Fatou and Julia Components of

Transversely Holomorphic Foliations

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In this paper we study foliations $\mathcal{F}$ on compact manifolds M, of real codimension 2, with a transversal holomorphic structure. We construct a decomposition of M into dynamically defined components, similar to the Fatou/Julia sets for iteration of rational functions, or the region of discontinuity/limit set partition for Kleinian groups in $PSL(2, \mathbb{C})$. All this in tune with Sullivan's well known dictionary between the different guises of conformal dynamics [14].

The tool we use to generate the partition of M is a subspace of "C^0-infinitesimal automorphisms" of $\mathcal{F}$, *i.e.* a subspace of the space of continuous vector fields of modulus of continuity $\epsilon \log \epsilon$ such that the flow they generate consists of homeomorphisms of M sending the leaves of $\mathcal{F}$ topologically onto the leaves of $\mathcal{F}$. One is not really interested in all the "infinitesimal automorphisms", since there are the "trivial" ones, which correspond to the vector fields tangent to the foliation. So, taking the infinitesimal automorphisms modulo these "trivial" vector fields, we obtain sections of the normal bundle $\nu^{1,0}$ of $\mathcal{F}$ which are constant along the leaves. More precisely, we denote by $H^0(M, \mathcal{C}_{\mathcal{F}}(\nu^{1,0}))$ the complex vector space of continuous sections X of $\nu^{1,0}$ which are constant along the leaves and that have distributional derivatives in L^2 with $\bar{\partial} X$ essentially bounded (see 1.2).

This opens a fundamental dichotomy for points on M:

1) The **Fatou set** $Fatou(\mathcal{F})$ of $\mathcal{F}$; it is the open subset formed by points $x \in M$ with $X(x) \neq 0$ for some $X \in H^0(M, \mathcal{C}_{\mathcal{F}}(\nu^{1,0}))$.

2) The **Julia set** $Julia(\mathcal{F})$ of $\mathcal{F}$; it is the closed subset of M defined by $X(x) = 0$ for all $X \in H^0(M, \mathcal{C}_{\mathcal{F}}(\nu^{1,0}))$.

Let $Fatou(\mathcal{F}) = \cup_k F_k$ be the decomposition of the Fatou set into connected components (open and $\mathcal{F}$-saturated). Denote by $\mathcal{F}_k$ the restriction of the foliation $\mathcal{F}$ to F_k. The elements of $H^0(M, \mathcal{C}_{\mathcal{F}}(\nu^{1,0}))$ may be lifted to vector fields on M which are uniquely integrable (see 1.2), giving rise to flows preserving the foliation. Since

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the codimension of the foliation is 1 over $\mathbb{C}$ and we can multiply by complex numbers of the form $e^{2i\pi\theta}$, we obtain that the foliation $\mathcal{F}_k$ is transitive, *i.e.* there are ambient leaf preserving isotopies sending any leaf in F_k to any other leaf of F_k. These facts allow us to apply Molino's theory of "transversely complete foliations" [12].

Theorem 1. *Let $\mathcal{F}$ be a transversely holomorphic foliation of complex codimension 1 on a compact connected manifold M and let $\mathcal{F}_k$ be the restriction of $\mathcal{F}$ to some connected component F_k of the Fatou set. Then there are three exclusive cases:*

1) **Wandering component***: the leaves of $\mathcal{F}_k$ are closed in F_k.*

2) **Semi-wandering component***: the closures of the leaves of $\mathcal{F}_k$ form a real codimension 1 foliation of F_k which has the structure of a fiber bundle over a 1-dimensional manifold.*

3) **Dense component***: the leaves of $\mathcal{F}_k$ are dense in F_k.*

We analyze the components separately to obtain:

Theorem 2 (wandering components). *1) Let F_k be a "wandering component" of the Fatou set. Then the leaf space of $\mathcal{F}_k$ is a finite Riemann surface Σ_k, i.e. it is Hausdorff and compact minus a finite number of points. The natural projection $F_k \to \Sigma_k$ has the structure of a locally trivial fiber bundle.*

2) There is a finite number of "wandering components" in the Fatou set, except possibly for those for which the leaf space is a sphere minus 1, 2 or 3 points.

Recall that if G is a Lie group, a foliation is called a *G-Lie foliation* if it is defined by a collection of submersions from open sets to G in such a way that any two of these submersions agree on the intersection of their domains of definitions modulo a post-composition by some left translation in G (see [12]). In dimension 2, there are only two simply connected Lie groups; $\mathbb{R}^2$ and the affine group $Aff(\mathbb{R})$ of the real line (consisting of maps $x \mapsto ax + b$ with $a \in \mathbb{R}^*_+, b \in \mathbb{R}$). Of course, $\mathbb{R}^2$ is identified with $\mathbb{C}$ so that an $\mathbb{R}^2$-Lie foliation is obviously transversely holomorphic. The group $Aff(\mathbb{R})$ acts holomorphically, freely and transitively, on the upper half space $\mathcal{H}$, so that one can identify $Aff(\mathbb{R})$ with the Riemann surface $\mathcal{H}$ in such a way that the left translations of $Aff(\mathbb{R})$ act holomorphically (but note that the right translations don't act holomorphically!). This means that an $Aff(\mathbb{R})$-Lie foliation is canonically a transversely holomorphic foliation.

Theorem 3 (semi-wandering components). *Let F_k be a "semi-wandering component" of the Fatou set. Then the closures of the leaves of $\mathcal{F}_k$ define a real analytic foliation $\bar{\mathcal{F}}_k$ given by a locally trivial fibration of F_k on the circle or an interval. The foliation $\mathcal{F}_k$ is a G-Lie foliation, where $G = \mathbb{C}$ or $Aff(\mathbb{R})$. The lift of $\mathcal{F}_k$ to the universal cover $\widetilde{F_k}$ is given by a locally trivial fibration of $\widetilde{F_k}$ onto some strip $\{z \in \mathbb{C} \mid \alpha < \Im(z) < \beta\}$ (with $-\infty \le \alpha < \beta \le +\infty$).*

Theorem 4 (dense components). *Let F_k be a "dense component" of the Fatou set. Then $\mathcal{F}_k$ is an ergodic foliation in F_k (with respect to the Lebesgue measure class of M). There are two possibilities:*

1) $\mathcal{F}_k$ is an $\mathbb{R}^2$-Lie foliation. The Julia set consists of a finite number of compact leaves and the Fatou set is connected. The foliation is defined by a meromorphic closed basic 1-form having poles on the Julia set.

2) $\mathcal{F}_k$ is an $Aff(\mathbb{R})$-Lie foliation.

The lift of $\mathcal{F}_k$ to the universal cover $\widetilde{F_k}$ of F_k is given by the fibers of a locally trivial fibration of $\widetilde{F_k}$ onto $\mathbb{C}$ (in case 1) or onto the upper half space (in case 2).

We may then further decompose the Julia set of $\mathcal{F}$ in the measurable category. An $\mathcal{F}$-invariant measurable set $J \subset M$ is said to be *recurrent in the measurable sense* if there is no transversal disc D containing a Borel set $B \subset J \cap D$ with positive (2-dimensional) Lebesgue measure and such that distinct points in B are in distinct leaves of $\mathcal{F}$.

Theorem 5. *Let $(M, \mathcal{F})$ be a transversely holomorphic foliated compact manifold such that the Lebesgue measure of the Julia set is positive. Then there is a (Lebesgue) measurable foliated partition of the Julia set $Julia(\mathcal{F}) = J_0 \cup \ldots \cup J_r$, $r \geq 0$ such that:*

1) For $k \geq 1$ the sets J_k have positive Lebesgue measure and $\mathcal{F}|_{J_k}$ is ergodic with respect to the Lebesgue measure class. The space of essentially bounded measurable basic Beltrami differentials on J_k is 1-dimensional.

2) J_0 is empty or it is a recurrent set in the measurable sense. There are no non-zero essentially bounded measurable basic Beltrami differentials on J_0.

In the absence of the regions of homogeneity given by the Fatou set, one obtains the decomposition of the manifold into a finite number of Lebesgue ergodic components plus the rigid recurrent set J_0. Note that if the leaves with nontrivial holonomy are dense, then $Fatou(\mathcal{F})$ is empty, so that indeed there are no Fatou components in this case.

In section 8, we describe a variety of examples that illustrate the previous theory. In particular, we show how it can be applied to holomorphic foliations in compact complex surfaces with "generic" singularities (see example 8.1). Indeed, by some "cut and paste" construction, we can easily construct non singular foliations from these singular foliations. In particular, we can generalize the definitions of the Fatou and Julia sets for such singular foliations. We obtain:

Theorem 6. *Let $\mathcal{F}$ be a holomorphic foliation in $\mathbb{C}P^2$ with Poincaré type singularities. Then:*

1) if the Fatou set of $\mathcal{F}$ is non-empty, then $\mathcal{F}$ restricted to its connected components is "homogeneous", as described in the above Theorems 1, 2, 3 and 4, *2) if the Julia set has positive Lebesgue measure then we may further decompose the Julia set into a finite number of measurable pieces like in Theorem 5.*

We note that the above Theorem is always giving information for an arbitrary $\mathcal{F}$ with Poincaré type singularities. Namely, either there are some "homogeneous regions" in $\mathcal{F}$, or in the absence of them, we have a unique finite measurable decomposition of $\mathbb{C}P^2$ into ergodic or recurrent components.

In the proof we use the solution of $\bar{\partial}$ with measurable coefficients and the finiteness of certain cohomology groups. It is through the solution of the $\bar{\partial}$ problem that we introduce a measurable point of view in the foliation. A. Haefliger observed that the theory here developed can be extended to compactly generated pseudogroups as defined in [9], and similar conclusions may be obtained in this more general context. We could have done so, but the exposition would have been harder to follow.

While doing this work, we have been inspired by the words and personalities of Lars Ahlfors, André Haefliger and Dennis Sullivan, to whom we are most thankful.

1. The Fatou and Julia Sets of Transversely Holomorphic Foliations

1.1. Transversely Holomorphic Foliations

Let M be a compact connected manifold of dimension $d+2$. A *transversely holomorphic foliation* $\mathcal{F}$ may be defined by an atlas $\{U_i, \varphi_i, \varphi_{ij}\}$. The $U_i \subset M$ are open sets covering M. The maps $\varphi_i: U_i \to \varphi_i(U_i) \subset \mathbb{C}$ are submersions with connected fibers and the *holomorphic* maps $\varphi_{ij}: \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ are such that $\varphi_{ij} \circ \varphi_j = \varphi_i$ on $U_i \cap U_j$. That is, $(M, \mathcal{F})$ is a transversely holomorphic foliation of complex codimension 1 (see for instance [7,8]). The foliation gives rise to a decomposition of M into leaves of dimension d; each connected component of the intersection of a leaf with U_i is a fiber of φ_i. Two transversely holomorphic foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ on M are equivalent if there is a foliated homeomorphism $\sigma: (M, \mathcal{F}_1) \to (M, \mathcal{F}_2)$, which is transversely holomorphic.

1.2. Infinitesimal Automorphisms of $(M, \mathcal{F})$ and their Flows

The transversely holomorphic foliation $(M, \mathcal{F})$ defines a subbundle τ of the tangent bundle T_M of M consisting of those vectors which are tangent to the leaves with quotient bundle $\nu^{1,0}$:

$$0 \to \tau \to T_M \xrightarrow{\pi} \nu^{1,0} \to 0. \tag{1.1}$$

We say that the germ of a section X of a vector bundle on M at a point $x \in M$ has *modulus of continuity* $\epsilon \log \epsilon$ if there is a positive constant C, a coordinate chart U containing x, and a trivialization of the bundle over U, such that for $x_1, x_2 \in U$ we have

$$|X(x_1) - X(x_2)| < -C|x_1 - x_2| \log(|x_1 - x_2|).$$

We will denote by $\mathcal{C}^{\epsilon\log\epsilon}(E)$ the sheaf of sections of the vector bundle E on M which are of modulus of continuity $\epsilon\log\epsilon$. They are fine sheaves, since we can use partitions of unity. The sheaves of sections of (1.1) of modulus of continuity $\epsilon\log\epsilon$ give rise to an exact sequence of fine sheaves on M:

$$0 \to \mathcal{C}^{\epsilon\log\epsilon}(\tau) \to \mathcal{C}^{\epsilon\log\epsilon}(T_M) \xrightarrow{\pi} \mathcal{C}^{\epsilon\log\epsilon}(\nu^{1,0}) \to 0. \tag{1.2}$$

The normal bundle $\nu^{1,0}$ to the foliation $\mathcal{F}$ defined in (1.1) is "flat along the leaves". This structure may be seen by observing that the normal bundle may be defined by the cocycle $\frac{\partial\varphi_{ji}}{\partial z}\circ\varphi_i : U_i \cap U_j \to \mathbb{C}^\star$ which is constant on the fibers of φ_i. Any time that we have a bundle which is flat along the leaves we can speak of sections which are constant along the leaves. We will call these sections *basic sections*. We can apply this to any bundle obtained by tensor algebra with the normal bundle, like $\nu^{1,0}\otimes\nu^{0,1^*}$.

The basic sections of the bundle $\nu^{1,0}\otimes\nu^{0,1^*}$ will be called *basic Beltrami differentials*. Denote by $\mathcal{L}^\infty_{\mathcal{F}}(\nu^{1,0}\otimes\nu^{0,1^*})$ the sheaf of germs of essentially bounded measurable basic Beltrami differentials on $(M,\mathcal{F})$. They are defined modulo sets of measure 0 with respect to the Lebesgue measure class of M, since they are equivalence classes of measurable sections.

The basic sections of the bundle $\nu^{1,0}$ will be called *basic normal vector fields*. We will consider continuous basic normal vector fields satisfying very weak differentiability conditions. Denote by $\mathcal{C}_{\mathcal{F}}(\nu^{1,0})$ the sheaf of germs of continuous basic sections of $\nu^{1,0}$ with distributional derivatives locally in L^2 with

$$\bar\partial\sigma \in \mathcal{L}^\infty_{\mathcal{F}}(\nu^{1,0}\otimes\nu^{0,1^*}).$$

Observe that the condition that distributional derivatives are locally in L^2 actually follows from the fact that $\bar\partial\sigma$ is essentially bounded: see lemma 3, page 90 of [2]. $\mathcal{C}_{\mathcal{F}}(\nu^{1,0})$ is a subsheaf of $\mathcal{C}^{\epsilon\log\epsilon}_{\mathcal{F}}(\nu^{1,0})$, since these vectors fields are of type $\epsilon\log\epsilon$ [11,13]. Define $\mathcal{C}^{\epsilon\log\epsilon}_{\mathcal{F}}(T_M)$ as the subsheaf $\pi^{-1}\mathcal{C}_{\mathcal{F}}(\nu^{1,0})$, giving rise to the exact sequence of sheaves:

$$0 \to \mathcal{C}^{\epsilon\log\epsilon}(\tau) \to \mathcal{C}^{\epsilon\log\epsilon}_{\mathcal{F}}(T_M) \to \mathcal{C}_{\mathcal{F}}(\nu^{1,0}) \to 0. \tag{1.3}$$

$\mathcal{C}^{\epsilon\log\epsilon}_{\mathcal{F}}(T_M)$ consists of germs of vector fields of type $\epsilon\log\epsilon$ which project to the normal bundle as a section σ which is constant along the leaves and which has distributional derivatives locally in L^2 with $\bar\partial\sigma$ locally essentially bounded.

The first sheaf in (1.3) is a fine sheaf, so it has no higher cohomology groups. The long exact sequence of cohomology groups of (1.3) gives rise to an exact sequence of global sections

$$0 \to H^0(M,\mathcal{C}^{\epsilon\log\epsilon}(\tau)) \to H^0(M,\mathcal{C}^{\epsilon\log\epsilon}_{\mathcal{F}}(T_M)) \to H^0(M,\mathcal{C}_{\mathcal{F}}(\nu^{1,0})) \to 0. \tag{1.4}$$

This implies that we may lift any basic normal vector field $X \in H^0(M,\mathcal{C}_{\mathcal{F}}(\nu^{1,0}))$ to a vector field on M of type $\epsilon\log\epsilon$. Vector fields of type $\epsilon\log\epsilon$ have the property of being uniquely integrable, in the sense that the differential equation $\dot{x} = X(x)$ has a unique solution for a given initial condition, and so defines a local flow [11,13]. Summarizing:

Lemma 1.1. *1) We may lift any basic normal vector field* $X \in H^0(M, \mathcal{C}_{\mathcal{F}}(\nu^{1,0}))$ *to a vector field in* $H^0(M, \mathcal{C}_{\mathcal{F}}^{\epsilon \log \epsilon}(T_M))$.

2) Any vector field $X \in H^0(M, \mathcal{C}_{\mathcal{F}}^{\epsilon \log \epsilon}(T_M))$ *in the compact manifold* M *gives rise to a global 1-parameter flow* $\Phi : M \times \mathbb{R} \to M$ *preserving the foliation* $\mathcal{F}$.

If $X \in H^0(M, \mathcal{C}^{\epsilon \log \epsilon}(\tau))$ then it is a "trivial" automorphism, since its flow is preserving each individual leaf. The elements in $H^0(M, \mathcal{C}_{\mathcal{F}}(\nu^{1,0}))$ represent then the "non-trivial infinitesimal symmetries of $\mathcal{F}$".

We note that if $Y \in H^0(M, \mathcal{C}_{\mathcal{F}}(\nu^{1,0}))$ then it has to vanish on every leaf that has non-trivial holonomy. This is so, since fixed points of conformal maps in $\mathbb{C}$ different from the identity are isolated. Hence the flow obtained by integrating Y preserves each of the leaves with non-trivial holonomy, which is a set with an at most countable number of leaves. If the union of these leaves is dense in M, then $H^0(M, \mathcal{C}_{\mathcal{F}}(\nu^{1,0})) = 0$.

We have continuous dependence of the solution with respect to variation of the initial conditions and the vector field [11].

1.3. The Fatou and Julia Sets of a Transversely Holomorphic Foliation

Definition 1.3. *The Julia set of the transversely holomorphic foliated compact manifold* $(M, \mathcal{F})$ *is the closed subset* $Julia(\mathcal{F})$ *of* M *where all the elements of* $H^0(M, \mathcal{C}_{\mathcal{F}}(\nu^{1,0}))$ *vanish:*

$$Julia(\mathcal{F}) = \{x \in M \mid X(x) = 0 \ , \ \forall X \in H^0(M, \mathcal{C}_{\mathcal{F}}(\nu^{1,0}))\}.$$

The Fatou set $Fatou(\mathcal{F}) = M \setminus Julia(\mathcal{F})$ *of* $(M, \mathcal{F})$ *is the complement of the Julia set, and hence open and* $\mathcal{F}$*-saturated. Its connected components will be called the Fatou components of* $(M, \mathcal{F})$.

2. Basic Beltrami Differentials

We have from [8] the following exact sequence of sheaves:

$$0 \to \Theta_{\mathcal{F}} \to \mathcal{C}_{\mathcal{F}}(\nu^{1,0}) \xrightarrow{\bar{\partial}} \mathcal{L}_{\mathcal{F}}^{\infty}(\nu^{1,0} \otimes \nu^{0,1^*}) \to 0 \tag{2.1}$$

where $\bar{\partial}$ is the operator $\partial/\partial\bar{z}$ in the transversal complex variable z. The kernel sheaf $\Theta_{\mathcal{F}}$ consists of those sections of the normal bundle which are constant along the leaves and are holomorphic. They correspond to the holomorphic infinitesimal automorphisms of $\mathcal{F}$. If $X \in H^0(M, \mathcal{C}^{\epsilon \log \epsilon}(T_M))$ with projection $\pi(X) \in H^0(M, \Theta_{\mathcal{F}})$, then the flow generated by X preserves $\mathcal{F}$ and is holomorphic in the transversal variables, providing a 1-parameter family of isomorphisms of the transversely holomorphic foliation $\mathcal{F}$.

The long exact sequence of cohomology groups of (2.1) is

$$0 \to H^0(M,\Theta_{\mathcal{F}}) \to H^0(M,\mathcal{C}_{\mathcal{F}}(\nu^{1,0})) \xrightarrow{\bar{\partial}} H^0(M,\mathcal{L}^{\infty}_{\mathcal{F}}(\nu^{1,0}\otimes\nu^{0,1^*})) \xrightarrow{\delta} H^1(M,\Theta_{\mathcal{F}}) \tag{2.2}$$

We endow $H^0(M,\mathcal{L}^{\infty}_{\mathcal{F}}(\nu^{1,0}\otimes\nu^{0,1^*}))$ with the essential supremum norm, where it becomes a Banach space. On $H^0(M,\mathcal{C}_{\mathcal{F}}(\nu^{1,0}))$ we introduce the norm

$$||\sigma|| = \max_{p\in M}|\sigma(p)| + \sup_{p\in M}|\bar{\partial}\sigma|$$

which endows it with a Banach space structure, where the middle map in (2.2) is continuous. Note that if a sequence of continuous functions f_n converges uniformly to a function f and if the distributions $\bar{\partial}f_n$ are L^∞ functions which converge in L^∞ to a function g then $\bar{\partial}f = g$ as a distribution; this guarantees the completeness of the normed space $H^0(M,\mathcal{C}_{\mathcal{F}}(\nu^{1,0}))$.

We have:

Proposition 2.1 ([7,8]). *If $(M,\mathcal{F})$ is a transversely holomorphic foliation in the compact manifold M, then the vector spaces $H^\star(M,\Theta_{\mathcal{F}})$ are finite dimensional and*

$$\bar{\partial}H^0(M,\mathcal{C}_{\mathcal{F}}(\nu^{1,0})) \subset H^0(M,\mathcal{L}^{\infty}_{\mathcal{F}}(\nu^{1,0}\otimes\nu^{0,1^*}))$$

is a closed subspace of finite codimension.

3. Classification of Components of the Fatou Set

Let $(M,\mathcal{F})$ be a transversely holomorphic foliation in the compact manifold M and let $Fatou(\mathcal{F}) = \cup_k F_k$ be the decomposition of the Fatou set into connected components (open and $\mathcal{F}$-saturated). Denote by $\mathcal{F}_k$ the restriction of the foliation $\mathcal{F}$ to F_k.

Let $X \in H^0(M,\mathcal{C}_{\mathcal{F}}(\nu^{1,0}))$ be a basic normal vector field such that it is not identically 0 on F_k. Choose $\widetilde{X},\widetilde{Y} \in H^0(M,\mathcal{C}^{\epsilon\log\epsilon}_{\mathcal{F}}(T_M))$ lifting X and iX respectively as in (1.3) and consider the map

$$\phi : M\times\mathbb{C} \to M \tag{3.1}$$

which associates to $(x,te^{i\theta})$ the point of M obtained by flowing with the vector field

$$\widetilde{X}_\theta = (\cos\theta)\widetilde{X} + (\sin\theta)\widetilde{Y}$$

a time $t>0$, and the identity for $t=0$. This map ϕ is continuous and satisfies the partial additivity property:

$$\phi(x,(t_1+t_2)e^{i\theta}) = \phi(\phi(x,t_1e^{i\theta}),t_2e^{i\theta}) \tag{3.2}$$

reflecting the fact that the restriction of ϕ to $M \times \{\mathbb{R}e^{i\theta}\}$ corresponds to the flow generated by the vector field $\widetilde{X}_\theta$ on M. The fact that X is basic implies that the maps

$$\phi(-, te^{i\theta}) : M \to M$$

are preserving $\mathcal{F}_k$.

Lemma 3.1. *Let $\mathcal{F}_k$ be the foliation in a Fatou component F_k of $(M, \mathcal{F})$ as above. Then:*

a) Given $x_1, x_2 \in F_k$ there is an $\mathcal{F}$ preserving homeomorphism of M sending x_1 to x_2.

b) The leaves $\mathcal{L}$ of $\mathcal{F}_k$ are either dense or nowhere dense.

c) The closures of the leaves

$$\{\mathcal{K}\} = \{\bar{\mathcal{L}} \mid \mathcal{L} \text{ is a leaf of } \mathcal{F}_k\} \tag{3.3}$$

form a partition of F_k. All the elements in the partition are ambient homeomorphic. The sets $\mathcal{K}$ are union of $\mathcal{F}_k$ leaves, and given two leaves in $\mathcal{K}$ there is a $\mathcal{F}_k$-leaf preserving homeomorphism of $\mathcal{K}$ sending one to the other.

d) If a leaf $\mathcal{L} \subset F_k$ is not closed, then it is recurrent, i.e. it accumulates on itself.

e) Let $X \in H^0(M, \mathcal{C}_{\mathcal{F}}(\nu^{1,0}))$ such that $X(x_1) \neq 0$ for some $x_1 \in F_k$. Then $X(x_2) \neq 0$ for $x_2 \in \bar{\mathcal{L}}_{x_1} \cap F_k$, where $\mathcal{L}_{x_1}$ is the leaf of $\mathcal{F}_k$ passing through x_1.

Proof. We follow some arguments from [12].

a) The existence of normal basic vector fields implies that we can move to nearby leaves with (3.1). Integrating vector fields tangent to the leaves, we may move to any point on the same leaf. A composition of maps of both types proves the assertion.

b) If the interior of $\bar{\mathcal{L}}$ is non-empty, then for a leaf $\mathcal{L}'$ in the topological boundary $\partial\bar{\mathcal{L}}$, its interior is empty, since it is contained in $\partial\bar{\mathcal{L}}$, contradicting the homogeneity of a).

c) The argument is similar to a), and rests on the existence of non-vanishing basic normal vector fields.

d) A non-recurrent leaf is open in its closure, and hence the leaf through an accumulation point of $\mathcal{L}$ cannot be dense in $\bar{\mathcal{L}}$ (it is contained in the boundary of $\mathcal{L} \subset \bar{\mathcal{L}}$), as required by the homogeneity properties in c).

e) If $x_2 \in \mathcal{L}_{x_1}$, we have $X(x_2) \neq 0$, since X is a basic vector field. If $X(x_2) = 0$ with $x_2 \in \bar{\mathcal{L}}_{x_1}$ then X would vanish on $\mathcal{L}_{x_2}$, which is dense in $\bar{\mathcal{L}}_{x_2} = \bar{\mathcal{L}}_{x_1}$, so we would also have $X(x_1) = 0$. $\square$

Proof of Theorem 1. Assume that the leaves $\mathcal{L}$ of $\mathcal{F}_k$ are not closed and are not dense. We have to show that possibility 2) is realized. Let $X \in H^0(M, \mathcal{C}_{\mathcal{F}}(\nu^{1,0}))$ be a basic normal vector field not identically zero on F_k, choose liftings $\widetilde{X}_\theta$ and obtain the map ϕ as in (3.1). Choose $x \in F_k$ with $X(x) \neq 0$, then ϕ restricted to $\{x\} \times \mathbb{C}$ gives a map $\phi_x : \mathbb{C} \to M$, transversal to the foliation $\mathcal{F}_k$. Since we are assuming that the leaf $\mathcal{L}_x$ is not closed, then it is recurrent by Lemma 3.1.d), so that we may choose a sequence of points in $\mathbb{C}$:

$$t_n e^{i\theta_n} \quad , \quad t_n \to 0 \quad , \quad \theta_n \to \theta_0 \quad , \quad \phi(\mathcal{L}_x, t_n e^{i\theta_n}) = \mathcal{L}_x. \tag{3.4}$$

We want to deduce from this, that for the above angle θ_0 and for any $t \in \mathbb{R}$ we have

$$\phi(\mathcal{L}_x, te^{i\theta_0}) \subset \bar{\mathcal{L}}_x. \tag{3.5}$$

Using the additivity property (3.2), we deduce from (3.4) that for any integer j we also have

$$\phi(\mathcal{L}_x, jt_n e^{i\theta_n}) = \mathcal{L}_x, \tag{3.6}$$

and in particular for any given $t > 0$ we may find a sequence of integers j_n with the property that $j_n t_n e^{i\theta_n}$ converges to $te^{i\theta_0}$. Taking the limits in (3.6) we obtain (3.5). Note that (3.5) implies that

$$\phi(\mathcal{L}_x \times \mathbb{R}e^{i\theta_0}) \subset \bar{\mathcal{L}}_x. \tag{3.7}$$

Let $F'_k \subset F_k$ be the open foliated set where $X \neq 0$. It is also a union of elements of the partition $\mathcal{K}$, by property e). We introduce the real codimension 1 topological foliation $\mathcal{G}$ in F'_k defined by the distribution

$$\tau \oplus \mathbb{R}\widetilde{X}_{\theta_0}.$$

Pulling back $\mathcal{G}$ via ϕ_x we obtain the foliation defined by a non-vanishing vector field Z with a leaf tangent to the line $\mathbb{R}e^{i\theta_0} \subset \mathbb{C}$. If $\mathcal{M}_x$ denotes the leaf of $\mathcal{G}$ which passes through x, we have $\bar{\mathcal{L}}_x = \bar{\mathcal{M}}_x$. The set $\phi_x^{-1}(\bar{\mathcal{M}}_x)$ consists of a closed set of solutions of Z. By hypothesis that $\mathcal{L}_x$ is not dense in F_k and the homogeneity property in Lemma 3.1.a, we deduce that it is a discrete or a Cantor set of Z-orbits. Assume it is a Cantor set. For arbitrary $e^{i\theta_1}$ different from $e^{i\theta_0}$ the line passing through 0 and direction $e^{i\theta_1}$ intersects $\phi_x^{-1}(\bar{\mathcal{M}}_x)$ in a Cantor set, so we may find a sequence

$$\lambda_n e^{i\theta_1} \to 0 \quad , \quad \mathcal{L}_{\phi(x, \lambda_n e^{i\theta_1})} = \mathcal{L}_x.$$

But repeating the initial argument of the proof to this new sequence, we obtain

$$\phi(x, \mathbb{R}e^{i\theta_1}) \subset \bar{\mathcal{L}}_x$$

that contradicts that $\mathcal{L}_x$ is not dense. This implies that the leaf $\mathcal{M}_x$ is closed in F'_k. Hence $\bar{\mathcal{L}}_x = \mathcal{M}_x$ is a topological $d+1$ dimensional manifold parametrized by (3.7). The bundle structure is deduced from the isotopies in (3.1). □

4. Fatou Components with Dense Leaves

We recall:

Theorem (Rado [6]). *Let f be a complex valued continuous function defined on the closed disc $\bar{D} \subset \mathbb{C}$ which is holomorphic in $\{z \in D \mid f(z) \neq 0\}$. Then f is a holomorphic function on D.*

Proof of Theorem 4. The leaves $\mathcal{L}$ of $\mathcal{F}_k$ are dense in F_k by Lemma 3.1.b. Let $X_k \in H^0(F_k, \mathcal{C}_{\mathcal{F}}(\nu^{1,0})) \setminus \{0\}$. The section X_k is nowhere vanishing on F_k, for otherwise it would have to vanish identically on some leaf (it is basic), but since the leaves are dense, it would have to vanish identically everywhere. Similarly, any two such normal vector fields have to be linearly $\mathbb{C}$-dependent, since they are linearly dependent over one point, and so a combination of them vanishes on this point, and hence identically. It follows that $H^0(F_k, \mathcal{C}_{\mathcal{F}}(\nu^{1,0}))$ is 1-dimensional.

The restriction map from M to F_k and the $\bar{\partial}$ operator define a commutative diagram

$$\begin{array}{ccc} \bar{\partial} : H^0(M, \mathcal{C}_{\mathcal{F}}(\nu^{1,0})) & \rightarrow & H^0(M, \mathcal{L}^{\infty}_{\mathcal{F}}(\nu^{1,0} \otimes \nu^{0,1^*})) \\ \downarrow & & \downarrow \\ \bar{\partial} : H^0(F_k, \mathcal{C}_{\mathcal{F}}(\nu^{1,0})) & \rightarrow & H^0(F_k, \mathcal{L}^{\infty}_{\mathcal{F}}(\nu^{1,0} \otimes \nu^{0,1^*})) \end{array}, \tag{4.1}$$

where the vertical arrows are surjective: the one on the left due to the fact that $H^0(F_k, \mathcal{C}_{\mathcal{F}}(\nu^{1,0}))$ is 1-dimensional and the hypothesis that $X(x) \neq 0$ for $x \in F_k$ and some $X \in H^0(M, \mathcal{C}_{\mathcal{F}}(\nu^{1,0}))$. The one on the right since we have a canonical decomposition

$$H^0(M, \mathcal{L}^{\infty}_{\mathcal{F}}(\nu^{1,0} \otimes \nu^{0,1^*})) = H^0(M \setminus F_k, \mathcal{L}^{\infty}_{\mathcal{F}}(\nu^{1,0} \otimes \nu^{0,1^*})) \oplus H^0(F_k, \mathcal{L}^{\infty}_{\mathcal{F}}(\nu^{1,0} \otimes \nu^{0,1^*}))$$

given by the extension by 0 on F_k or $M \setminus F_k$ respectively. Here we are using in a significant way that we are working with measurable sections. By Proposition 2.1 we have that the top map in (4.1) has finite codimension, hence the space of basic measurable Beltrami differentials $H^0(F_k, \mathcal{L}^{\infty}_{\mathcal{F}}(\nu^{1,0} \otimes \nu^{0,1^*}))$ on F_k is finite dimensional.

The space of basic Beltrami differentials is a module over the space of basic measurable functions. The finite dimensionality of the space of essentially bounded basic measurable Beltrami differentials on F_k implies that we may find a basis of the form $\rho_1\mu, \ldots, \rho_m\mu$ where ρ_j are characteristic functions of $\mathcal{F}$-saturated measurable sets $V_j \subset F_k$ and $\mu \in H^0(F_k, \mathcal{L}^{\infty}_{\mathcal{F}}(\nu^{1,0} \otimes \nu^{0,1^*}))$ having support on $\cup V_j$. Since we can move the measurable sets V_j with the flow generated by X_k (Lemma 1.1), we see that $j = 1$ and $V_1 = F_k$. Note that the flow generated by X_k is transversely quasiconformal in the transverse direction, hence absolutely continuous with respect to the transverse Lebesgue measure. In particular, this flow maps a measurable $\mathcal{F}$-saturated set with zero Lebesgue measure to another set with the same property. This shows that $\mathcal{F}_k$ is ergodic.

We now construct a 1-form $\omega_k : TF_k \to \mathbb{C}$ defining $\mathcal{F}_k$. Let $v_x \in T_xF_k$ be a tangent vector at x. Define the complex number $\omega_k(v_x)$ by

$$\omega_k(v_x)X_k(x) = \pi(v_x),$$

where π is as in (1.1). It is a $\mathbb{C}$-linear form and basic, since X_k is. The kernel of ω_k defines the distribution τ associated to the foliation in (1.2). In a foliated chart ω_k is the pull-back of a 1-form $A(z)dz$ by a submersion defining the foliation locally. This function A is continuous and, as a distribution, $\partial A/\partial\bar{z}$ is essentially bounded. The 2-forms $d\omega_k = \partial A/\partial\bar{z}\ d\bar{z}\wedge dz$ and $\omega_k\wedge\bar{\omega}_k$ are basic, so by ergodicity they are linearly dependent *i.e.* $d\omega_k = \alpha\ \omega_k\wedge\bar{\omega}_k$ for some constant α.

If $\alpha = 0$, then ω_k is a closed 1-form which is holomorphic on F_k, since $d\omega_k = \bar{\partial}\omega_k = \partial A/\partial\bar{z}\ d\bar{z}\wedge dz = 0$. Let $X\in H^0(M,\mathcal{C}_{\mathcal{F}}(\nu^{1,0}))$ be a non-zero normal vector field on F_k and let D be a small disc transversal to $\mathcal{F}$ at a point in the boundary of F_k. In D the normal vector field X induces a vector field $B(z)\partial/\partial z$ where B is holomorphic in $D\cap F_k$ and is 0 on $D\cap Julia(\mathcal{F})$.

We now show that $D\cap F_k$ is dense in D. Assume it is not dense, then define a function $C(z)$ as B on $D\cap F_k$ and 0 on the complement. It is a continuous function on D which is holomorphic where non-zero. So by Rado's Theorem it is holomorphic, but then it is 0 since it vanishes on $D\setminus F_k$. This contradicts that X was non-zero on F_k. Hence $D\cap F_k$ is dense in D.

Since $D\cap F_k$ is dense, B is holomorphic where non-vanishing, so by Rado's Theorem again, B is holomorphic on D. Hence the boundary of F_k in D consists of isolated points and the normal basic vector field X is transversely holomorphic on M. This proves part 1) of Theorem 4.

If $\alpha\neq 0$, we can assume that $\alpha = i/2$ replacing X by some of its multiples. Consider the 1-form $\eta = \Im(z)^{-1}\ dz$ (with complex values but not holomorphic) on the upper half space $\mathcal{H} = \{z\in\mathbb{C}\mid \Im(z) > 0\}$. This 1-form η is invariant under the action of $Aff(\mathbb{R})$ on $\mathcal{H}$ and satisfies the same relation as ω_k, *i.e.* $d\eta = i/2\ \eta\wedge\bar{\eta}$. It follows that in each open set $U_i\cap F_k$ of the foliation atlas of $\mathcal{F}_k$, one can find a map $f_i : U_i\cap F_k\to\mathcal{H}$ such that $f_i^\star\eta = \omega_k$; this is indeed a very special (and elementary) case of Maurer-Cartan theory. Note that these maps f_i are holomorphic (submersions) since their differentials are $\mathbb{C}$-linear in the transversal direction. Observe also that a local diffeomorphism of some open connected set in $\mathcal{H}$ preserves the form η if and only if it is the restriction of some element of $Aff(\mathbb{R})$. It follows that these maps f_i are unique up to post-composition by some element of $Aff(\mathbb{R})$. Summarizing, we showed that $\mathcal{F}_k$ admits a structure of an $Aff(\mathbb{R})$-Lie foliation.

The rest of Theorem 4 follows from the general description of G-Lie foliations; there is a submersion of the universal cover $\widetilde{F_k}$ in G whose fibers define the lifted foliation [12]. Moreover, since we assume that there are non trivial complete basic vector fields, we can use the flows they generate to show that this submersion is actually a *locally trivial fibration onto* G. $\square$

The reader will find in [5] a complete description of the holonomy pseudogroups which can occur in case 1) of Theorem 4.

5. Fatou Components with Semi-wandering Leaves

We prove here a result stronger than Theorem 3:

Theorem 5.1. *Let $F_k \subset M$ be a "semi-wandering component". Then the closures of the leaves of $\mathcal{F}_k$ define a real analytic foliation $\bar{\mathcal{F}}_k$ given by a locally trivial fibration of F_k on the circle $\mathbb{S}^1$ (case a) or an interval (case b). The foliation $\mathcal{F}_k$ is a G-Lie foliation, where $G = \mathbb{C}$ (case I) or $Aff(\mathbb{R})$ (case II). The lift of $\mathcal{F}_k$ to the universal cover $\widetilde{F_k}$ is given by a locally trivial fibration $\pi : \widetilde{F_k} \to V \subset \mathbb{C}$ which is equivariant under some $h : \pi_1(F_k) \to \Gamma \subset G$ (i.e. $\pi(\gamma.x) = h(\gamma)(\pi(x))$ for all $\gamma \in \pi_1(F_k)$ and $x \in \widetilde{F_k}$) and such that:*

case Ia) $V = \mathbb{C}$ and the closure $\bar{\Gamma}$ of $\Gamma \subset \mathbb{C}$ is $\mathbb{R} \times i\mathbb{Z} \subset \mathbb{C}$;

case Ib) V is a strip $\{z \in \mathbb{C} \mid \alpha < \Im(z) < \beta\}$ with $-\infty \leq \alpha < \beta \leq \infty$ and the closure $\bar{\Gamma}$ is $\mathbb{R} \subset \mathbb{C}$.

case IIa) V is the upper half plane $\mathcal{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$ and the closure $\bar{\Gamma}$ is $\{x \mapsto \lambda^n x + b \ , \ n \in \mathbb{Z}, b \in \mathbb{R}\}$ for some $\lambda > 0$,

case IIb) does not occur, or more precisely, coincides with case Ib.

Proof. Pick a point $x_0 \in F_k$ and a transversal disc D containing x_0 and identified with some disc in $\mathbb{C}$ (using the transversely holomorphic structure).

Let X be a basic normal vector field inducing a continuous vector field on D (still denoted by X). Restricting D if necessary, we may assume that X does not vanish on D. Let $D_1 \subset D$ be an open disc containing x_0 and such that $\bar{D}_1 \subset D$. There is some $\varepsilon > 0$ such that for any $x \in D_1$ the flows ϕ^t and ψ^t of X and iX on D are defined at x for $|t| \leq \varepsilon$. Choose D_1 so small that each element of D_1 can be written in a unique way in the form $\phi^t \circ \psi^s(x_0)$ with $|t| \leq \epsilon \ , \ |s| \leq \epsilon$.

Let $\mathcal{H}ol$ be the holonomy pseudogroup restricted to D. We know by section 3 that we may choose X such that the closure of the orbit of x_0 by $\mathcal{H}ol$ is the orbit of X through x_0.

Lemma 5.2. *There is a non-vanishing* **holomorphic** *vector field Z in D_1 such that the orbits of Z (considered as a real vector field) are the closures of the orbits of $\mathcal{H}ol$ in D_1. This vector field is unique up to multiplication by a real constant.*

Proof. Let $\mathcal{H}ol_0 \subset \mathcal{H}ol$ be the subset consisting of elements γ defined at x_0 and such that $\gamma(x_0) \in D_1$. For each $\gamma \in \mathcal{H}ol_0$ there is a real number $\tau(\gamma_0)$ with $|\tau(\gamma_0)| \leq \epsilon$ such that $\gamma(x_0) = \phi^{\tau(\gamma)}(x_0)$.

Since X and iX are basic, each $\gamma \in \mathcal{H}ol_0$ is the restriction of some other element $\hat{\gamma}$ of $\mathcal{H}ol_0$ defined on all D_1:

$$\hat{\gamma}(\phi^t\psi^s(x_0)) = \phi^t\psi^s(\gamma(x_0)) = \phi^t\psi^s\phi^{\tau(\gamma)}(x_0). \tag{5.1}$$

Let τ be small enough so that $\phi^\tau(x_0) \in D_1$. We know that there is a sequence $\gamma_n \in \mathcal{H}ol_0$ such that $\gamma_n(x_0)$ converges to $\phi^\tau(x_0)$. Formula (5.1) shows that $\hat{\gamma}_n$

converges uniformly to the map

$$h(\tau) \ : \ \phi^t \psi^s(x_0) \in D_1 \mapsto \phi^t \psi^s \phi^\tau(x_0) \in D.$$

Being a uniform limit of the holomorphic maps $\hat{\gamma}_n$, this last map is holomorphic. Hence, for each real τ small enough, we constructed a holomorphic embedding $h(\tau)$ of D_1 in D. We claim that $h(\tau_1 + \tau_2) = h(\tau_1) \circ h(\tau_2)$ for small τ_1, τ_2 (where it makes sense). Indeed, $h(\tau_1 + \tau_2) \circ h(\tau_2)^{-1} \circ h(\tau_1)^{-1}$ fixes x_0 and commutes with ϕ^t and ψ^s (since it is a limit of elements of $\mathcal{H}ol_0$). Hence it is the identity on its domain of definition. We therefore constructed a local flow, hence a vector field Z. A vector field whose local flow consists of holomorphic maps is a holomorphic vector field so that Z is holomorphic. The maps $h(\tau)$ being limits of elements of $\mathcal{H}ol$ send orbit closures to orbit closures so that the orbits of the local flow $h(\tau)$ are the orbit closures of $\mathcal{H}ol_0$. This proves the existence part of the Lemma.

As for uniqueness, it is obvious that another choice would be of the form gZ, where g has to be real and holomorphic, hence constant. $\square$

Consider the local complex flow of the vector field Z. This gives a parametrization of D_1 by some open set in $\mathbb{C}$. Modifying Z by a constant, or changing $x_0 \in D_1$, amounts to a modification of the parameter z by some affine map $z \mapsto az + b$ with $a \in \mathbb{R}^*$ and $b \in \mathbb{C}$. It follows that $(F_k, \mathcal{F})$ is equipped with a canonical transversely affine structure, modeled on the action of the 3 dimensional group $G_{\mathbb{C}} = \{az + b \ , \ a \in \mathbb{R}^*, b \in \mathbb{C}\}$ on $\mathbb{C}$.

According to the general theory of such foliations (see [12]), there is a global holomorphic submersion

$$\pi : \widetilde{F_k} \to V \subset \mathbb{C}$$

of the universal cover $\widetilde{F_k}$ of F_k onto some open domain $V \subset \mathbb{C}$ whose fibers define the lifted foliation $\widetilde{\mathcal{F}}$. Moreover, there is a homomorphism $h : \pi_1(F_k) \to \Gamma \subset G_{\mathbb{C}}$ such that $\pi(\gamma.x) = h(\gamma)(\pi(x))$ for all $\gamma \in \pi_1(F_k)$ and $x \in \widetilde{F_k}$.

Of course, V has to be Γ-invariant. Note also that for each point $z \in V$ there is a complete vector field X on V such that $X(z) \neq 0$, which is Γ-invariant and which can be lifted to a complete vector field on $\widetilde{F_k}$. It follows that π is a locally trivial fibration over V (paths can be lifted to the total space).

Let $\bar{\Gamma}$ be the closure of Γ in $G_{\mathbb{C}}$. We know that for each $z \in V$ the orbit $\bar{\Gamma}.z$ intersects a neighborhood of z on a horizontal segment (since $\partial/\partial z$ corresponds to the local holomorphic vector field Z that we constructed on D_1). It follows that V is a strip $\{z \in \mathbb{C} \mid \alpha < \Im(z) < \beta\}$ with $-\infty \leq \alpha < \beta \leq \infty$.

Note that a Γ-invariant vector field is also $\bar{\Gamma}$-invariant and that a vector field on $\mathbb{C}$ invariant by a homothety $z \mapsto az + b \ , \ a \neq 1$, has to vanish at the fixed point. Hence, $\bar{\Gamma}$ acts freely on V. We now consider several cases:

1) If $V = \mathbb{C}$, then Γ has to act by translations (since $az + b$ with $a \neq 1$ would have a fixed point). This is case I. As for $\bar{\Gamma}$, it is a closed subgroup of dimension 1

of the group of translations. Hence it is (up to conjugation by a homothety) $\mathbb{R}\times i\mathbb{Z}$ (case a) or $\mathbb{R}\subset\mathbb{C}$, (case b).

2) If V is a strip different from all $\mathbb{C}$, after conjugating by a homothety it can be transformed to $\{z\in\mathbb{C}\mid 0<\Im(z)<1\}$ or $\{z\in\mathbb{C}\mid\Im(z)>0\}$. The set of affine maps $az+b$ preserving $\{z\in\mathbb{C}\mid 0<\Im(z)<1\}$ and with no fixed point in the strip contains only real translations. This case is Ib.

The group of affine maps $az+b$ preserving $\mathcal{H}=\{z\in\mathbb{C}\mid\Im(z)>0\}$ is the real affine group $\{az+b\ ,\ a\in\mathbb{R}_+^\star, b\in\mathbb{R}\}$. This is case II. As for the closure $\bar\Gamma$ it is a 1-dimensional closed subgroup of $Aff(\mathbb{R})$. It can be either of the form

$$\{\lambda^n z+b\ ,\ n\in\mathbb{Z}, b\in\mathbb{R}\}\text{ for some }\lambda>0,$$

(this is case IIa), or of the form $\{x+b\ ,\ b\in\mathbb{R}\}$, if it consists only of translations. In this case, the foliation is not only transversely affine, but transversely $\mathbb{C}$, so that case IIb is really a special case of Ib. Of course, the assertion that $\bar{\mathcal{F}}_k$ is a locally trivial fibration over $\mathbb{S}^1$ or $\mathbb{R}$ should be clear by now. □

6. Fatou Components with Wandering Leaves

Let $(M,\mathcal{F})$ be a transversely holomorphic foliated compact manifold. A point $x\in M$ is *wandering* if there is a small disc D transverse to $\mathcal{F}$ such that two distinct points of D are in distinct leaves of $\mathcal{F}$. A leaf is *wandering* if it has one wandering point (and hence all its points are wandering). Let $\Omega_{\mathcal{F}}\subset M$ be the wandering set consisting of all wandering points. A component F_k of the Fatou set is a wandering component of the Fatou set if $F_k\cap\Omega_{\mathcal{F}}\neq\emptyset$, and then by Lemma 3.1 we have that all its points are wandering $F_k\subset\Omega_{\mathcal{F}}$. A leaf in the Fatou set is wandering if and only if it is closed in $Fatou(\mathcal{F})$, by Lemma 3.1.d.

We say that a point $x_1\in\Sigma$ in a non-Hausdorff Riemann surface is a *non-Hausdorff point* if there is a point $x_2\in\Sigma\setminus\{x_1\}$ such that any neighborhood of x_1 intersects every neighborhood of x_2.

Lemma 6.1. *Let Ω_k be a connected component of the wandering set of $\mathcal{F}$ and let S_k be its leaf space. Then:*

1) S_k is possibly a non-Hausdorff Riemann surface and the set of non-Hausdorff points of S_k has empty interior.

2) There exists a component F_k of the Fatou set contained in Ω_k.

Proof. By definition, the leaf space S_k is a connected Riemann surface, possibly non-Hausdorff. Cover S_k by countably many discs D_j such that there are transversal discs $\widetilde{D}_j$ to the foliation that project injectively to D_j. For any j_1, j_2 consider the topological boundary

$$\partial(D_{j_1}\cap D_{j_2})=\overline{D_{j_1}\cap D_{j_2}}\setminus D_{j_1}\cap D_{j_2}.$$

This is a closed set with empty interior. Let Bd be the union of all these topological boundaries. By Baire's Theorem (which obviously applies to non-Hausdorff manifolds) Bd has empty interior. In order to prove the first part of the Lemma, it suffices to prove that any non-Hausdorff point of S_k is necessarily in Bd.

Let $x_1\in S_k\setminus Bd$, $x_2\in S_k\setminus\{x_1\}$ and let us show that they have disjoint neighborhoods. Let $x_1\in D_{j_1}$ and $x_2\in D_{j_2}$. Since $x_1\notin Bd$ then it is not in $D_{j_1}\cap(\bar D_{j_2}\setminus D_{j_2})$. Hence, there are two possibilities:

i) $x_1\in D_{j_1}\setminus\bar D_{j_2}$. Then there is a neighborhood of x_1 which is disjoint from D_{j_2} which is a neighborhood of x_2, or

ii) $x_1\in D_{j_1}\cap D_{j_2}$, then x_1 and x_2 are both in D_{j_2}, which is a disc. Hence there are disjoint neighborhoods of x_1 and x_2.

This proves part 1. To prove part 2 it suffices to prove that there are basic normal vector fields $X\in H^0(M,\mathcal{C}_{\mathcal{F}}(\nu^{1,0}))$ which are non-identically 0 on Ω_k. Take a transversal disc D to the foliation that injects to the leaf space. Any measurable Beltrami coefficient on D may be extended to a basic Beltrami coefficient on the $\mathcal{F}$-saturation of D and then as 0 on the complement. Since we may construct an infinite vector space of such, by Lemma 3.1, we may find one, say μ with $\mu=\bar\partial X$. This X is non-zero on Ω_k, and so $\Omega_k\cap Fatou(\mathcal{F})\neq\emptyset$. By the homogeneity properties of the foliation in the Fatou component F_k in Lemma 2.1, we obtain that $F_k\subset\Omega_k$. □

Proof of Theorem 2. Let F_k be a component of the Fatou set with wandering leaves, and Σ_k its leaf space. The locally trivial bundle structure of the leaves is obtained by restricting the map (3.1) to $\mathcal{L}_0\times\mathbb{C}\to F_k$, which is a covering map. This also shows that the leaf space Σ_k is a (possibly non-Hausdorff) Riemann surface.

The set of non-Hausdorff points of Σ_k is obviously invariant under any homeomorphism of Σ_k. But by Lemma 3.1 we have a transitive group of homeomorphisms of Σ_k being induced from homeomorphisms of M preserving $\mathcal{F}$. Hence there are no special points on Σ_k. The set of non-Hausdorff points is empty, and Σ_k is a Hausdorff Riemann surface.

The restriction map from M to F_k and the $\bar\partial$ operator define a commutative diagram

$$\begin{array}{ccc}\bar\partial: H^0(M,\mathcal{C}_{\mathcal{F}}(\nu^{1,0})) & \to & H^0(F_k,\mathcal{L}^\infty_{\mathcal{F}}(\nu^{1,0}\otimes\nu^{0,1*}))\oplus H^0(M\setminus F_k,\mathcal{L}^\infty_{\mathcal{F}}(\nu^{1,0}\otimes\nu^{0,1*}))\\ \downarrow Res_1 & & \downarrow Res_2\\ \bar\partial: H^0(F_k,\mathcal{C}_{\mathcal{F}}(\nu^{1,0})) & \to & H^0(F_k,\mathcal{L}^\infty_{\mathcal{F}}(\nu^{1,0}\otimes\nu^{0,1*}))\end{array}\tag{6.1}$$

The top horizontal map has closed range of finite codimension by Proposition 2.1 and the right vertical map is surjective, hence $Res_2\circ\bar\partial$ has closed range of finite codimension. Note that

$$H^0(M,\mathcal{L}^\infty_{\mathcal{F}}(\nu^{1,0}\otimes\nu^{0,1*}))=H^0(F_k,\mathcal{L}^\infty_{\mathcal{F}}(\nu^{1,0}\otimes\nu^{0,1*}))\oplus H^0(M\setminus F_k,\mathcal{L}^\infty_{\mathcal{F}}(\nu^{1,0}\otimes\nu^{0,1*})),$$

since we are working there on the measurable category.

Since basic sections of bundles on F_k correspond to sections on corresponding bundles on Σ_k, we have that the horizontal lower map in (6.1) corresponds to the map

$$\bar{\partial} : H^0(\Sigma_k, \mathcal{C}(\tau^{1,0}_{\Sigma_k})) \to H^0(\Sigma_k, \mathcal{L}^{\infty}_{\Sigma_k}(\tau^{1,0}_{\Sigma_k} \otimes \tau^{0,1*}_{\Sigma_k})), \tag{6.2}$$

where $\tau^{1,0}_{\Sigma_k}$ is the tangent bundle of the Riemann surface Σ_k.

If Σ_k is compact or isomorphic to the sphere minus 1 or 2 points, then it is a finite Riemann surface, and we are done. So we can assume that the Riemann surface Σ_k is non-compact and hyperbolic, *i.e.* covered by the unit disc D. First, recall that then Σ_k is a Stein manifold, so that in particular the map (6.2) is surjective, since the following group in the long exact sequence would be $H^1(\Sigma_k, \tau^{1,0}_{\Sigma_k})$, which is then 0 by Cartan's Theorem B. The classical infinitesimal Teichmüller Theory indicates that one should not look at all vector fields on Σ_k, but only at those which give rise to a complete flow. Let $\mathcal{M}_{\Sigma_k} \subset H^0(\Sigma_k, \mathcal{C}(\tau^{1,0}_{\Sigma_k}))$ be the set of vector fields on Σ_k which have the property that when lifted to the universal covering space of Σ_k, and after identifying this covering space with the unit disc, extend as 0 to the unit circle. Then the map

$$\bar{\partial} : \mathcal{M}_{\Sigma_k} \to H^0(\Sigma_k, \mathcal{L}^{\infty}_{\Sigma_k}(\tau^{1,0}_{\Sigma_k} \otimes \tau^{0,1*}_{\Sigma_k})) \tag{6.3}$$

has closed range and the cokernel of the map is the tangent space to Teichmüller space, or equivalently, its dual consists of the integrable quadratic differentials (see [1], Lemmas 8 and 9).

Lemma 6.2. *The restriction map* $Res_1 : H^0(M, \mathcal{C}_{\mathcal{F}}(\nu^{1,0})) \to H^0(\Sigma_k, \mathcal{C}(\tau^{1,0}_{\Sigma_k}))$ *has image in* $\mathcal{M}_{\Sigma_k}$.

Assuming the Lemma, we have shown that in diagram (6.1) $Res_2 \circ \bar{\partial}$ has closed range of finite codimension, and since the diagram (6.1) commutes and the above Lemma, we have that (6.3) has closed range of finite codimension, and so by Theorem 1 in [1], Σ_k is a finite Riemann surface.

Proof of Lemma 6.2. Let $X \in H^0(M, \mathcal{C}_{\mathcal{F}}(\nu^{1,0}))$ and choose two liftings $\widetilde{X}, \widetilde{Y}$ in $H^0(M, \mathcal{C}^{\epsilon \log \epsilon}_{\mathcal{F}}(T_M))$ of X and iX as in (1.3). Let X' be the vector field in $H^0(\Sigma_k, \mathcal{C}(\tau^{1,0}_{\Sigma_k}))$ induced from X and let $\widetilde{X}'$ be the lift of X' to the unit disc D via the uniformization of Σ_k. Since X is a complete vector field, so are X' and $\widetilde{X}'$. Let $\Phi : D \times \mathbb{R} \to D$ denote the flow of $\widetilde{X}'$, which now is a group action. Using the fact that this is a 1-parameter family of quasiconformal maps inducing bijections of the disc that extend continuously to the boundary, we see that it has to preserve the boundary of D, but since the same argument applies to the vector field $i\widetilde{X}'$, we conclude that $\widetilde{X}'$ vanishes on the boundary, and hence the Lemma. □

There can only be a finite number of components that have non-trivial deformations, since each contributes with a positive constant to the finite codimensional space (6.1). So we only have to worry about rigid Riemann surfaces: spheres minus 0, 1, 2, or 3 points. The sphere may appear only once:

Lemma 6.3. *If the leaf space of a wandering Fatou component is a sphere, then there is only one Fatou component.*

Proof. For $\Sigma_k = \bar{\mathbb{C}}$, we may find a finite number $X_1, \dots, X_r \in H^0(M, \mathcal{C}_{\mathcal{F}}(\nu^{1,0}))$ in such a way that for every point in Σ_k the induced vector fields $Y_1, \dots, Y_r$ do not vanish simultaneously. Let Z be a holomorphic vector field on Σ_k. There is a constant $C > 0$ such that

$$|Z(x)| \leq C \ \max\{|Y_1(x)|, \dots, |Y_r(x)|\}.$$

But then if $\widetilde{Z}$ denotes the normal vector field on F_k obtained by lifting Z, we have

$$|\widetilde{Z}(x)| \leq C \max\{|X_1(x)|, \dots, |X_r(x)|\}.$$

Since these last ones have a continuous extension to ∂F_k as 0, so does $\widetilde{Z}$. Extend $\widetilde{Z}$ as 0 on $M \setminus F_k$. Apply now Rado's Theorem, to see that $\widetilde{Z}$ is (locally) holomorphic, so that in particular the Julia set is at most a finite number of leaves and the Fatou set is connected. □

This finishes the proof of Theorem 2. □

Proposition 6.4. *There is a one to one correspondence between the connected components of the wandering set $\Omega_{\mathcal{F}}$ and the Fatou components with wandering leaves, with $F_k \subset \Omega_k$. We have that $\Omega_k \setminus F_k$ consists of a discrete set of leaves, and each of these leaves is associated to a puncture of the leaf space Σ_k of the Fatou component F_k. If two of these leaves are associated to the same puncture, then the corresponding points in the leaf space S_k of Ω_k may not be separated.*

Proof. We have $\Sigma_k \subset S_k$, where Σ_k is a finite Riemann Surface. Let $\bar{\Sigma}_k$ be the compact Riemann surface obtained by adjoining the punctures to Σ_k. By the Riemann-Roch Theorem, we may find a meromorphic function on $\bar{\Sigma}_k$ which has poles at exactly $\Lambda_k = \bar{\Sigma}_k \setminus \Sigma_k$. Let $x_n \in \Sigma_k$ be a sequence of points converging to a point $x_\infty \in S_k$. We claim that $\lim_{n\to\infty} f(x_n) = \infty$. Suppose otherwise. By taking a subsequence we may assume that $\lim_{n\to\infty} f(x_n) = a$. Take small discs around the points $f^{-1}(a)$ in Σ_k. The points x_n are not in these discs for large n, since they tend to $S_k \setminus \Sigma_k$, but this is a contradiction, since $\lim_{n\to\infty} f(x_n) = a$. Extend f to S_k by defining it to be ∞ in $S_k \setminus \Sigma_k$. By our previous argument, it follows that f is continuous so that Rado's theorem implies that f is indeed holomorphic. It follows that $S_k \setminus \Sigma_k$ consists of isolated points (maybe not separated). □

7. Measurable Decomposition of the Julia set

Proof of Theorem 5. Since the sections of $\mathcal{L}^\infty_{\mathcal{F}}(\nu^{1,0} \otimes \nu^{0,1^*})$ are measurable, the space of global sections admits a splitting as a sum of closed subspaces:

$$H^0(M,\mathcal{L}^\infty_{\mathcal{F}}(\nu^{1,0}\otimes\nu^{0,1^*})) = H^0(Julia(\mathcal{F}),\mathcal{L}^\infty_{\mathcal{F}}(\nu^{1,0}\otimes\nu^{0,1^*}))\oplus H^0(Fatou(\mathcal{F}),\mathcal{L}^\infty_{\mathcal{F}}(\nu^{1,0}\otimes\nu^{0,1^*}))$$

where we can think of sections in each factor as being global measurable sections, extended by 0 on the complementary set.

We claim that for the $\bar\partial$ operator in (2.2)

$$H^0(M,\mathcal{C}_{\mathcal{F}}(\nu^{1,0}))\xrightarrow{\bar\partial} H^0(M,\mathcal{L}^\infty_{\mathcal{F}}(\nu^{1,0}\otimes\nu^{0,1^*})),$$

we have

$$\bar\partial H^0(M,\mathcal{C}_{\mathcal{F}}(\nu^{1,0}))\cap H^0(Julia(\mathcal{F}),\mathcal{L}^\infty_{\mathcal{F}}(\nu^{1,0}\otimes\nu^{0,1^*}))=0.$$

Assume that $\bar\partial X$ is an element of the intersection. Hence X is a basic normal vector field which is holomorphic on the Fatou set and vanishes on the Julia set. Take a trivializing chart $\mathbb{R}^d\times D$ around a point in $Julia(\mathcal{F})$, where X will be represented by a continuous function $f(z)$ that is holomorphic on the complement of the Julia set. In particular f is continuous in D and holomorphic in $D\setminus\{f=0\}$. Hence by Rado's Theorem, f is holomorphic in D, so that $\bar\partial X=0$ on all of M. This proves our claim.

By Proposition 2.1 we have then that $H^0(Julia(\mathcal{F}),\mathcal{L}^\infty_{\mathcal{F}}(\nu^{1,0}\otimes\nu^{0,1^*}))$ is a finite dimensional vector space. It is a module over the basic measurable functions of $Julia(\mathcal{F})$, and by taking quotients of its elements, we obtain such basic measurable functions. The only way that such quotients form a finite dimensional space is that there are saturated measurable sets $J_1,\dots,J_r$ with characteristic functions $\rho_1,\dots,\rho_r$ such that a basis is obtained with $\rho_1\mu,\dots,\rho_r\mu$, for a suitably chosen μ. We then define J_0 as $Julia(\mathcal{F})\setminus\cup_{j=1}^r J_j$. This proves Theorem 5. □

8. Examples

Consider a codimension 2 transversely oriented foliation $\mathcal{F}$ on a compact manifold M and assume that the normal bundle is equipped with a Riemannian metric which is conformally invariant under the holonomy pseudogroup. Then local transversals are equipped with a conformal structure and hence can be locally parametrized by open sets in $\mathbb{C}$ (any Riemannian metric in dimension 2 is locally conformally flat). Using these local parameters on transversals, one sees that $\mathcal{F}$ can also be considered as a transversely holomorphic foliation in complex codimension 1. Conversely, any transversely holomorphic foliation in complex codimension 1 can be equipped with such a Riemannian metric so that the two notions actually coincide.

There are many examples of transversely holomorphic foliations. We would like to describe some of them emphasizing the dichotomy Fatou/Julia set.

Example 8.1. Consider a vector field in $\mathbb{C}^2$

$$X=P(z,w)\frac{\partial}{\partial z}+Q(z,w)\frac{\partial}{\partial w},$$

where P and Q are two polynomials without common factors. This defines by integration a holomorphic foliation in $\mathbb{C}^2 \setminus \{P = Q = 0\}$ that is tangent to the plane field $\mathbb{C}X$ and it extends in a canonical way to a holomorphic foliation of the complex projective plane $\mathbb{C}P^2$ minus a finite number of points, called the *singularities* of the foliation. For a generic polynomial vector field the singular points are of Poincaré type, meaning that the eigenvalues of the linear part $DX(x)$ at a singular point x are $\mathbb{R}$-linearly independent. We may then apply Poincaré's Linearization Theorem (see for instance [3]) that tells us that in local coordinates around the singular point $x = (0, 0)$ the vector field X is linear and diagonal.

The condition of $\mathbb{R}$-linear independence of the eigenvalues implies that the leaves of the foliation are transversal to the spheres $\mathbb{S}^3 \subset \mathbb{C}^2$ centered at the origin. The intersection of the linear foliation with the sphere $\mathbb{S}^3$ gives a foliation of $\mathbb{S}^3$ by leaves of dimension 1. The leaves of this foliation consist of two closed orbits which correspond to the two axes, and all other leaves are born in one of these closed orbit and die at the other, having then a simple attractor-repellor dynamics.

If all the singular points on $\mathbb{C}P^2$ of the foliation defined by X are of Poincaré type we may remove a small ball around each of the singular points $M_1 = \mathbb{C}P^2 \setminus \cup B_i$. The compact manifold with boundary M_1 carries a foliation that is arriving transversely at each boundary component. We may now apply to M_1 the double construction; M is obtained with two copies of M_1, reversing the orientation of the leaves of one of them, and gluing along the boundary components with the identity. The manifold M has a transversely holomorphic foliation $\mathcal{G}$ (the transversal holomorphic structure can be seen from Schwarz reflection principle).

Now let us translate our definitions and conclusions from M back to the singular foliation in $\mathbb{C}P^2$. Since Poincaré type singularities have a cone-like structure, what follows is independent of the size of the small balls removed. Intersect the Fatou set of the foliation $\mathcal{G}$ of M with M_1 and extend this open set using the cone structure to the foliation inside the balls B_i. This gives an open set in $\mathbb{C}P^2$ that we may call the Fatou set of the singular foliation $\mathcal{F}$ on $\mathbb{C}P^2$. In the same way, we define the Julia set of $\mathcal{F}$ as the closed set in $\mathbb{C}P^2$ obtained by adding the singular points of $\mathcal{F}$ to the saturation of $Julia(\mathcal{G}) \cap M_1$ (note that the singular points are in the closure of the separatrices, which are contained in the Julia set since they have non-trivial holonomy). The decomposition obtained in this way of $\mathbb{C}P^2$ has open Fatou components and a closed Julia set. One sees then that Theorem 6 follows directly from Theorems 1 to 5.

Example 8.2. Let $\mathcal{F}$ be a transversely holomorphic foliation in a connected compact manifold M such that it has a non-trivial basic *holomorphic* normal vector field, *i.e.* $H^0(M, \Theta_{\mathcal{F}}) \neq 0$, where $\Theta_{\mathcal{F}}$ is defined in (2.1).

If $H^0(M, \Theta_{\mathcal{F}})$ has dimension at least 2, let X_1 and X_2 be two linearly independent elements. The subsets defined by

$$\{x \in M \mid (\lambda_1 X_1 + \lambda_2 X_2)(x) = 0\}$$

consist of a finite number of leaves of the foliation, since X_1 and X_2 are basic and holomorphic. The leaves can come with a multiplicity, if the expression vanishes of higher order than 1 in some leaf. Call $L_{(\lambda_1,\lambda_2)}$ the set obtained by canceling the common factors (if any, and with multiplicities). The holomorphic map to the 1-dimensional projective line

$$\phi = (X_1 : X_2) : M \to \mathbb{C}P^1$$

has fibers precisely $L_{(\lambda_1,\lambda_2)}$. Hence such a foliation will have all leaves compact and it can be considered as well known.

Let now $\mathcal{F}$ be a transversely holomorphic foliation in the connected compact manifold M such that $H^0(M, \Theta_{\mathcal{F}})$ has dimension 1, and let X be a generator. The Julia set of $\mathcal{F}$ is $\{x \in M \mid X(x) = 0\}$, and hence it is empty, if X is non-vanishing, or consists of a finite number of leaves of $\mathcal{F}$. This example corresponds to type 1) in Theorem 4.

We can think that the foliations arising from this example are of special type, that is, one expects that a generic foliation has no transverse holomorphic automorphisms: $H^0(M, \Theta_{\mathcal{F}}) = 0$.

Example 8.3. Any codimension 2 *Riemannian foliation* on a compact manifold is of course transversely conformal. This kind of foliation is well understood (see [12]). A *Seifert foliation* $\mathcal{F}$ on a compact manifold M is a foliation such that all leaves are compact and have finite holonomy. This implies that $\mathcal{F}$ is Riemannian. In the codimension 2 oriented case, such a Seifert foliation is defined by a locally trivial fibration of M on some "orbifold Riemann surface" Σ. There is a finite number of leaves with non trivial holonomy which correspond to the singular points of Σ. Any flow preserving the foliation has to preserve these "singular fibers". Away from these fibers, the foliation is homogeneous so that in this situation the Julia set consists of finitely many compact leaves. This example shows some of the limits of our definition since from the dynamical point of view, one may have preferred to consider these singular fibers in the Fatou set; after all their holonomy group is finite so that the dynamics in the neighborhood of these leaves is not very "chaotic".

Example 8.4. Let B be a compact connected manifold and $h : \pi_1(B) \to PSL(2, \mathbb{C})$ be any homomorphism from its fundamental group to the group of projective transformations of the Riemann sphere $\mathbb{C}P^1$. Using the classical *suspension* method, one constructs a foliation $\mathcal{F}$ on some compact manifold M which fibers over B and whose transversal structure is given by the group $\Gamma = h(\pi_1(B)) \subset PSL(2, \mathbb{C})$ acting on the Riemann sphere. Basic normal vector fields correspond to Γ-invariant vector fields on $\mathbb{C}P^1$. If there is such a non trivial invariant vector field, Γ acts freely on the open set where it is non zero. It is easy to see that it is only possible if Γ is either discrete or abelian or conjugate to a subgroup of the affine group of the real line $Aff(\mathbb{R})$, considered as a subgroup of $PSL(2, \mathbb{C})$.

If Γ *is discrete*, we are in the classical situation of Kleinian groups. The Riemann sphere is the disjoint union of the limit set Λ and the discontinuity domain Ω. The limit set yields a compact part in M which is obviously contained in the Julia set of $\mathcal{F}$. It is not exactly true that the discontinuity domain corresponds to the Fatou set. Indeed, the action of Γ on Ω is proper but might have some non trivial isolated fixed points so that every continuous Γ-invariant vector field on $\mathbb{C}P^1$ has to vanish at these fixed points. However, it is not difficult to check that for every point in Ω with trivial stabilizer there is a Γ-invariant continuous vector vector field X on $\mathbb{C}P^1$ which is not vanishing at this point. One can moreover choose X such that $\bar{\partial}X$ is essentially bounded. Hence the Fatou set is precisely the open set in M corresponding to points of Ω with trivial stabilizer under the Γ action.

If Γ *is non discrete and abelian*, it is contained in a one parameter subgroup of $PSL(2,\mathbb{C})$. Hence there is a holomorphic Γ-invariant vector field on $\mathbb{C}P^1$. In this elementary situation the Fatou set corresponds to the complement of the one or two singularities of this vector field. The Julia set consists of one or two compact leaves.

Suppose now that Γ *is non discrete, non abelian and contained in* $Aff(\mathbb{R})$. Consider the action of the group $Aff(\mathbb{R})$ on the Riemann sphere $\mathbb{C}P^1 \simeq \mathbb{C} \cup \{\infty\}$. It has four orbits; the upper and lower half planes, the real axis and the point at infinity. Restricted to the upper half plane, this action is conjugated to left translations of $Aff(\mathbb{R})$ on itself, hence commuting with right translations. These right translations define an action of $Aff(\mathbb{R})$ on $\mathcal{H}$ which is not holomorphic but quasiconformal. One checks easily that this action, extended by the identity in $\mathbb{C}P^1 \setminus \mathcal{H}$, is a continuous action of $Aff(\mathbb{R})$ on $\mathbb{C}P^1$. Moreover the one parameter subgroup corresponding to translations is indeed associated to a vector field X_+ with $\bar{\partial}X_+$ locally integrable. In formula, this vector field is $X_+ = y\frac{\partial}{\partial x}$ for $y > 0$ and $X_+ = 0$ for $y \leq 0$ and one verifies that X_+ has the required regularity also in the neighborhood of ∞. Of course, one can make exactly the same computations for the lower half plane so that we get another continuous vector field X_- on $\mathbb{C}P^1$ which is invariant under the action of $Aff(\mathbb{R})$ on $\mathbb{C}P^1$ and which now vanishes on the upper half plane.

The decomposition of $\mathbb{C}P^1$ in four orbits defines a decomposition of M in four parts. The two vector fields X_+, X_- give rise to normal basic vector fields so that the two open sets in M corresponding to upper and lower half planes are contained in the Fatou set. Since Γ is non abelian, every continuous Γ-invariant vector field on $\mathbb{C}P^1$ has to vanish on the real axis so that the two parts corresponding to the real axis and the point ∞ are in the Julia set. In these examples, the Julia set is a real codimension 1 hypersurface in M. All leaves in this Julia set are dense in the Julia set except the compact leaf corresponding to the point ∞. The Fatou set contains two open components. In each one, the foliation is transversely modeled on the action of $h(\pi_1(B))$ on $Aff(\mathbb{R}) \simeq \mathcal{H}$. If Γ is dense in $Aff(\mathbb{R})$, then leaves are dense in each Fatou component. Otherwise, the closure of Γ is the group of transformations $\{\lambda^n x + b, n \in \mathbb{Z}, b \in \mathbb{R}\}$, for some $\lambda > 0$ and the leaf closures in each Fatou component define a locally trivial fibration over the circle.

Recall that the dynamics of Riccati's equations in $\mathbb{C}P^2$ is equivalent to the dynamics of a finitely generated subgroup of $PSL(2,\mathbb{C})$. Hence the examples that we described can also be considered in the context of (singular) foliations in $\mathbb{C}P^2$ or $\mathbb{C}P^1 \times \mathbb{C}P^1$, as in example 8.1.

Example 8.5. The affine group $Aff(\mathbb{R})$ acts isometrically, freely and transitively, on the upper half plane $\mathcal{H}$ equipped with the Poincaré metric. Choose some Lie group G such that there exists a surjection $\pi : G \to Aff(\mathbb{R})$ and some discrete subgroup Γ in G with $\Gamma\backslash G$ compact. The foliation on G given by the fibers of π is invariant under left translations by Γ so that it defines a foliation $\mathcal{F}$ on $\Gamma\backslash G$. The transversal structure of $\mathcal{F}$ is given by the left action of $\pi(\Gamma)$ on $Aff(\mathbb{R}) \simeq \mathcal{H}$. In particular, $\mathcal{F}$ is a transversely holomorphic codimension 1 foliation which is also a Riemannian foliation. Since any right invariant vector field X on $Aff(\mathbb{R})$ defines a basic normal vector field, we see that the Fatou set of $\mathcal{F}$ is the whole manifold $\Gamma\backslash G$. If $\pi(\Gamma)$ is dense in $Aff(\mathbb{R})$, all leaves are dense. Otherwise, one can show that the closures of the leaves are codimension 1 manifolds in $\Gamma\backslash G$ which constitute the fibers of some fibration on the circle.

Let us describe some explicit examples of such G and Γ that will be useful later. Let $\mathfrak{K}$ be a totally real number field of degree n over the rationals and let ι denote one of the n embeddings of $\mathfrak{K}$ in $\mathbb{R}$. Let $\mathfrak{O} \subset \mathfrak{K}$ the ring of algebraic integers in $\mathfrak{K}$ and $\mathfrak{U}$ be the group of units in $\mathfrak{O}$. Denote by $\mathfrak{U}_+$ the group of totally positive units, *i.e.* those which are mapped in $\mathbb{R}_+^\star$ by all embeddings in $\mathbb{R}$. As it is well known, $\mathfrak{O}$ is a free $\mathbb{Z}$-module of rank n and $\mathfrak{U}_+$ is a free abelian group of rank $n-1$ (Dirichlet's units theorem). Since $\mathfrak{U}_+$ obviously acts by multiplication on $\mathfrak{O}$, one can define the semi-product Γ of $\mathfrak{U}_+$ and $\mathfrak{O}$. "Tensoring" by the reals, we get a linear action of $\mathfrak{U}_+ \otimes \mathbb{R} \simeq \mathbb{R}^{n-1}$ on $\mathfrak{O} \otimes \mathbb{R} \simeq \mathbb{R}^n$ and the corresponding semi-direct product is a $2n-1$ dimensional Lie group G which obviously contains a copy of Γ as a discrete cocompact subgroup. Now the embedding ι defines a homomorphism π from Γ into $Aff(\mathbb{R})$, sending the integer $x \in \mathfrak{O}$ to the translation by $\iota(x)$ and the unit $u \in \mathfrak{U}_+$ to the multiplication by $\iota(u)$. This homomorphism extends to a surjection π from G to $Aff(\mathbb{R})$. In this way we get explicit examples of codimension 1 transversely holomorphic foliations for which the Fatou set is the whole manifold $\Gamma\backslash G$. The leaves are dense precisely when $\iota(\mathfrak{U}_+)$ is dense in $\mathbb{R}_+^\star$, *i.e.* if and only if $n > 2$.

Example 8.6. Let us now consider Riemannian examples in the same spirit for which the Fatou set is empty. Consider a discrete torsion free subgroup Δ of $PSL(2,\mathbb{R})^n$ such that the quotient $PSL(2,\mathbb{R})^n/\Delta$ is compact. For $n \geq 2$, there are examples such that the projection $\pi(\Delta)$ of Δ in the first factor of $PSL(2,\mathbb{R})^n$ is a dense subgroup (see [4]). The action of Δ on the n^{th}-power $\mathcal{H}^n$ of the upper half plane is free and cocompact and obviously preserves the complex codimension 1 foliation given by the projection on the first factor. In this way, we get a compact complex manifold equipped with a codimension 1 transversely holomorphic foliation $\mathcal{F}$ whose transversal structure is given by the action of $\pi(\Delta)$ on $\mathcal{H}$. This action has dense orbits and there is no non trivial invariant vector field since, by density, such a vector field would be invariant under the full group $PSL(2,\mathbb{R})$. *This foliation $\mathcal{F}$ is therefore Riemannian and the Fatou set is empty.*

Example 8.7. Let us come back to our totally real number field $\mathfrak{K}$ of degree n over the rationals and keep the notations of example 8.5. Let $\Delta = PSL(2,\mathfrak{O}) \subset PSL(2,\mathfrak{K})$. Using the n embeddings of $\mathfrak{K}$ in $\mathbb{R}$, we get an embedding of Δ in $PSL(2,\mathbb{R})^n$ as a discrete subgroup. However, strictly speaking, Δ is not of the form described in example 8.6 for two reasons. Firstly, Δ might contain elements of finite order, so we replace it by some of its finite index torsion free subgroups. Secondly, Δ is *not* cocompact in $PSL(2,\mathbb{R})^n$. The complex manifold $M = \mathcal{H}^n/\Delta$ has a finite number of "cusps" which can be analyzed in detail (see [10]). Each end has a neighborhood of the form $S \times \mathbb{R}^+$ where S is a real codimension 1 submanifold in M (hence of real dimension $2n-1$) which is transverse to the foliation $\mathcal{F}$. Moreover, the foliation induced on this submanifold S is conjugated, up to some finite cover, to one of the examples 8.5. That will allow us, later on, to do some surgery on these examples in order to produce new examples. For the time being, note that after deleting these cusps neighborhoods, we get a foliation on a compact manifold with boundary, transversal to the boundary. The double of this manifold provides an example with empty Fatou set.

In the special case $n = 2$, the complex surface $M = \mathcal{H}^2/\Delta$ can be made into a compact complex singular surface by adding one point at each cusp, and then desingularizing the resulting singular complex surface. The result is a smooth compact (algebraic) surface S equipped with a foliation which is isomorphic, outside of some exceptional divisor, to the horizontal foliation on $\mathcal{H}^2/\Delta$. According to Hirzebruch, for some very explicit choices of the quadratic number field $\mathfrak{K}$, this surface S is birationaly equivalent to the complex projective plane [10]. In these cases, transporting the foliation by the birational isomorphism, we finally get an example of a foliation $\mathcal{F}$ on $\mathbb{C}P^2$, hence given by a polynomial differential equation like in example 8.1.

Example 8.8. In example 8.4, we choose a manifold B of the form $\Gamma\backslash G$ as in example 8.5 and we choose h to be the natural homomorphism $\pi : \Gamma \to Aff(\mathbb{R}) \subset PSL(2,\mathbb{C})$. We still denote by $\mathcal{F}$ the corresponding suspended foliation, on a manifold M which fibers over $B = \Gamma\backslash G$ with fibers diffeomorphic to $\mathbb{C}P^1$. Let us analyze the Fatou component corresponding to the upper half plane. This can be described as the quotient of $G \times Aff(\mathbb{R})$ by the subgroup of elements of the form $(\gamma, \pi(\gamma))$ with $\gamma \in \Gamma$ and the foliation is given by the projection on the second factor. Now, consider the subgroup of $G \times Aff(\mathbb{R})$ consisting of elements (g_1, g_2) with $\pi(g_1) = g_2$. This is invariant under the Γ action and defines a submanifold W embedded into the Fatou component. Of course, W is diffeomorphic to $\Gamma\backslash G$ and is transversal to the foliation in the Fatou component. In other words, if we choose properly B and h in example 8.4, we can find a codimension 2 submanifold W embedded in one of the Fatou components, transversely to the foliation, in such a way that the induced foliation on W is an Example 8.5. Dig out a small tubular neighborhood of W in M in order to get a compact manifold $\overline{M}$ with a boundary diffeomorphic to $W \times \mathbb{S}^1$ and transverse to the foliation. We can then take the double of $\overline{M}$. The result is a new compact boundaryless manifold M_1 equipped with a transversely holomorphic foliation $\mathcal{F}_1$ with the following properties.

The Fatou set of $\mathcal{F}_1$ consists of three components; indeed the Fatou component of $\mathcal{F}$ which does not contain W has been doubled, giving rise to two disjoint components and the Fatou component containing W gives rise to a single component. The Julia set consists of the disjoint union of two codimension 1 submanifolds, each containing a compact leaf.

Inside M_1, we can again find another copy of W embedded in a Fatou component, we can delete a tubular neighborhood of this copy and glue $\overline{M}$ on its boundary. Iterating this process, we get a family of *examples of transversely holomorphic foliations on compact manifolds whose Fatou set consists of an arbitrary number of connected components and whose Julia set is the disjoint union of codimension 1 submanifolds.*

Example 8.9. Start with the manifold with boundary $\overline{M}$ that we described in example 8.8. Instead of taking a double, we may glue some other piece with the same boundary. We have seen such a piece in example 8.7. Indeed, to a totally real number field $\mathfrak{K}$, we associated a discrete subgroup Δ and a complex manifold $\mathcal{H}^n/\Delta$. In order to simplify our puzzle game, assume that the number field is such that $\mathcal{H}^n/\Delta$ has only one cusp (this means that $\mathfrak{O}$ is a principal ring). Since we know that each cusp has a neighborhood bounded by a hypersurface isomorphic to an example 8.5, if we delete this neighborhood to $\mathcal{H}^n/\Delta$, we get a compact manifold with boundary equipped with a foliation transversal to the boundary inducing a foliation conjugated to some example 8.5. Crossing with a circle, we get a compact manifold $\widehat{M}$ with boundary which has the same boundary as $\overline{M}$ with the same foliation induced on the boundary. Gluing these two manifolds, we get an example of a transversely holomorphic foliation on a compact manifold with the following properties:

The Julia set consists of two parts. The first is the Julia set in $\overline{M}$, *i.e.* a compact codimension 1 submanifold containing a compact leaf. The second is the union of $\widehat{M}$ and of the component of the Fatou component in $\overline{M}$ which contains the boundary. Indeed we know that there is no non trivial basic vector fields in $\mathcal{H}^n/\Delta$ hence in $\widehat{M}$. Note in particular that *the Julia set might have non empty interior without being the whole manifold* (contrary to the situation of rational maps or Kleinian groups). Of course, we can play the same game again; digging out tubular neighborhoods of W, assembling these jigsaw puzzles together, using also copies of $\widehat{M}$.

Summarizing: we constructed examples of foliations of compact manifolds M containing finitely many disjoint codimension 1 real submanifolds $T_1, \ldots, T_m$ and such that the Fatou set consists of some of the connected components of $M \setminus \cup T_i$ and the Julia set consists of the T_i together with the other connected components of the complement. Moreover, the combinatorics of this situation can be arbitrarily prescribed. More precisely, consider the graph whose vertices are the connected components of $M \setminus \cup T_i$ and whose edges correspond to adjacent components. This is a finite graph with two types of vertices since a component may be in the Fatou or Julia set. The reader may verify that our construction is flexible enough to provide examples with an arbitrary finite graph and that one can prescribe arbitrarily the set of vertices corresponding to Fatou components.

In these examples, note that the foliation restricted to the Julia set is not ergodic as soon as there are at least two open components in the Julia set. Note that in these examples the ergodic components of the Julia set are open sets of the Julia set.

Example 8.10. Let us indicate another way of modifying the previous examples using ramified covers. Start with an example 8.4 associated to a homomorphism $h : \pi_1(B) \to Aff(\mathbb{R}) \subset PSL(2, \mathbb{C})$. Choose h such that the ambient manifold M is diffeomorphic to a product $B \times \mathbb{C}P^1$. There is a compact leaf corresponding to ∞ whose holonomy group, changing coordinates to look in the neighborhood of the origin, consists of germs of maps of the form $z \mapsto az/1+bz$ with $a \in \mathbb{R}^\star_+$ and $b \in \mathbb{R}$. We can now consider the Hopf fibration $\mathbb{S}^3 \to \mathbb{C}P^1$. Pulling back, we get a foliation on $M' = B \times \mathbb{S}^3$ with a compact leaf L which is B times a fiber of the Hopf fibration. In particular, there is a non trivial homomorphism from the fundamental group of the complement $M' \setminus L$ onto $\mathbb{Z}$. It follows that one can find a connected finite cover of $M' \setminus L$ with an arbitrary number n of sheets. In other words, there is a finite ramified covering of M' ramified along L. One can pull-back the foliation $\mathcal{F}$ on the total space of this cover M'_n; one easily checks that the corresponding foliation $\mathcal{F}_n$ can be equipped with a transversal holomorphic structure. This foliation $\mathcal{F}_n$ has a compact leaf whose holonomy group has been ramified, *i.e.* consisting of germs of the form $z \mapsto (az^n/1 + bz^n)^{1/n}$ with $a \in \mathbb{R}^\star_+$ and $b \in \mathbb{R}$. This foliation $\mathcal{F}_n$ again has a unique compact leaf and two Fatou components. The Julia set is not a submanifold anymore; it consists of a finite number of codimension 1 submanifolds meeting along the compact leaf.

Non Examples 8.11. We give two examples of non transversely holomorphic foliations displaying some features that cannot occur in the holomorphic case by our results.

Consider a projective transformation $f : \mathbb{R}P^2 \to \mathbb{R}P^2$ associated to a 3×3 real matrix which is diagonalizable over the reals with distinct eigenvalues. It is a diffeomorphism with three fixed points and all other points are wandering under iteration of f. However, the quotient of the complement of the three fixed points in $\mathbb{R}P^2$ by the iterates of f is a *non* Hausdorff surface having a curve of non Hausdorff points. The suspension of f gives a 1-dimensional foliation on a compact 3-manifold with three compact leaves and such that all other leaves are wandering but such that the space of non wandering leaves is non Hausdorff. This shows that Theorem 2 is not true for non transversely holomorphic foliations.

Let g be a linear Anosov automorphism of the 2-dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$. It is an area preserving diffeomorphism which is ergodic. Consider now a small perturbation g' of g which is such that the origin is still a fixed point but in such a way that the Jacobian at the origin is different from 1. The classical theory of Anosov diffeomorphisms shows that g' is now totally dissipative with respect to the Lebesgue measure; there is a Borel set $B \subset \mathbb{R}^2/\mathbb{Z}^2$ which is disjoint from all its iterates $(g')^n(B), n \in \mathbb{Z}$ and such that the union of these $(g')^i(B)$ covers almost all the torus (with respect to the Lebesgue measure). Note that, by structural stability, the two diffeomorphisms g, g' are conjugate by some (non absolutely continuous!) homeomorphism of the torus. The suspension of g' gives a codimension 2 non holomorphic foliation with no non trivial basic vector field which is totally dissipative with respect to the Lebesgue measure. Theorem 5 shows that this is not possible in the holomorphic case.

9. Some questions

We finish with a list of a few questions that arose in this study of the dynamical behavior of transversely holomorphic foliations.

Wandering components

Is the number of wandering Fatou components finite? In order to answer this question, one should understand those wandering Fatou components whose leaf space is the sphere minus 1, 2 or 3 points.

Is the space of wandering leaves Hausdorff? We saw in 6.4 that this space is not quite contained in the Fatou set and that there might be isolated non Hausdorff points. However we know no example of such a situation.

Semi-wandering components

These semi-wandering components are analogous to Siegel discs or Herman rings for rational maps. However, the examples that we gave in section 8 are not so interesting from that point of view since their boundaries are very regular. Is it always the case for a general semi-wandering component?

We saw that the transverse structure in such a semi-wandering component is given by some group Γ acting affinely on some strip. Is this group Γ finitely generated?

Is the number of semi-wandering components finite? It is easy to see that the components of type a) have a modulus so that there is a finite number of them.

Dense components

The most interesting case is the $Aff(\mathbb{R})$ case. It is easy to show that there is a finite number of such components since there is an invariant Beltrami differential on the upper half space invariant under $Aff(\mathbb{R})$. Can one describe further this type of component? In particular, is it the case that their boundaries are regular? Is the corresponding group $\Gamma \subset Aff(\mathbb{R})$ finitely (*resp.* compactly) generated?

Julia set

It would be interesting to understand better the topological and ergodic dynamics inside the Julia set.

Are the ergodic components in $Julia(\mathcal{F})$ open in $Julia(\mathcal{F})$?

Is the Julia set the closure of leaves with non-trivial holonomy?

Generic situation

For a generic polynomial vector field in the complex projective plane, is it true that the semi-wandering and dense parts of the Fatou set are empty? Can one describe the wandering part?

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