk}(\mu_1, \mu_2) > 0$ for every $\mu_1, \mu_2 \in \mathcal{M}$.

Theorem : *If X is left handed then any finite collection of periodic orbits of X is the boundary of a Birkhoff section.*

Example. Let H denote the Hopf flow on $\mathbb{S}^3$. We choose n complex numbers $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ and define a set $B \subset \mathbb{S}^3$ by

$$B = \{(z_1, z_2) : (z_1 - \alpha_1 z_2) \cdots (z_1 - \alpha_n z_2) \in \mathbb{R}^+\}.$$

It can be verified that B is a Birkhoff section.

Lecture IX

Left handed flows

Let M be a 3-manifold. A bi-form Ω of degree $(1,1)$ is a symmetric bilinear map $\Omega_{x,y}(v_x, v_y)$, defined whenever $v_x \in T_xM, v_y \in T_yM$ and $x \neq y$. Let Ω be a bi-form on $\mathbb{S}^3$. It is called a *Gauss form* if for any two disjoint loops $\gamma_1, \gamma_2 : S^1 \to S^3$ we have

$$lk(\gamma_1, \gamma_2) = \int \int \Omega_{\gamma_1(t_1), \gamma_2(t_2)}(\gamma_1^{'}(t_1), \gamma_2^{'}(t_2)) dt_1 dt_2.$$

Theorem : *Let X be a non-singular vector field on $\mathbb{S}^3$. Then the following two conditions are equivalent :*
a) For any $\mu, \nu \in \mathcal{M}$, $lk(\mu, \nu) > 0$.
b) There exists a Gauss linking form Ω such that $\Omega_{x,y}(X_x, X_y) > 0$ for all x, y.

Now suppose X is a left handed vector field on $\mathbb{S}^3$, and γ is a periodic orbit. We claim that X admits a Birkhoff section. To see this we first define a 1-form ω by

$$\omega_x(v) = \int \Omega_{(x, \gamma(t))}(v, \gamma^{'}(t)) dt.$$

It is easy to see that ω is a closed form. Hence it defines an element of $H^1(\mathbb{S}^3 - \gamma, \mathbb{Z})$. Now we fix a base point b in $\mathbb{S}^3$. For any x in $\mathbb{S}^3 - \gamma$ we choose a curve α joining b to x, and define

$$h(x) = \int_b^x \omega.$$

Clearly h is a well defined map from $\mathbb{S}^3 - \gamma$ to $\mathbb{R}/\mathbb{Z}$. The inverse image of $\{0\}$ defines a Birkhoff section with γ as the boundary.

This construction can be generalized to invariant measures. Suppose μ is an element of $\mathcal{M}$. We define a 1-form ω by

$$\omega_x(v) = \int \Omega_{(x,y)}(v, X_y) d\mu(y).$$

It can be shown that ω is a closed form on $\mathbb{S}^3 - Supp(\mu)$. As before ω induces a map from $\mathbb{S}^3 - Supp(\mu)$ to $\mathbb{R}/\mathbb{Z}$. This map h defines a foliation on $\mathbb{S}^3 - Supp(\mu)$ that is transversal to ϕ. It is called the Birkhoff foliation. For example, if μ arises from a finite collection of periodic orbits, then this gives rise to a Birkhoff section with the union of those orbits as the boundary.

Corollary : *Let $X^\infty(\mathbb{S}^3)$ denote the set of all C^∞-vector fields on $\mathbb{S}^3$. Then the collection of all left handed vector fields forms an open subset of $X^\infty(\mathbb{S}^3)$.*

Since the Hopf vector field H on $\mathbb{S}^3$ is C^∞ and left handed, this immediately implies that any vector field sufficiently close to H is left handed. As another consequence we obtain the following :

Corollary : *If X is a left handed vector field in $\mathbb{S}^3$ then any periodic orbit of X is a fibered knot.*

Example : Let g be a Riemannian metric on $\mathbb{S}^2$, and let ϕ be the geodesic flow on $T_1\mathbb{S}^2$. We identify $T_1\mathbb{S}^2$ with $SO(3)$ and lift ϕ to a flow $\widehat{\phi}$ on $\mathbb{S}^3$. If g is the canonical metric then $\widehat{\phi}$ is the Hopf flow. We deduce that if g is sufficiently close to a metric with constant positive curvature then $\widehat{\phi}$ is left handed.

Conjecture : If g is positively curved then $\widehat{\phi}$ is left handed.

Suppose M is a compact 3-manifold such that $H_1(M, \mathbb{Z}) = 0$. Then we can define left handedness of vector fields in a similar fashion, and the analogue of the previous theorem holds. For example suppose Γ is a co-compact Fuchsian group. Then $M = PSL(2, \mathbb{R})/\Gamma$ is a homology sphere. Let $\{g_t\}$ be the one parameter subgroup of $PSL(2,\mathbb{R})$ defined by

$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix},$$

and let ϕ be the induced geodesic flow on M.

Conjecture. These flows are left handed.

We mention one interesting special case. Suppose p, q, r are positive integers with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. Let $\Gamma \subset PSL(2,\mathbb{R})$ be the group corresponding to the triangle with angles $\frac{\pi}{p}, \frac{\pi}{q}$ and $\frac{\pi}{r}$. Then $M = PSL(2,\mathbb{R})/\Gamma$ is a homology sphere.

Theorem : (Pierre Dehornoy) *If* $(p,q,r) = (2,3,7)$ *then the geodesic flow on* M *is left handed.*

Six open problems :

Problem 1 : Hilbert's 16'th problem. (very hard !) Let X be a polynomial vector field on $\mathbb{R}^2$ of degree d. Prove that the number of limit cycles of X is bounded by some function $f(d)$. It would be even better to find some explicit bound for $f(d)$.

Problem 2 : Quantitative Seifert theorem. Find a large Neighborhood U of the Hopf vector field H on $\mathbb{S}^3$ such that every element of U has at least one periodic orbit close to a Hopf fiber. The question is probably easy if one looks for some neighborhood. It is probably more difficult to explicitly construct some kind of maximal open set U. What is happening on the boundary of U ?

Problem 3 : Minimal flows on 3-manifolds. Which 3-manifolds admit minimal flows ? This is a widely open question. Some examples are known, often with low smoothness regularity. Any progress is welcome, even after adding strong assumptions on the flow or on the manifold.

Problem 4 : Volume preserving flows on $\mathbb{S}^3$ without periodic orbits. K. Kuperberg's example is of class C^∞, but it is not volume preserving. G. Kuperberg's example is not C^2 but it is volume preserving. Can one find a volume preserving C^∞-vector field on $\mathbb{S}^3$ with no periodic orbits ?

Problem 5 : Geodesic flows on positively curved surfaces. Let g be a Riemannian metric on $\mathbb{S}^2$ with positive curvature. Is it true that the geodesic flow of g, acting on $T_1\mathbb{S}^2$, is left handed ? If not, can one characterize metrices g for which this is the case ?

Problem 6 : Left handed vector fields on homology spheres. Let Γ be a co-compact discrete subgroup of $PSL(2,\mathbb{R})$ such that $M = PSL(2,\mathbb{R})/\Gamma$ is a homology sphere. Is it true that the geodesic flow acting on M is left handed ? Pierre Dehornoy proved it for some specific examples.

References

[1] DENJOY A. Sur les courbes définies par les équations différentielles à la surface du tore. *J. Math. Pures Appl.* **11** (1932), 333-375.

[2] E. GHYS Construction de champs de vecteurs sans orbite périodique *Séminaire Bourbaki* **785** (1994), 283307.

[3] KATOK A., HASSELBLATT B. Introduction to the Modern Theory of Dynamical Systems *Cambridge University Press*

[4] G. KUPERBERG A volume-preserving counterexample to the Seifert conjecture *Comment. Math. Helv.* **71** (1996), no.1 7097.

[5] K. KUPERBERG A smooth counterexample to the Seifert conjecture in dimension three *Ann. of Math.* **140** (1994), 723-732.

[6] H. ROSENBERG Un contre-exemple à la conjecture de Seifert (d'après P. Schweitzer) *Séminaire Bourbaki* **434** (1972-73), Springer Lecture Notes in Maths 383.

[7] P. A. SCHWEITZER Counterexamples to the Seifert Conjecture and opening closed leaves of foliations *Ann. of Math.* **100** (1974), 386400.

[8] H. SEIFERT Closed integral curves in 3-space and two-dimensional deformations. *Proc. Amer. Math. Soc.* **1** (1950), 287302.

[9] F. W. WILSON On the minimal sets of non singular vector fields. *Ann. of Math.* **84** (1966), 529536.

Additional references.

Since these Panorama lectures, (at least) two preprints appeared which contain substantial contributions to this topic.

P. Dehornoy : Geodesic flow, left-handedness, and templates, arXiv:1112.6296 (April 2012).

S. Hurder, A. Rechtman, The dynamics of generic Kuperberg flows, arXiv:1306.5025 (June 2013).