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GROUPS ACTING ON THE CIRCLE

by Étienne GHYS

1. INTRODUCTION

The classical theory of dynamical systems studies the orbit structure of a homeomorphism or of a flow on a manifold, *i.e.* of actions of the group **Z** or **R**. This theory can be generalized to actions of an arbitrary group Γ on a manifold. These notes propose a survey of some results concerning the case where the group Γ is quite arbitrary and the manifold is the circle or the real line.

This paper covers a very small part of the theory. We decided to discuss only the topological aspect; this is a pity since the theory of groups of smooth diffeomorphisms is so rich! For instance, we would have liked to discuss the so called "level theory" around Sacksteder's theorem or problems related to structural stability. Even in the restricted domain of topological dynamics, these notes are incomplete; we should have discussed at least the remarkable classification of convergence groups due to Tukia, Casson-Jungreis, Gabai [15, 24]... The author hopes that in the near future he will be able to write a reasonably complete survey on this area.

Our main goal is to provide a motivation for our paper on actions of higher rank lattices on the circle [26]. Section 3 describes some important examples of group actions on the circle. Section 4 reviews some of the main topological and algebraic properties of the group of homeomorphisms of the circle. In Sections 5 and 6 we describe the interplay between the classical rotation number and the cohomological invariant given by the Euler class. Finally, in Section 7 we discuss recent results concerning actions of lattices on the circle. Subsection 7.2 is essentially an extract from [26].

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I would like to thank John Crisp, Bruno Sévennec and Dave Witte for interesting comments on this text.

CONTENTS

2. SOME CLASSICAL DEFINITIONS

We begin with some very general definitions concerning group actions. For an introduction to this subject, we refer to [42].

Let Γ be any group and X be any topological space. An *action* of Γ on X is a homomorphism ϕ from Γ to the group Homeo(X) of homeomorphisms of X. An element $\gamma \in \Gamma$ and a point $x \in X$ produce the point $\gamma \cdot x = \phi(\gamma)(x)$. Conversely a map

$$(\gamma, x) \in \Gamma \times X \mapsto \gamma \cdot x \in X$$

comes from an action if for every γ, the point $\gamma \cdot x$ depends continuously on x and if for every γ_1, γ_2 we have $\gamma_1 \cdot (\gamma_2 \cdot x) = (\gamma_1\gamma_2) \cdot x$ and $e.x = x$ (e denotes the identity element in Γ).

Two actions ϕ_1 and ϕ_2 of Γ on X_1 and X_2 are *conjugate* if there exists a homeomorphism h from X_1 to X_2 such that for every $\gamma \in \Gamma$, one has $\phi_2(\gamma) = h\phi_1(\gamma)h^{-1}$.

An action ϕ is *faithful* if it is injective, *i.e.* if non trivial elements in the group act non trivially on the space. This is a minor assumption since we can always consider the associated faithful action of the quotient group $\Gamma/\ker(\phi)$.

The *orbit* of a point x is the set $\mathcal{O}(x) = \{\phi(\gamma)(x) \mid \gamma \in \Gamma\} \subset X$. The main object of topological dynamics is to study the topological properties of the partition of X into orbits. An action is *transitive* if there is only one orbit. We say in this case that X is *homogeneous* under the action of Γ. Of course, these transitive actions are quite trivial from the topological dynamics point of view but this does not mean that the geometrical study of homogeneous spaces is not interesting !

The *stabilizer* of the point x is the subgroup

$$Stab(x) = \{\gamma \in \Gamma \mid \phi(\gamma)(x) = x\} \subset \Gamma .$$

There is a natural bijection between the quotient $\Gamma/Stab(x)$ and the orbit $\mathcal{O}(x)$. Note that the stabilizers of two points in the same orbit are conjugate subgroups in Γ. An action is *free* if the stabilizer of every point is trivial, *i.e.* if the action of a non trivial element of Γ has no fixed point.

In some cases, Γ might be a topological group. In these cases, we frequently consider *continuous actions* such that $\gamma \cdot x$ is a continuous function on $\Gamma \times X$. The orbit map bijection from $\Gamma/Stab(x)$ to $\mathcal{O}(x)$ is continuous but is usually not a homeomorphism when $\mathcal{O}(x)$ is equipped with the induced topology from X. The easiest non trivial example is the case where $\Gamma = \mathbf{R}$, *i.e.* of a topological flow: if the stabilizer of a point x is trivial, the orbit $\mathcal{O}(x)$ is the image of a continuous bijection $\mathbf{R} \to \mathcal{O}(x) \subset X$ but in many cases this orbit might be recurrent (for instance dense in X) and this bijection is not a homeomorphism. There is however a special case in which this map is indeed a homeomorphism and we use this fact constantly (and sometimes implicitely) in these notes. Consider a Lie group G acting continuously and transitively on a manifold M and denote by H the stabilizer of a point. Then H is a closed subgroup of G, hence a closed Lie subgroup, and the quotient space G/H is naturally a smooth manifold. In this case, the orbit map from G/H to M is a homeomorphism.

3. Two basic examples

Up to homeomorphism, the circle is the only compact connected 1-dimensional manifold: this is probably the reason why we meet so many circles in mathematics... We can think of the circle $\mathbf{S}^1$ in many ways. We can first consider it as the unit circle in $\mathbf{R}^2$ but we can also see it as the abstract 1-dimensional manifold which is the quotient of the real line $\mathbf{R}$ by the subgroup of integers $\mathbf{Z}$. From this point of view, $\mathbf{S}^1$ can be thought of as being an abelian group, isomorphic to $\mathrm{SO}(2, \mathbf{R})$ or to the 1-dimensional torus. The circle can also be considered as the real projective line $\mathbf{RP}^1$ consisting of lines in $\mathbf{R}^2$ going through the origin (identified with $\mathbf{R} \cup \{\infty\}$ by taking the slope of a line).

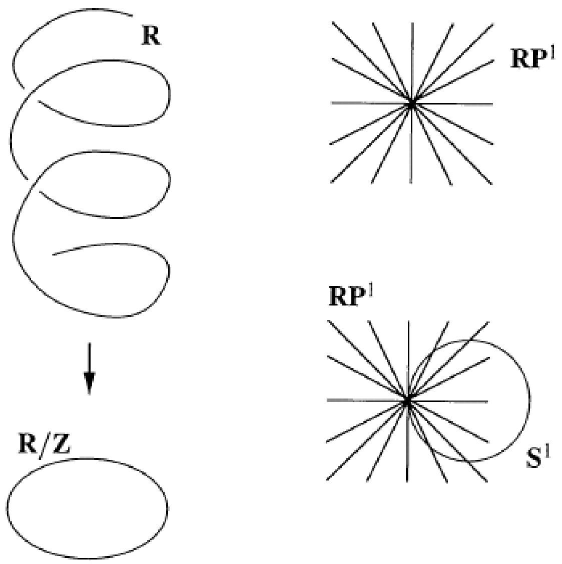

Figure 1

Going from one point of view to another is easy, the identifications being given by:

$$t \in \mathbf{R}/\mathbf{Z} \mapsto (\cos(2\pi t), \sin(2\pi t)) \in \mathbf{S}^1 \subset \mathbf{R}^2$$

$$t \in \mathbf{R}/\mathbf{Z} \mapsto \tan(\pi t) \in \mathbf{R} \cup \{\infty\} = \mathbf{RP}^1$$

$$s \in \mathbf{RP}^1 \mapsto (\frac{1 - s^2}{1 + s^2}, \frac{2s}{1 + s^2}) \in \mathbf{S}^1 .$$

In this first section, we would like to give two very basic examples of groups acting on the circle which will play a central role in these lecture notes. The properties of these examples will be detailed in this text and we could say that a major theme of research would be to show that many groups acting on the circle can be reduced to them.

3.1 THE PROJECTIVE GROUP

The linear group $\mathrm{GL}(2,\mathbf{R})$ consists of 2×2 real invertible matrices. Its center is the group of scalar matrices and the quotient of $\mathrm{GL}(2,\mathbf{R})$ by this center is denoted by $\mathrm{PGL}(2,\mathbf{R})$ and called the *projective group*. There is a natural (projective) action of $\mathrm{PGL}(2,\mathbf{R})$ on the circle (seen as $\mathbf{RP}^1$). Indeed, $\mathrm{GL}(2,\mathbf{R})$ acts linearly on $\mathbf{R}^2$ and induces an action on the set of lines in $\mathbf{R}^2$ going through the origin, which is $\mathbf{RP}^1$ by definition. A formula for the action is given by:

$$\left(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right], x\right) \in \mathrm{PGL}(2,\mathbf{R}) \times \mathbf{RP}^1 \mapsto \frac{ax+b}{cx+d} \in \mathbf{RP}^1 .$$

We use a (square) bracket to denote the equivalence class modulo scalar matrices. Note that the group $\mathrm{PGL}(2,\mathbf{R})$ has two connected components given by the sign of the determinant. The component of the identity is isomorphic to $\mathrm{PSL}(2,\mathbf{R})$, which is the quotient of the unimodular group $\mathrm{SL}(2,\mathbf{R})$ by its center which consists of $\pm\mathrm{Id}$. The action of an element of $\mathrm{PGL}(2,\mathbf{R})$ on the circle preserves or reverses orientation according to the sign of its determinant.

An important feature of this action is that it extends to the disc. The real projective line $\mathbf{RP}^1$ sits naturally inside the complex projective line $\mathbf{CP}^1 \simeq \mathbf{C}\cup\{\infty\}$ which is the Riemann sphere. In the same way, the real projective group $\mathrm{PGL}(2,\mathbf{R})$ is a subgroup of the complex projective group $\mathrm{PGL}(2,\mathbf{C})$ which acts on the Riemann sphere by Mœbius transformations.

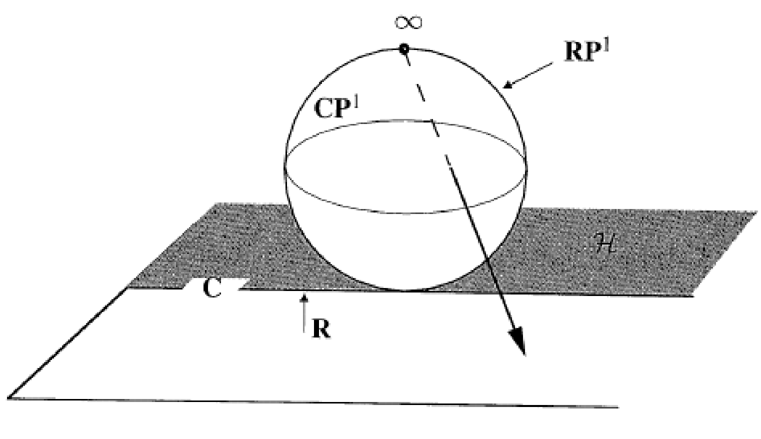

FIGURE 2

Hence we get an action of $\mathrm{PGL}(2,\mathbf{R})$ on the Riemann sphere $\mathbf{CP}^1$ preserving the circle $\mathbf{RP}^1$. The complement of this circle in this sphere consists of two discs which are preserved or permuted by an element of $\mathrm{PGL}(2,\mathbf{R})$ according to the sign of the determinant. In the obvious coordinates, the circle $\mathbf{RP}^1$ is the real axis in $\mathbf{CP}^1 \simeq \mathbf{C}\cup\{\infty\}$ plus the point at infinity. Denote by $\mathcal{H} \subset \mathbf{C} \subset \mathbf{CP}^1$ the upper half space, *i.e.* the set of complex numbers z with positive imaginary part: this is one of the two components of the complement of $\mathbf{RP}^1$ in $\mathbf{CP}^1$. We get an action of $\mathrm{PSL}(2,\mathbf{R})$ on $\mathcal{H}$ which extends the action of $\mathrm{PSL}(2,\mathbf{R})$ on the boundary $\mathbf{RP}^1$. This extension is holomorphic and is actually an isometric action when we equip $\mathcal{H}$ with its Poincaré metric (see for instance [67]).

The group $\mathrm{PSL}(2,\mathbf{R})$ and the rotation group $\mathrm{SO}(3,\mathbf{R})$ are the only simple Lie groups of real dimension 3 and there is no non trivial simple Lie group of lower dimension [58]. This may explain why several versions of these groups occur in mathematics. We give one of them, showing a different aspect of the action of $\mathrm{PSL}(2,\mathbf{R})$ on the circle.

Consider the quadratic form $Q = x_1^2+x_2^2-x_3^2$ on $\mathbf{R}^3$. Its group of isometries is denoted by $\mathrm{O}(2,1)$. This group has four connected components (see for example [54]) and it turns out that the component of the identity is isomorphic to $\mathrm{PSL}(2,\mathbf{R})$. A simple way to check this fact is to consider the action of $\mathrm{GL}(2,\mathbf{R})$ on the space $\mathrm{M}(2,\mathbf{R})$ of 2×2 matrices given by conjugation:

$$(A,M) \in \mathrm{GL}(2,\mathbf{R}) \times \mathrm{M}(2,\mathbf{R}) \mapsto AMA^{-1}.$$

Note that this action factors through an action of $\mathrm{PGL}(2,\mathbf{R})$ since the center acts of course trivially. We can moreover restrict this action to the invariant 3-dimensional vector space E consisting of matrices whose trace is 0. Finally, we observe that the determinant of M provides an invariant quadratic form on E. It is easy to check that the signature of this quadratic form is $(-,-,+)$ so that, using suitable coordinates, we get an injection of $\mathrm{PGL}(2,\mathbf{R})$ in $\mathrm{O}(2,1)$. This injection gives the promised identification between $\mathrm{PSL}(2,\mathbf{R})$ and the connected component of the identity in $\mathrm{O}(2,1)$. Figure 3 shows the orbits of this linear action on E.

Since $\mathrm{O}(2,1)$ acts linearly on $\mathbf{R}^3$, it acts projectively on the projective plane $\mathbf{RP}^2$ consisting of lines in $\mathbf{R}^3$. The zero locus of Q in $\mathbf{R}^3$ is a cone which projects to a conic C in $\mathbf{RP}^2$ invariant under $\mathrm{O}(2,1)$. As any non degenerate conic in the projective plane can be rationally parametrized by $\mathbf{RP}^1$, we get an action of $\mathrm{O}(2,1)$ on the circle $\mathbf{RP}^1$. The reader will easily check that we get, up to conjugacy and identifications, the same action of $\mathrm{PSL}(2,\mathbf{R})$ on $\mathbf{RP}^1$ that we described earlier.

The conic C bounds two domains in $\mathbf{RP}^2$. One of them is homeomorphic to a disc and is the projection of the set of points for which $Q<0$: it

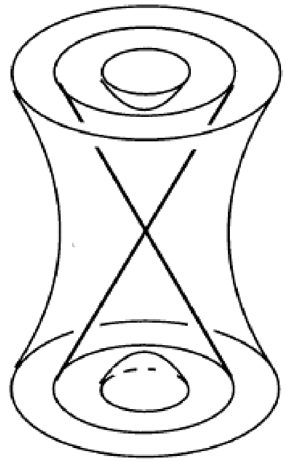

FIGURE 3

is called its interior and is denoted by D. The exterior is homeomorphic to a Mœbius band. Hence we can think of the circle $C \simeq \mathbf{RP}^1 \simeq \mathbf{S}^1$ as the boundary of a disc $D \subset \mathbf{RP}^2$ on which $\mathrm{PGL}(2,\mathbf{R})$ acts projectively. We have extended the action of $\mathrm{PGL}(2,\mathbf{R})$ on the circle to an action on the disc. This is the Klein model.

Of course, the two extensions of the action of $\mathrm{PSL}(2,\mathbf{R})$ on a disc are conjugate, even though they don't quite look the same. The first one is conformal in one complex variable and the second one is projective in two real variables. There are several ways of describing a conjugacy between these two actions [67]. The following one is nice and not so well known. Consider the linear action of $\mathrm{PSL}(2,\mathbf{R})$ on the 3-dimensional vector space of polynomials of the second degree $aX^2 + bXY + cY^2$ by linear change of coordinates. The discriminant $b^2 - 4ac$ defines an invariant quadratic form of signature $(+,+,-)$. Hence, we can identify this linear action with the linear action of the identity component of $\mathrm{O}(2,1)$ that we considered above. Now any polynomial in the negative cone of the discriminant defines a polynomial $aX^2 + bX + c$ with two complex conjugate roots. Hence, we can define a map from the disc D to the upper half plane $\mathcal{H}$ sending the line through the polynomial to the unique root in $\mathcal{H}$. This map is obviously a conjugation between the two actions of $\mathrm{PSL}(2,\mathbf{R})$ on $\mathcal{D}$ and $\mathcal{H}$. Note however that the two actions of $\mathrm{PSL}(2,\mathbf{R})$ on $\mathbf{RP}^2$ and $\mathbf{CP}^1$ that we constructed are not conjugate since $\mathbf{RP}^2$ and $\mathbf{CP}^1$ are not homeomorphic !

The action of $\mathrm{PSL}(2,\mathbf{R})$ on the circle that we described is well known and there is not much to say about its dynamics since it has only one orbit ! In order to get examples which are interesting from the dynamical point of view, we should restrict it to suitable subgroups of $\mathrm{PSL}(2,\mathbf{R})$. We mention the *fuchsian groups* which are by definition the discrete subgroups of $\mathrm{PSL}(2,\mathbf{R})$. These groups come from many parts of mathematics, in particular from number theory. For instance, the modular group $\mathrm{PSL}(2,\mathbf{Z})$ is fundamental in the study of quadratic forms in two variables over the integers and its action on $\mathbf{RP}^1$ or on $\mathcal{H}$ is one of the main tools to understand it. Gauss began its analysis in his famous *Disquisitiones* and the modular group might be the first non-commutative group to have been been studied in the history of mathematics. As another example, consider a quadratic form in three variables with integral coefficients and signature $(+,+,-)$; the group of its isometries with integer coefficients is of course a fuchsian group. This was another motivation for Poincaré when he studied these groups [60]. We also want to emphasize that not only the discrete groups of $\mathrm{PSL}(2,\mathbf{R})$ might be interesting, even from the number theoretical point of view. Examples can be given by taking a number field k embedded in $\mathbf{R}$ and looking at the ring of integers $\mathcal{O}$ in this field (for instance $\mathbf{Z}[\sqrt{2}]$ in $\mathbf{Q}(\sqrt{2})$). The group $\mathrm{PSL}(2,\mathcal{O})$ of elements of $\mathrm{PSL}(2,\mathbf{R})$ with entries in $\mathcal{O}$ is a very important one (even though it is dense in $\mathrm{PSL}(2,\mathbf{R})$ if k is not the field of rational numbers).

3.2 Piecewise linear groups

Our second example is a much bigger group: the group of piecewise linear homeomorphisms of the circle $\mathbf{S}^1$, considered here as $\mathbf{R}/\mathbf{Z}$. A homeomorphism f of the real line $\mathbf{R}$ is called *piecewise linear* if there is an increasing sequence of real numbers x_i parametrized by $i \in \mathbf{Z}$ such that $\lim_{\pm\infty} x_i = \pm\infty$ and such that the restriction of f to each interval $[x_i, x_{i+1}]$ coincides with an affine map. If such a homeomorphism satisfies $f(x+1) = f(x)+1$ for all x, then it induces a homeomorphism of the circle $\mathbf{S}^1 \simeq \mathbf{R}/\mathbf{Z}$. Such a homeomorphism of $\mathbf{S}^1$ is called a piecewise linear homeomorphism of the circle. Note that, by our definition, we are only considering orientation preserving homeomorphisms of the circle. The collection of these homeomorphisms is a group, denoted by $\mathrm{PL}_+(\mathbf{S}^1)$.

Again, this group is acting transitively on the circle so there is not much to say about its orbits... However $\mathrm{PL}_+(\mathbf{S}^1)$ contains some very interesting subgroups which will provide good examples of some dynamical phenomena on the circle. We shall mention only one of them.

The *Thompson group*, denoted by G, is a countable subgroup of $\mathrm{PL}_+(\mathbf{S}^1)$ which has been studied quite a lot recently and deserves more attention. Some of its properties will be mentioned in these notes, in particular as a source of (counter)-examples. To define it, we consider first the group $\widetilde{G}$ consisting of piecewise linear homeomorphisms f of $\mathbf{R}$ which have the following four properties.

- The sequence x_i can be chosen in such a way that x_i and $f(x_i)$ consist of dyadic rational numbers (*i.e.* of the form $p2^q$, $p, q \in \mathbf{Z}$).
- The set of dyadic rational numbers is preserved by f.
- The derivatives of the restrictions of f to $]x_i, x_{i+1}[$ are powers of 2 (*i.e.* of the form 2^q, $q \in \mathbf{Z}$).
- One has $f(x+1) = f(x) + 1$ for all x.

The elements of $\widetilde{G}$ induce homeomorphisms of the circle $\mathbf{S}^1 \simeq \mathbf{R}/\mathbf{Z}$. The collection of these homeomorphisms is the Thompson group G. Figure 4 shows the graphs of two typical elements of G.

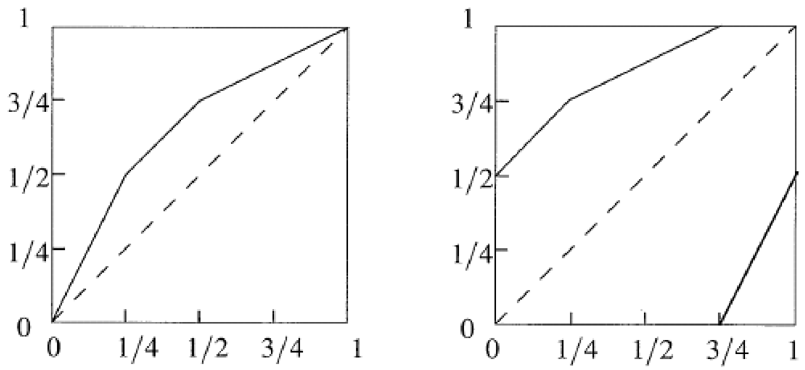

FIGURE 4

Among the nice properties of G, we mention first the fact that G is an *infinite finitely presented simple group*. This was the main motivation for Thompson: indeed G was the first example of such a group (recall that a group is called simple if it contains no proper normal subgroup).

We also mention a connection with the modular group $\mathrm{PSL}(2, \mathbf{Z})$ acting on $\mathbf{RP}^1$. Consider the group of homeomorphisms of $\mathbf{RP}^1$ which are piecewise-$\mathrm{PSL}(2, \mathbf{Z})$, *i.e.* for which one can partition $\mathbf{RP}^1$ as a finite union of intervals with rational endpoints in such a way that on each of these intervals, the homeomorphism coincides with an element of $\mathrm{PSL}(2, \mathbf{Z})$. It turns out that there is a homeomorphism h from $\mathbf{R}/\mathbf{Z}$ to $\mathbf{RP}^1$ mapping the dyadic points in $\mathbf{R}/\mathbf{Z}$ to the rational points of $\mathbf{QP}^1$ and conjugating the Thompson group G with this group of piecewise-$\mathrm{PSL}(2, \mathbf{Z})$!

Somehow, we could say that G sits inside $\mathrm{PL}_+(\mathbf{S}^1)$ like a fuchsian group sits inside $\mathrm{PSL}(2, \mathbf{R})$. For more information concerning this group, see [13, 28].

4. THE GROUP OF HOMEOMORPHISMS OF THE CIRCLE

We denote by $\mathrm{Homeo}_+(\mathbf{S}^1)$ the group of orientation preserving homeomorphisms of the circle $\mathbf{S}^1$. In this section, we want to describe the main properties of this group.

Denote by $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ the group of homeomorphisms $\widetilde{f}$ of the real line $\mathbf{R}$ which satisfy $\widetilde{f}(x+1) = \widetilde{f}(x) + 1$ for all x, *i.e.* which commute with integral translations. Every element of $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ defines a homeomorphism of the circle which is orientation preserving, so that we get a homomorphism p from $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ to $\mathrm{Homeo}_+(\mathbf{S}^1)$. The kernel of p consists of integral translations of the real line. Moreover p is onto: any orientation preserving homeomorphism of the circle lifts to a homeomorphism of its universal covering space, which is the line $\mathbf{R}$, commuting with integral translations. In other words, we have an exact sequence:

$$0 \to \mathbf{Z} \to \widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1) \to \mathrm{Homeo}_+(\mathbf{S}^1) \to 1\,.$$

We say that $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ is a central extension of $\mathrm{Homeo}_+(\mathbf{S}^1)$.

Equipped with the topology of uniform convergence, these groups are naturally topological groups.

We would like to turn these groups into infinite dimensional Lie groups. It is not so easy to do so for many reasons. One of the difficulties is that it is not true that an element of $\mathrm{Homeo}_+(\mathbf{S}^1)$ close to the identity lies on a 1-parameter subgroup (see 5.10). For an excellent survey on infinite dimensional Lie groups, we refer to [53]. In any case, it is customary to think of these homeomorphism groups as "some kind of infinite dimensional Lie groups". For a recent study of the "Lie algebra" of $\mathrm{Homeo}_+(\mathbf{S}^1)$, see [47].

Even though $\mathrm{Homeo}_+(\mathbf{S}^1)$ is not quite a Lie group, it shares many properties with finite dimensional Lie groups. More precisely, we shall try to show that $\mathrm{Homeo}_+(\mathbf{S}^1)$ is a kind of infinite dimensional analogue of $\mathrm{PSL}(2,\mathbf{R})$.

Lie groups admit a maximal compact subgroup K, unique up to conjugacy, and the embedding of K in the Lie group is a homotopy equivalence (see for instance [58]). In case of $\mathrm{PSL}(2,\mathbf{R})$, the maximal compact subgroup is $\mathrm{SO}(2,\mathbf{R})/\{\pm \mathrm{Id}\} \simeq \mathbf{R}/\mathbf{Z}$ and the quotient of $\mathrm{PSL}(2,\mathbf{R})$ by its maximal compact subgroup is contractible since it is identified with the upper half space $\mathcal{H}$ (we remark that $\mathrm{PSL}(2,\mathbf{R})$ acts transitively on $\mathcal{H}$ and that the stabilizer of a point is a maximal compact subgroup).

The same result is true for the group of homeomorphisms of the circle:

PROPOSITION 4.1. *Up to conjugacy, the rotation group* $\mathrm{SO}(2,\mathbf{R})$ *is the only maximal compact subgroup of* $\mathrm{Homeo}_+(\mathbf{S}^1)$.

Proof. Let K be a compact subgroup in $\mathrm{Homeo}_+(\mathbf{S}^1)$ and λ its Haar probability measure, *i.e.* the unique probability measure on K which is invariant under left (and right) translations in K (see for instance [69]). Each element k of K sends the Lebesgue measure dx of the circle $\mathbf{R}/\mathbf{Z}$ to a probability measure $k_\star dx$ on the circle. Averaging using λ, we get a probability measure $\mu = \int_K (k_\star dx)\,d\lambda$ on the circle which is invariant under the action of K. This measure μ obviously has no atom and is non zero on non empty open sets. It follows that there is an orientation preserving homeomorphism h of the circle such that $h_\star\mu = dx$. This is a very special case of a theorem which is valid in any dimension but the proof is very easy on the circle. Indeed, fix a point x_0 on the circle (for instance $0 \mod \mathbf{Z}$) and define $h(x)$ to be the unique point such that the μ-measure of the positive interval from x_0 to x is equal to the Lebesgue measure of the positive interval from x_0 to $h(x)$. The existence and continuity of h follow from the properties of μ and the fact that h sends μ to the Lebesgue measure is obvious from the definition. Now, after conjugating K by h, we get a group of orientation preserving homeomorphisms of the circle preserving the Lebesgue measure, *i.e.* a group of rotations. Hence, some conjugate of K is contained in $\mathrm{SO}(2,\mathbf{R})$. □

Note in particular that the proposition implies that *any finite subgroup of* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *is cyclic and is conjugate to a cyclic group of rotations.*

PROPOSITION 4.2. *The embedding of* $\mathrm{SO}(2,\mathbf{R})$ *in* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *is a homotopy equivalence.*

Proof. Observe first that the group of orientation preserving homeomorphisms of the line $\mathbf{R}$ is a convex set since it is the set of strictly increasing functions from $\mathbf{R}$ to $\mathbf{R}$ tending to $\pm\infty$ as the variable tends to $\pm\infty$. Consider the group $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$. An element $\widetilde{f}$ of this group can be written in the form $\widetilde{f}(x) = x + t(x)$ where t is a $\mathbf{Z}$-periodic function. Now, any such periodic function can be written in a unique way in the form $c_0 + t_0$ where c_0 is a constant and t_0 is a periodic function whose average value over a period is 0. If $0 \leq s \leq 1$ is a parameter, we define $\widetilde{f}_s$ by $\widetilde{f}_s(x) = x + c_0 + (1-s)t_0(x)$. We have $\widetilde{f}_0 = \widetilde{f}$ and $\widetilde{f}_1$ is a translation. It follows from our preliminary observation that for each s in $[0,1]$, $\widetilde{f}_s$ is a homeomorphism in $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$. Hence we get a continuous retraction of $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ on the subgroup of translations of $\mathbf{R}$, isomorphic to $\mathbf{R}$. Note also that this retraction commutes with integral translations, since the average value of $t(x)+1$ over a period is of course c_0+1. In other words, we can define a continuous retraction from the quotient group $\mathrm{Homeo}_+(\mathbf{S}^1) = \widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)/\mathbf{Z}$ onto the group of rotations $\mathrm{SO}(2,\mathbf{R}) \simeq \mathbf{R}/\mathbf{Z}$. This is the homotopy equivalence that we were looking for. Observe that we actually proved something a little bit stronger: $\mathrm{Homeo}_+(\mathbf{S}^1)$ is homeomorphic to the product of $\mathrm{SO}(2,\mathbf{R})$ and a convex set. □

We should not only consider $\mathrm{Homeo}_+(\mathbf{S}^1)$ as a kind of Lie group but as an analogue of a *simple* Lie group (as for example $\mathrm{PSL}(2,\mathbf{R})$) for which there is a well developed and wonderful theory (see for instance [58]).

THEOREM 4.3. *The group* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *is simple.*

Proof. Recall that if Γ is any group, its *first commutator subgroup* $\Gamma' \subset \Gamma$ is the subgroup generated by commutators $[\gamma_1,\gamma_2] = \gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}$ of elements γ_1,γ_2 in Γ (see [46]). A group is called *perfect* if it is equal to its first commutator group, *i.e.* if every element is a product of commutators.

We shall establish later that $\mathrm{Homeo}_+(\mathbf{S}^1)$ is perfect (see 5.11). (Note that the corresponding statement for diffeomorphism groups is also true and much harder to prove but we decided not to discuss groups of diffeomorphisms...) We now show that this implies quite formally the simplicity of $\mathrm{Homeo}_+(\mathbf{S}^1)$. The reader will find in [18] a general theorem stating that a perfect group of homeomorphisms of a manifold which is acting "sufficiently transitively on finite sets" is necessarily a simple group (see also [2]). The proof we present here is an adaptation of this argument.

Recall that the *support* of a homeomorphism is the closure of the set of points which are not fixed. Let N be a non trivial normal subgroup in $\mathrm{Homeo}_+(\mathbf{S}^1)$ and suppose f is some element of $\mathrm{Homeo}_+(\mathbf{S}^1)$ whose support is contained in some compact interval $I \subset \mathbf{S}^1$. Let n_0 be a non trivial element of N and choose some closed interval in $\mathbf{S}^1$ which is disjoint from its image under n_0. Observe that $\mathrm{Homeo}_+(\mathbf{S}^1)$ acts transitively on closed intervals in the circle. Conjugating n_0 by a suitable element of $\mathrm{Homeo}_+(\mathbf{S}^1)$, we can therefore find an element n in N such that $n(I)$ is disjoint from I. Consider now the commutator $g = n^{-1}f^{-1}nf$. It is an element of N since N is a normal subgroup. Moreover, it is clear that g agrees with f on I, with $n^{-1}f^{-1}n$ on $n^{-1}(I)$, and with the identity in the complement of these two disjoint intervals.

Consider now two elements f_1 and f_2 of $\mathrm{Homeo}_+(\mathbf{S}^1)$ whose supports are contained in the same interval I. We can find elements n_1 and n_2 of N such that the intervals I, $n_1^{-1}(I)$ and $n_2^{-1}(I)$ are disjoint. Then the two elements

$g_1 = n_1^{-1}f_1^{-1}n_1f_1$ and $g_2 = n_2^{-1}f_2^{-1}n_2f_2$ are in N and their commutator is equal to the commutator of f_1 and f_2. So we showed that *the commutator of two elements of* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *having support contained in the same interval is in* N.

Cover the circle by three intervals I_1, I_2, I_3 with empty triple intersection but with non empty intersection two by two. Let G_1, G_2, G_3 be the subgroups of homeomorphisms with supports in I_1, I_2 and I_3 respectively and denote by G the subgroup of $\mathrm{Homeo}_+(\mathbf{S}^1)$ that they generate. If a group is generated by a subset, its first commutator subgroup is generated by conjugates of commutators of elements in this subset. It follows that the first commutator subgroup of G is generated by conjugates of commutators of elements in $G_1 \cup G_2 \cup G_3$. Since the union of two of the intervals I_1, I_2, I_3 is not the full circle, it is contained in some compact interval. Hence we can use the above argument to conclude that the commutator of two elements in $G_1 \cup G_2 \cup G_3$ is in N. It follows that N *contains the first commutator subgroup of* G.

We finally prove that G coincides with $\mathrm{Homeo}_+(\mathbf{S}^1)$ and this will prove the proposition since we know that $\mathrm{Homeo}_+(\mathbf{S}^1)$ is equal to its first commutator subgroup (actually we postponed the proof to 5.11 !). Let $x_{1,2}, x_{2,3}, x_{3,1}$ be points in the interiors of $I_1 \cap I_2, I_2 \cap I_3, I_3 \cap I_1$ respectively. Let f be an element of $\mathrm{Homeo}_+(\mathbf{S}^1)$ close enough to the identity so that $f(x_{1,2}), f(x_{2,3}), f(x_{3,1})$ are in the interiors of $I_1 \cap I_2, I_2 \cap I_3, I_3 \cap I_1$ respectively. Then, we can find (commuting) elements g_1, g_2, g_3 of $\mathrm{Homeo}_+(\mathbf{S}^1)$ with supports in $I_1 \cap I_2, I_2 \cap I_3, I_3 \cap I_1$ respectively, agreeing with f in neighbourhoods of $x_{1,2}, x_{2,3}, x_{3,1}$. Hence $g_1^{-1}g_2^{-1}g_3^{-1}f$ is the identity in neighbourhoods of $x_{1,2}, x_{2,3}, x_{3,1}$ and is therefore a product of three elements of $G_1 \cup G_2 \cup G_3$. This shows that every element of $\mathrm{Homeo}_+(\mathbf{S}^1)$ close enough to the identity is an element of G. The general case follows from the well known fact that a connected topological group is generated by any neighbourhood of the identity. □

As a corollary of Proposition 4.2, the fundamental group of $\mathrm{Homeo}_+(\mathbf{S}^1)$ is $\mathbf{Z}$ so that for each integer $k \geq 1$ there is a unique connected covering space $\mathrm{Homeo}_{k,+}(\mathbf{S}^1)$ of $\mathrm{Homeo}_+(\mathbf{S}^1)$ with k sheets. In the same way, there is a unique connected covering space $\mathrm{PSL}_k(2,\mathbf{R})$ of $\mathrm{PSL}(2,\mathbf{R})$ with k sheets. It is easy to construct these coverings explicitly. Consider a k-fold cover of the circle onto itself. Any element of $\mathrm{Homeo}_+(\mathbf{S}^1)$ can be lifted to exactly k homeomorphisms of the circle and $\mathrm{Homeo}_{k,+}(\mathbf{S}^1)$ consists of the collection of all the lifts of all homeomorphisms. Another way of expressing the same thing is the following. Consider the finite cyclic group of rotations of order k acting on the circle $\mathbf{R}/\mathbf{Z}$. Then we can consider the subgroup of $\mathrm{Homeo}_+(\mathbf{S}^1)$

consisting of homeomorphisms commuting with this cyclic group: this is a group isomorphic to $\mathrm{Homeo}_{k,+}(\mathbf{S}^1)$. This presentation has the advantage of expressing $\mathrm{Homeo}_{k,+}(\mathbf{S}^1)$ as a subgroup of $\mathrm{Homeo}_+(\mathbf{S}^1)$. Analogously, we can realize $\mathrm{PSL}_k(2,\mathbf{R})$ as the group of lifts of elements of $\mathrm{PSL}(2,\mathbf{R})$ to the k-fold cover of $\mathbf{RP}^1$. This k-fold cover is homeomorphic to a circle so that $\mathrm{PSL}_k(2,\mathbf{R})$ can be realized as a subgroup of $\mathrm{Homeo}_{k,+}(\mathbf{S}^1)$ (of course up to conjugacy). Note however that $\mathrm{PSL}_k(2,\mathbf{R})$ cannot be realized as a subgroup of $\mathrm{PSL}(2,\mathbf{R})$.

Summing up, for each integer $k \geq 1$, we have well defined conjugacy classes of subgroups $\mathrm{PSL}_k(2,\mathbf{R})$ and $\mathrm{Homeo}_{k,+}(\mathbf{S}^1)$ in $\mathrm{Homeo}_+(\mathbf{S}^1)$. The first ones are finite dimensional and the second ones are very close to the full group of homeomorphisms, something like "finite codimension subgroups".

PROBLEM 4.4. *Let Γ be a closed subgroup of* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *acting transitively on the circle. Is Γ conjugate to one of the subgroups* $\mathrm{SO}(2,\mathbf{R})$, $\mathrm{PSL}_k(2,\mathbf{R})$ *or* $\mathrm{Homeo}_{k,+}(\mathbf{S}^1)$ *?*

Informally, this problem asks whether there is an interesting geometry on the circle besides projective geometry. For instance, the analogous question for the group of homeomorphisms of the 2-sphere would have a negative answer: besides finite dimensional Lie groups acting on the 2-sphere, there is the group of area preserving homeomorphisms which acts transitively and is "much smaller" than the full group of homeomorphisms of the 2-sphere.

Let F_2 be the free group on two generators. It is very easy to construct explicit examples of embeddings of F_2 in $\mathrm{SL}(2,\mathbf{R})$ (see for instance [31]). It follows that for a generic choice of two elements of $\mathrm{SL}(2,\mathbf{R})$, the subgroup that they generate is free. Indeed, let f and g be two elements in $\mathrm{SL}(2,\mathbf{R})$. There is a homomorphism i from F_2 to $\mathrm{SL}(2,\mathbf{R})$ sending the first and the second generator to f and g respectively. In practice, if w is a non trivial element of F_2 seen as a word in the two generators and their inverses, $i(w) = w(f,g)$ is obtained by substituting f and g to the two generators of F_2 in w. Let $X_w \subset \mathrm{SL}(2,\mathbf{R}) \times \mathrm{SL}(2,\mathbf{R})$ be the set of (f,g) such that $w(f,g) = \mathrm{Id}$. This is a real algebraic subset of the algebraic irreducible affine variety $\mathrm{SL}(2,\mathbf{R}) \times \mathrm{SL}(2,\mathbf{R})$ which is not everything since otherwise $\mathrm{SL}(2,\mathbf{R})$ would not contain any free subgroup on two generators. Therefore the set of couples (f,g) which generate a free subgroup is the complement of a countable union of proper algebraic submanifolds of $\mathrm{SL}(2,\mathbf{R}) \times \mathrm{SL}(2,\mathbf{R})$. Hence for a generic choice of (f,g) (in the sense of Baire), the group generated by (f,g) is free.

We now prove the analogous statement for the group of homeomorphisms (which is not an algebraic group).

PROPOSITION 4.5. *For a generic set of pairs* (f, g) *of elements of* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *(in the sense of Baire), the group generated by* (f, g) *is a free group on two generators.*

Proof. First observe that the topology of $\mathrm{Homeo}_+(\mathbf{S}^1)$ can be defined by a complete metric. It follows that $\mathrm{Homeo}_+(\mathbf{S}^1) \times \mathrm{Homeo}_+(\mathbf{S}^1) \times \mathbf{S}^1$ is a Baire space, *i.e.* a countable intersection of dense open sets is dense. For each non trivial $w \in F_2$, consider the closed set $X_w \subset \mathrm{Homeo}_+(\mathbf{S}^1) \times \mathrm{Homeo}_+(\mathbf{S}^1) \times \mathbf{S}^1$ consisting of those (f, g, x) such that $w(f, g)(x) = x$. We shall show that all these closed sets have empty interior. It will follow in particular that for each non trivial word w, the set of (f, g) such that $w(f, g) = id$ has empty interior so that, by Baire's theorem, for a generic choice of (f, g) in $\mathrm{Homeo}_+(\mathbf{S}^1) \times \mathrm{Homeo}_+(\mathbf{S}^1)$ there is no non trivial relation of the form $w(f, g) = id$ and the group generated by f and g is indeed free.

Assume by contradiction that some X_w has non empty interior and let w be a word of minimal length k for which this is the case (note that $k > 1$). Let U be some non empty open set of $\mathrm{Homeo}_+(\mathbf{S}^1) \times \mathrm{Homeo}_+(\mathbf{S}^1) \times \mathbf{S}^1$ contained in X_w. For each pair of words w_1, w_2 of length strictly less than k, consider the closed set of triples (f, g, x) such that $w_1(f, g) w_2(f, g)(x) = w_2(f, g)(x)$: this is the image of X_{w_1} by $(f, g, x) \mapsto (f, g, w_2^{-1}(f, g)(x))$ and therefore has an empty interior. Choose a triple (f, g, x) which is in U but not in these (finitely many) closed sets with empty interiors. Write $w = a_1.a_2. \cdots .a_k$ where each a_i is one of the generators or its inverse. Write $w(f, g)$ as $f_1 f_2 \cdots f_k$ where each f_i is one of f, f^{-1}, g, g^{-1}.

Finally, consider the sequence of points $x_1, \ldots, x_{k-1}$ defined by $x_1 = f_1(x)$, $x_2 = f_2(x_1)$, $\ldots$, $x_{k-1} = f_{k-1}(x_{k-2})$. Since we know that $(f, g, x) \in U$, we have $f_k(x_{k-1}) = x$. We claim that the points $x, x_1, \ldots, x_{k-1}$ are different. Indeed, the contrary would mean that some word w_1 of length strictly less than k would fix one of the points x_i. Since each point x_i has the form $w_2(f, g)(x)$ for some w_2 of length strictly less than k, the triple (f, g, x) would be in one of these closed sets with empty interior that we excluded.

We slightly modify (f, g) in $(\bar{f}, \bar{g})$ in such a way that the corresponding $\bar{f}_1, \ldots, \bar{f}_k$ still satisfy $x_1 = \bar{f}_1(x), x_2 = \bar{f}_2(x_1), \ldots, x_{k-1} = \bar{f}_{k-1}(x_{k-2})$ but also such that $\bar{f}_k(x_{k-1}) \neq x$. This is possible since x is different from $x_1, \ldots, x_{k-1}$. It follows that $w(\bar{f}, \bar{g})(x) \neq x$. This contradicts the definition of U. □

Note that this proof works equally well for the homeomorphism group of manifolds of any dimension.

Brin and Squier have discovered the remarkable fact that the situation is completely different in groups of piecewise linear homeomorphisms [10].

THEOREM 4.6 (Brin-Squier). *The group* $\mathrm{PL}_+([0,1])$ *of piecewise linear homeomorphisms of the interval* $[0,1]$ *does not contain any non abelian free subgroup.*

Proof. If f is any homeomorphism, we denote by $Supp_0(f)$ its "open support", *i.e.* the set of non fixed points. Suppose by contradiction that there exist two elements f and g of $\mathrm{PL}_+([0,1])$ which generate a free subgroup F_2 on the generators f and g. The union $I = Supp_0(f) \cup Supp_0(g)$ is a union of a finite number of open intervals $I_1, \ldots, I_n$. Note that the commutator $fgf^{-1}g^{-1}$ has an open support whose closure is contained in I since near the boundary of I, the maps f and g are linear and therefore commute.

Among the non trivial elements h in F_2 such that the closure of $Supp_0(h)$ is contained in I, consider an element h such that $Supp_0(h)$ meets the least possible number of the n components of I. Let $]a,b[$ be one of these components and let $[c,d]$ be a interval contained in the interior of $]a,b[$ and containing $Supp_0(h) \cap]a,b[$. If x is in $]a,b[$ then the orbit of x under the group F_2 is contained in $]a,b[$ and its upper bound is a common fixed point of f and g so that it has to be b. It follows that there exists an element l in the group sending c (and hence $[c,d]$) to the right of d. In particular the restrictions to $[a,b]$ of h and lhl^{-1} commute and generate a group isomorphic to $\mathbf{Z}^2$. Of course h and lhl^{-1} don't commute in the free group generated by f and g since otherwise they would generate a group isomorphic to $\mathbf{Z}$. Consider now the commutator of h and lhl^{-1}. It is a non trivial element whose support does not intersect $]a,b[$ and therefore intersects strictly fewer components of I than h did. This contradicts our choice of h. □

Finding a group which does not contain any non abelian free subgroups is not very difficult: consider for example an abelian group! However, the interesting feature of $\mathrm{PL}_+([0,1])$ is that it contains no non abelian free subgroups *and satisfies no law.* This means that for every non trivial word w, we can find two elements (f,g) in $\mathrm{PL}_+([0,1])$ such that $w(f,g) \neq \mathrm{Id}$ (this is not difficult: see [10]). Abelian groups, on the contrary, satisfy the law that $fgf^{-1}g^{-1}$ is always the identity element.

Remark also that the proposition is *not* claiming that the group $\mathrm{PL}_+(\mathbf{S}^1)$

does not contain any non abelian free subgroups. Indeed, it is very easy to find free subgroups on two generators in $\mathrm{PL}_+(\mathbf{S}^1)$ using for instance the classical "Klein ping-pong lemma" (see [31] or Section 5.2). Later in this paper, we shall prove that "most subgroups" of $\mathrm{Homeo}_+(\mathbf{S}^1)$ contain free subgroups (5.14).

4.1 Locally compact groups acting on the circle

Recall that a very important (and difficult) theorem of Montgomery and Zippin states that a locally compact group is a Lie group if and only if there is a neighbourhood of the identity which does not contain a non trivial compact subgroup [40, 56]. We know the structure of compact subgroups of $\mathrm{Homeo}_+(\mathbf{S}^1)$: they are conjugate to subgroups of $\mathrm{SO}(2,\mathbf{R})$ and therefore they are either finite cyclic groups or conjugate to $\mathrm{SO}(2,\mathbf{R})$. None of these subgroups can be in a small neighbourhood of the identity. Indeed, consider the neighbourhood U of the identity in $\mathrm{Homeo}_+(\mathbf{R}/\mathbf{Z})$ consisting of those homeomorphisms f such that the distance between x and $f(x)$ is less than $1/3$ for all x in $\mathbf{R}/\mathbf{Z}$. Every element f in U has a unique lift $\widetilde{f}$ in $\widetilde{\mathrm{Homeo}}_+(\mathbf{R}/\mathbf{Z})$ which is such that $|\widetilde{f}(\widetilde{x})-\widetilde{x}| \leq 1/3$ for all $\widetilde{x}$ in $\mathbf{R}$. Of course, if f, g and fg are in U, we have $\widetilde{fg} = \widetilde{f}\widetilde{g}$. In particular, if there were a non trivial subgroup H contained in U this subgroup H would lift as a subgroup of $\widetilde{\mathrm{Homeo}}_+(\mathbf{R}/\mathbf{Z})$. Since $\widetilde{\mathrm{Homeo}}_+(\mathbf{R}/\mathbf{Z})$ is a torsion free group and since any compact subgroup of $\mathrm{Homeo}_+(\mathbf{R}/\mathbf{Z})$ contains elements of finite order, it follows that no non trivial compact subgroup of $\mathrm{Homeo}_+(\mathbf{R}/\mathbf{Z})$ can lift to $\widetilde{\mathrm{Homeo}}_+(\mathbf{R}/\mathbf{Z})$. In particular U contains no non trivial compact subgroup. We deduce:

THEOREM 4.7. *A locally compact subgroup of* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *is a Lie group.*

It would be interesting to prove this theorem by elementary means, *i.e.* without the use of the Montgomery-Zippin theorem.

Consider a *connected Lie group G acting continuously and faithfully* on the circle by a homomorphism $\phi\colon G \to \mathrm{Homeo}_+(\mathbf{S}^1)$. Our objective is to determine all such actions. Orbits of the action are connected, so they can be of three kinds: the full circle, an open interval or a point. In other words, there is a closed set $F \subset \mathbf{S}^1$ (which might be empty) consisting of fixed points for the action, and the orbits which are not fixed points are the connected components of $\mathbf{S}^1 - F$. So, in order to understand the action, it is basically sufficient to understand it on each 1-dimensional orbit (homeomorphic to $\mathbf{R}$ or $\mathbf{S}^1$). Note that the action of G on one orbit is not necessarily faithful anymore but, taking the quotient by the kernel, we are led to study transitive and faithful actions of a connected Lie group G on $\mathbf{R}$ or $\mathbf{S}^1$.

Denote by H the stabilizer of a point in such an orbit. This is a closed subgroup of G, hence a Lie subgroup of codimension 1 and G acts smoothly on the 1-dimensional manifold G/H. The Lie algebra $\mathfrak{G}$ will therefore induce a finite dimensional Lie algebra of smooth vector fields on G/H. Since G acts transitively on G/H, for any point on G/H there is an element of this Lie algebra which does not vanish at this point.

Consider the case of the projective action of $\mathrm{PSL}(2,\mathbf{R})$ on $\mathbf{RP}^1$. The Lie algebra $\mathfrak{sl}(2,\mathbf{R})$ is the algebra of 2×2 real matrices with trace 0. Taking the differential of the action at the identity, one easily checks that the corresponding Lie algebra of vector fields is the algebra of vector fields of the form $u(x)\partial/\partial x$ where u is a polynomial of degree at most 2; thus we get the following identification of algebras:

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}(2,\mathbf{R}) \mapsto (b + 2ax - cx^2)\frac{\partial}{\partial x}\,.$$

Denote by $\mathfrak{Vect}$ the Lie algebra of germs of smooth vector fields of $\mathbf{R}$ in the neighbourhood of 0. The subspace $\mathfrak{Vect}_k$ of vector fields $u(x)\partial/\partial x$ where u vanishes at the origin together with its first k derivatives is an ideal in $\mathfrak{Vect}$ and the quotient Lie algebra $\mathfrak{Vect}/\mathfrak{Vect}_k$ can be identified, as a vector space with the space $\mathfrak{P}_k$ of vector fields of the form $u(x)\partial/\partial x$ where u is a polynomial of degree at most k.

Note however that $\mathfrak{P}_k$ is a subalgebra of $\mathfrak{Vect}$ if and only if $k = 0, 1$ or 2. One can therefore think of $\mathfrak{P}_0, \mathfrak{P}_1, \mathfrak{P}_2$ at the same time as subalgebras of $\mathfrak{Vect}$ and as quotient algebras of $\mathfrak{Vect}$.

The general situation was analyzed a long time ago by Lie, who found all the possibilities [45]:

THEOREM 4.8 (Lie). *Let $\mathfrak{G}$ be a non trivial finite dimensional Lie algebra consisting of germs of smooth vectors fields in the neighbourhood of* 0 *in* $\mathbf{R}$. *Assume that not all these vector fields vanish at the origin. Then the dimension of $\mathfrak{G}$ is at most 3. More precisely, in suitable coordinates $\mathfrak{G}$ consists of all germs of the form $u(x)\partial/\partial x$ where u is a polynomial of degree less than or equal to k for $k = 0, 1$ or 2.*

Proof. Since one element of $\mathfrak{G}$ does not vanish at the origin, we can find a suitable local coordinate x such that the germ of this element is $\partial/\partial x$. Let $\mathcal{E}$ be the finite dimensional vector space of germs of functions u such that $u(x)\partial/\partial x$ belongs to $\mathfrak{G}$. Of course $\mathcal{E}$ contains the constants and is stable under the operation of taking derivatives, since the bracket $[\partial/\partial x, u(x)\partial/\partial x]$ equals $u'(x)\partial/\partial x$. The successive iterates of the linear operator induced by the derivative must be linearly dependent. This shows that there exists a linear differential equation with constant coefficients which is satisfied by all elements in $\mathcal{E}$. It follows that all elements in $\mathcal{E}$ are real analytic functions. Every non trivial element u of $\mathcal{E}$ therefore has a convergent Taylor expansion of the form $u(x) = a_i x^i + \dots$ with $a_i \neq 0$. Moreover, this integer i is bounded since a solution of a linear differential equation with constant coefficients which vanishes at a point together with its derivatives of orders up to the degree of the equation has to vanish identically. Choose an element u for which the integer i is maximal. Now the algebra $\mathfrak{G}$ contains $[u(x)\partial/\partial x, u'(x)\partial/\partial x] = a_i(x^{2i-2} + \dots)\partial/\partial x$. It follows that $2i-2 \leq i$, so that $i \leq 2$.

For each element of $\mathfrak{G}$, consider the Taylor expansion of degree 2 of the associated vector field, considered as an element of $\mathfrak{P}_2 \simeq \mathfrak{sl}(2,\mathbf{R})$. This produces a linear map $j_2 \colon \mathfrak{G} \to \mathfrak{P}_2$ which is clearly an algebra homomorphism and which is injective by the previous argument.

If the image of j_2 is 1-dimensional, then $\mathfrak{G}$ consists only of constant multiples of $\partial/\partial x$. In this case, G is (locally) isomorphic to $\mathbf{R}$ and the Lie algebra of H is trivial, which means that H is discrete.

Suppose that the image of j_2 is 3-dimensional, *i.e.* that j_2 is an isomorphism. Consider the element $X = \partial/\partial x$ of $\mathfrak{P}_2$. Note that the linear operator $ad^3(X) \colon \mathfrak{P}_2 \to \mathfrak{P}_2$ is trivial. The vector field $j_2^{-1}(X)$ does not vanish at the origin so that we could have used it at the beginning when we chose a local coordinate x. In other words, there is a local coordinate x such that $\partial/\partial x$ belongs to $\mathfrak{G}$ and such that the linear operator induced by taking bracket with $\partial/\partial x$ is nilpotent of order 3. This means that the third derivative of every element of $\mathcal{E}$ vanishes. In suitable coordinates $\mathfrak{G}$ coincides with polynomial vector fields of degree at most 2. In this case, G is locally isomorphic to $\mathrm{SL}(2,\mathbf{R})$ and H is locally isomorphic to the group of upper triangular matrices.

Suppose finally that the image of j_2 is 2-dimensional. This means that the Taylor expansion of order 1 is an isomorphism $j_1 \colon \mathfrak{G} \to \mathfrak{P}_1$ and one can reproduce the above proof with the nilpotent operator of order 2 induced by $\partial/\partial x$. In this case, G is locally isomorphic to the 2-dimensional group of upper triangular matrices in $\mathrm{SL}(2,\mathbf{R})$ and H is locally isomorphic to the 1-dimensional subgroup of unipotent matrices. □

This theorem gives a complete *local* description of transitive actions of a Lie group. It is not difficult to deduce the complete classification of transitive and faithful actions of connected Lie groups on 1-manifolds. Up to conjugacy, the list is the following.

- The action of $\mathbf{R}$ on itself.
- The action of the circle $\mathbf{R}/\lambda\mathbf{Z}$ on itself (for $\lambda > 0$).
- The action of the affine group $\mathrm{Aff}_+(\mathbf{R})$ on $\mathbf{R}$.
- The action of the k-fold cover $\mathrm{PSL}_k(2,\mathbf{R})$ of $\mathrm{PSL}(2,\mathbf{R})$ on the circle, described in Section 4 (for $k \geq 1$).
- The action of the universal cover $\widetilde{\mathrm{SL}}(2,\mathbf{R})$ of $\mathrm{SL}(2,\mathbf{R})$ on the universal cover of $\mathbf{S}^1$.

Loosely speaking, we could say that there are three geometries of finite type on 1-manifolds: euclidean, affine and projective.

The full description of faithful non transitive actions of a connected Lie group G on the circle is now easy in principle. We should choose a closed set $F \subset \mathbf{S}^1$ consisting of fixed points and for each connected component I of the complement of F, the action is described by some surjection from G to $\mathbf{R}$, $\mathrm{Aff}_+(\mathbf{R})$, $\mathrm{PSL}_k(2,\mathbf{R})$ or $\widetilde{\mathrm{SL}}(2,\mathbf{R})$.

As a trivial example, we get the description of *topological flows on the circle*, *i.e.* of actions of $\mathbf{R}$ on the circle. If such a flow is transitive, it is conjugate to the action of $\mathbf{R}$ on the circle $\mathbf{R}/\lambda\mathbf{Z}$ for some $\lambda > 0$ (the "period" of the flow). If it is not transitive, it has a non empty set of fixed points $F \subset \mathbf{S}^1$ and the conjugacy class is completely described by the orientation: on each component of $\mathbf{S}^1 - F$, the flow is positive or negative.

Finally, we should describe the actions of non connected Lie groups G. Let G_0 be the connected component of the identity in G so that we already understand the action of G_0. Observe that G_0 is a normal subgroup of G so that the action of G preserves F and permutes the connected components of $\mathbf{S}^1 - F$. It is not easy to fully analyze this situation but it is quite clear that when G_0 is non trivial, its normalizer is usually very small. We leave to the reader the details of this analysis. Of course, when G_0 is trivial, *i.e.* when G is discrete, the previous discussion has no content. Hence among locally compact groups acting on the circle, the most interesting ones are the discrete groups.

5. Rotation numbers

5.1 Dynamics of a single homeomorphism

The main invariant of homeomorphisms of the circle was introduced by H. Poincaré (it is still very interesting to read [59]).

Let us start with an element $\widetilde{f}$ of $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$, *i.e.* a homeomorphism of $\mathbf{R}$ which commutes with integral translations. Observe that if two points x, x' in $\mathbf{R}$ differ by at most 1, the same is true for their images by $\widetilde{f}$. It follows that for any two points x, x', the two numbers $\widetilde{f}(x) - x$ and $\widetilde{f}(x') - x'$ differ by at most 1. Let us define $T(\widetilde{f}) = \widetilde{f}(0) = \widetilde{f}(0) - 0$. If $\widetilde{f}_1$ and $\widetilde{f}_2$ are two elements of $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$, we have $T(\widetilde{f}_1\widetilde{f}_2) = (\widetilde{f}_1(\widetilde{f}_2(0)) - \widetilde{f}_2(0)) + (\widetilde{f}_2(0) - 0)$ so that $|T(\widetilde{f}_1\widetilde{f}_2) - T(\widetilde{f}_1) - T(\widetilde{f}_2)|$ is bounded by 1. Let us formalize this notion:

Definition 5.1. Let Γ be a group. A quasi-homomorphism from Γ to $\mathbf{R}$ is a map $F\colon \Gamma \to \mathbf{R}$ such that there is a constant D such that for every γ_1, γ_2 in Γ we have $|F(\gamma_1\gamma_2) - F(\gamma_1) - F(\gamma_2)| \leq D$.

The following is an easy exercise left to the reader.

Lemma 5.2. *Let $F\colon \mathbf{Z} \to \mathbf{R}$ be a quasi-homomorphism. Then, there exists a unique real number τ such that the sequence $F(n) - n\tau$ is bounded.*

As we shall see later, this lemma is far from being true if we replace the group $\mathbf{Z}$ by a more general group Γ.

Let us restrict the quasi-homomorphism T to the group generated by a homeomorphism $\widetilde{f}$, *i.e.* let us consider the sequence $T\big(\widetilde{f}^{\,n}\big)$. According to the lemma, there is a unique number $\tau(\widetilde{f})$ such that $T\big(\widetilde{f}^{\,n}\big) - n\tau$ is bounded. This number $\tau(\widetilde{f})$ is by definition the *translation number* of $\widetilde{f}$. It follows from the definition that if we compose $\widetilde{f}$ with an integral translation, the translation number increases by an integer. If we consider an element f in $\mathrm{Homeo}_+(\mathbf{S}^1)$, the translations numbers of its lifts in $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ differ by integers so that the element $\rho(f) = \tau(\widetilde{f}) \mod \mathbf{Z} \in \mathbf{R}/\mathbf{Z}$ is well defined. This is called the *rotation number* of the homeomorphism f.

These definitions show that τ is a quasi-homomorphism from $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ to $\mathbf{R}$ and that it has been "normalized" so that it is a homomorphism on each one generator subgroup, *i.e.* we have $\tau\big(\widetilde{f}^{\,n}\big) = n\tau(\widetilde{f})$ for every $\widetilde{f}$ and every integer n.

Of course, it is an easy matter to check that the translation number of the translation by τ in $\mathbf{R}$ is τ and that the rotation number of the rotation $x \in \mathbf{R}/\mathbf{Z} \mapsto x+\rho \in \mathbf{R}/\mathbf{Z}$ of "angle" ρ on the circle is indeed ρ as it should be !

The next proposition is easy but is a justification for introducing these numbers.

PROPOSITION 5.3. *The translation number and the rotation number are invariant under conjugation in* $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ *and* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *respectively.*

Proof. This follows formally from the fact that τ is a quasi-homomorphism and is a homomorphism on one generator groups. Indeed,

$$\tau\big(\widetilde{f}^{\,n}\big) = n\tau(\widetilde{f})$$

and

$$\tau\big(\widetilde{h}\widetilde{f}^{\,n}\widetilde{h}^{-1}\big) = \tau\big((\widetilde{h}\widetilde{f}\widetilde{h}^{-1})^n\big) = n\tau(\widetilde{h}\widetilde{f}\widetilde{h}^{-1})$$

differ by a bounded amount, independent of n, so that they must be equal. This shows that the translation number is a conjugacy invariant. The assertion concerning the rotation number follows immediately. □

Let us give some universal characterization of the translation number.

PROPOSITION 5.4 ([4]). *The translation number is the unique quasi-homomorphism* $\tau\colon \widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1) \to \mathbf{R}$ *which is a homomorphism when restricted to one generator groups and which takes the value* 1 *on the translation by* 1.

Proof. An easy generalization of Lemma 5.2 shows that any quasi-homomorphism $\mathbf{Z}^2 \to \mathbf{R}$ differs from a homomorphism $\mathbf{Z}^2 \to \mathbf{R}$ by a bounded amount. This implies that if a quasi-homomorphism $\mathbf{Z}^2 \to \mathbf{R}$ is a homomorphism when restricted to one generator groups, it is a homomorphism $\mathbf{Z}^2 \to \mathbf{R}$ (note that a bounded homomorphism is necessarily trivial).

Let t be another quasi-homomorphism satisfying the conditions of the proposition. It follows from our first observation that t is a homomorphism when restricted to the (commutative) group generated by one element $\widetilde{f}$ and the integral translations. Consider now the difference $r = \tau - t$. Its value on an element $\widetilde{f}$ depends only on the projection of $\widetilde{f}$ in $\mathrm{Homeo}_+(\mathbf{S}^1)$ so that we get a quasi-homomorphism $\overline{r}\colon \mathrm{Homeo}_+(\mathbf{S}^1) \to \mathbf{R}$ which is a homomorphism on one generator groups. We claim that $\overline{r}$ must be trivial. This will follow from a property of $\mathrm{Homeo}_+(\mathbf{S}^1)$ that we shall prove later (see 5.11): any homeomorphism f in $\mathrm{Homeo}_+(\mathbf{S}^1)$ can be written as a commutator $[f_1,f_2] = f_1 f_2 f_1^{-1} f_2^{-1}$. (In fact we only prove in 5.11 that any homeomorphism is a product of *two* commutators but this is enough for the proof which follows.) Assuming this result, we see that any quasi-homomorphism from $\mathrm{Homeo}_+(\mathbf{S}^1)$ has to be bounded. Indeed, up to a bounded amount, the value of the quasi-homomorphism $\bar{r}$ on $f = [f_1,f_2]$ is equal to the sum of its values on $f_1,f_2,f_1^{-1},f_2^{-1}$ which is bounded (since $\bar{r}(f_1)+\bar{r}(f_1^{-1})$ is bounded). Now, a bounded quasi-homomorphism which is a homomorphism on one generator groups is trivial so that $\bar{r}$ is zero. □

We mention a very interesting problem coming from [37]:

PROBLEM 5.5 (Jankins-Neumann). *Let $\mathcal{R} \subset (\mathbf{R}/\mathbf{Z})^3$ be the set of triples (ρ_1,ρ_2,ρ_3) such that there exist three elements f_1,f_2,f_3 of* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *such that $f_1 f_2 f_3 = \mathrm{Id}$ and whose rotation numbers are (ρ_1,ρ_2,ρ_3). Can one describe this set $\mathcal{R}$ explicitly?*

In [37], the authors show that $\mathcal{R}$ has a fractal structure. First, they explicitly describe the set $\mathcal{R}_0 \subset (\mathbf{R}/\mathbf{Z})^3$ of triples (ρ_1,ρ_2,ρ_3) such that there exist three elements f_1,f_2,f_3 of some $\mathrm{PSL}_k(2,\mathbf{R})$ such that $f_1 f_2 f_3 = \mathrm{Id}$ and whose rotation numbers are (ρ_1,ρ_2,ρ_3). Of course, $\mathcal{R}_0 \subset \mathcal{R}$ and they conjecture that these two sets are equal. As a motivation for their conjecture, they find an explicit set $\mathcal{R}_1$ such that $\mathcal{R}_0 \subset \mathcal{R} \subset \mathcal{R}_1$ and such that $\mathcal{R}_1 - \mathcal{R}_0$ is "small": the Lebesgue measure of $\mathcal{R}_1 - \mathcal{R}_0$ is indeed $0.0010547\ldots$ and the Lebesgue measure of $\mathcal{R}_0$ is $25/8+3\zeta(2)+3\zeta(3)-6\zeta(2)\zeta(3)/\zeta(5) \simeq 0.224649208402\ldots$ (where ζ is the Riemann ζ-function). As Jankins and Neumann write, their conjecture is therefore 99.9 % proved!

We shall show that the "number" $\rho(f)$ contains a lot of information on the topological dynamics of f. Let us begin by explaining the main possibilities for the dynamics of an arbitrary group of homeomorphisms.

PROPOSITION 5.6. *Let Γ be any subgroup of* $\mathrm{Homeo}_+(\mathbf{S}^1)$. *Then there are three mutually exclusive possibilities.*

1) *There is a finite orbit.*

2) *All orbits are dense.*

3) *There is a compact Γ-invariant subset $K \subset \mathbf{S}^1$ which is infinite and different from $\mathbf{S}^1$ and such that the orbits of points in K are dense in K. This set K is unique, contained in the closure of any orbit and is homeomorphic to a Cantor set.*

Proof. Let us consider the collection of compact sets in $\mathbf{S}^1$ which are non empty and Γ-invariant, ordered by inclusion. By Zorn's lemma, there is a minimal set in this collection. Choose such a *minimal set* K. Note that the closure of the orbit of any point in K is a closed non empty Γ-invariant set contained in K so that it must coincide with K by minimality: the orbit of a point in K is dense in K. Observe now that the topological boundary $\partial K = K - interior(K)$ and the set K' of accumulation points of K are closed and Γ-invariant. Hence, we have the following possibilities.

1) K' is empty. In this case, K is finite and we found a finite orbit.

2) ∂K is empty, so that K is the full circle. In this case, all orbits are dense.

3) $K' = K$ and $\partial K = K$, so that K is a compact perfect set in the circle with empty interior: this is one definition of a Cantor set.

In order to prove the uniqueness of K in the last case, we show that K is contained in the closure of any orbit. The complement of K in the circle is a disjoint union of a countable family of open intervals. Let x be a point in the complement of K, lying in some interval I and let a be the origin of I (note that I is oriented). Finally, let y be any point in K. Since we know that the orbit of any point of K is dense in K and that K has no isolated point, there is a sequence of elements γ_n such that $\gamma_n(a)$ consists of distinct points and converges to y. The intervals $\gamma_n(I)$ are therefore disjoint so that the distance between $\gamma_n(a)$ and $\gamma_n(x)$ converges to zero. It follows that $\gamma_n(x)$ converges to y. This proves that K is contained in the closure of every orbit and the uniqueness of the minimal set K follows immediately. □

Case 3 looks strange at first sight: it is called the *exceptional minimal set* case for this reason. We reduce it to case 2, using the notion of *semi-conjugacy*. Consider a map $\widetilde{h}$ from $\mathbf{R}$ to $\mathbf{R}$ which is continuous, increasing (if $x \leq y$ then $\widetilde{h}(x) \leq \widetilde{h}(y)$) and which commutes with integral translations. We stress the fact that $\widetilde{h}$ might be non injective: typically it might be constant on some intervals. Such a map defines a map h from the circle to itself. We call such a map an *increasing continuous map of degree* 1 *from the circle to itself.*

DEFINITION 5.7. Let Γ be a group and ϕ_1, ϕ_2 be two homomorphisms from Γ to $\mathrm{Homeo}_+(\mathbf{S}^1)$. We say that ϕ_1 is *semi-conjugate* to ϕ_2 if there is an increasing continuous map h of degree 1 from the circle to itself such that for every γ in Γ, we have $\phi_2(\gamma)h = h\phi_1(\gamma)$.

Observe that this notion is not symmetric: ϕ_2 is not necessarily semi-conjugate to ϕ_1.

PROPOSITION 5.8. *Let Γ be a group and ϕ be a homomorphism from Γ to* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *such that $\phi(\Gamma)$ has an exceptional minimal set K. Then there is a homomorphism $\overline{\phi}$ from Γ to* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *such that ϕ is semi-conjugate to $\overline{\phi}$ and $\overline{\phi}(\Gamma)$ has dense orbits on the circle.*

Proof. The complement of K in the circle is a countable union of open intervals. For each of these intervals, collapse its closure to a point. The resulting quotient space is homeomorphic to a circle. In other words, there is an increasing continuous map h of degree 1 from the circle to itself such that $h(K) = \mathbf{S}^1$ and such that the fibers $h^{-1}(x)$ are either points or the closed intervals which are the closures of the connected components of the complement of K. Since $\phi(\Gamma)$ acts on the circle and preserves K, it also acts on the "collapsed" circle so that we can define another homomorphism $\overline{\phi}$ which satisfies the conditions of the proposition (we know that orbits of points in K are dense in K). □

The main object of these notes is to discuss the dynamics of "big groups" Γ acting on the circle. However, we first restrict ourselves to the "easy" case where Γ is generated by one element so that we really study the dynamics of one homeomorphism of the circle. Of course, we allow ourselves to say that a homeomorphism f_1 is semi-conjugate to f_2 if the corresponding homomorphisms from $\mathbf{Z}$ to $\mathrm{Homeo}_+(\mathbf{S}^1)$ are semi-conjugate. The following result shows that the rotation number of a homeomorphism contains a lot of information on the dynamics.

THEOREM 5.9 (Poincaré). *Let f be an element of* $\mathrm{Homeo}_+(\mathbf{S}^1)$. *Then f has a periodic orbit if and only if the rotation number $\rho(f)$ is rational, i.e. belongs to $\mathbf{Q}/\mathbf{Z}$. If the rotation number $\rho(f)$ is irrational, then f is semi-conjugate to the rotation on the circle of angle $\rho(f) \in \mathbf{R}/\mathbf{Z}$. This semi-conjugacy is actually a conjugacy if the orbits of f are dense.*

Proof. Choose a lift $\widetilde{f}$ of f in $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$. We know that the numbers $\widetilde{f}^n(x) - n\tau(\widetilde{f})$ are uniformly bounded (independently of $n \in \mathbf{Z}$). Define $\widetilde{h}(x) = \sup_n \big(\widetilde{f}^n(x) - n\tau(\widetilde{f})\big)$. The following properties of $\widetilde{h}$ are obvious:

1) $\widetilde{h}$ is increasing (it is left continuous but not necessarily continuous).
2) $\widetilde{h}(x+1) = \widetilde{h}(x) + 1$.
3) $\widetilde{h}(\widetilde{f}(x)) = \widetilde{h}(x) + \tau(\widetilde{f})$.

If $\widetilde{h}$ were continuous, that would lead to a semi-conjugacy between f and the rotation by an angle $\tau(\widetilde{f})$ mod $\mathbf{Z} = \rho(f) \in \mathbf{R}/\mathbf{Z}$. The structure of an increasing function like $\widetilde{h}$ from $\mathbf{R}$ to $\mathbf{R}$ is not difficult to analyze. First, the fibers $\widetilde{h}^{-1}(x)$ are either points or intervals. There is at most a countable number of these intervals: call the union of the interior of these intervals the plateau set $Plat(\widetilde{h})$ of $\widetilde{h}$; it is empty if and only if $\widetilde{h}$ is injective. Second, the image $\widetilde{h}(\mathbf{R})$ is the complement of the union of at most countably many disjoint intervals: call the union of the interior of these intervals the jump set $Jump(\widetilde{h})$ of $\widetilde{h}$; it is empty if and only if $\widetilde{h}$ is continuous and onto. In our situation it is clear from 2 that $Plat(\widetilde{h})$ and $Jump(\widetilde{h})$ are open sets which are invariant under integral translations, so that they define open sets in the circle. Moreover, property 3 shows that $Plat(\widetilde{h})$ is invariant under $\widetilde{f}$ and $Jump(\widetilde{h})$ is invariant under the translation by $\tau(\widetilde{f})$ so that the corresponding open sets in the circle are invariant under f and the rotation by an angle $\rho(f)$ respectively.

We can now prove the theorem. Assume first that $\tau(\widetilde{f})$ is irrational so that all the orbits of the rotation of angle $\rho(f)$ are dense. It follows that $Jump(\widetilde{h})$ has to be empty so that $\widetilde{h}$ defines a semi-conjugacy between f and the rotation by the angle $\rho(f)$. If the orbits of f are dense, then $Plat(\widetilde{h})$ has to be empty and $\widetilde{h}$ defines an actual conjugacy between f and the rotation. Also, since a semi-conjugacy maps finite orbit to finite orbit, f cannot have any periodic orbits in the case $\rho(f)$ irrational.

Assume that $\tau(\widetilde{f})$ is a rational number of the form p/q. The we know that the element $\widetilde{l}$ defined by $\widetilde{l}(x) = \widetilde{f}^q - p$ has a vanishing translation number so that the orbit of any point x in $\mathbf{R}$ under $\widetilde{l}$ is bounded. The upper bound of any orbit is a fixed point of $\widetilde{l}$. Since $\widetilde{l}$ projects to f^q in $\mathrm{Homeo}_+(\mathbf{S}^1)$, we have found a fixed point for f^q, hence a periodic orbit for f. This establishes the theorem. Note that in this last case, we showed something more: if $\rho(f) = p/q$ mod $\mathbf{Z}$, then there is a periodic orbit whose "cyclic ordering" is the same as a rotation of angle p/q. This means that there is a homeomorphism h in $\mathrm{Homeo}_+(\mathbf{S}^1)$ whose *restriction* to the periodic orbit is a conjugacy between f and the rotation of angle p/q. $\square$

We can now describe the dynamics of a homeomorphism f in $\mathrm{Homeo}_+(\mathbf{S}^1)$ quite precisely.

Suppose first that $\rho(f)$ is *irrational*: we have two possibilities.

1) If all orbits are dense, then f is conjugate to the rotation of angle $\rho(f)$.

2) If there is an exceptional minimal set $K \subset \mathbf{S}^1$ then f is semi-conjugate to the rotation of angle $\rho(f)$. The connected components of the complement of K are wandering intervals, *i.e.* disjoint from all their iterates.

It is not difficult to construct examples of the second type. Start with an irrational rotation of angle ρ on the circle and choose a (dense) orbit $\mathcal{O} \subset \mathbf{S}^1$. Then "blow up" each point in $\mathcal{O}$ to replace it by an interval. In other words, consider a continuous increasing map h of degree 1 such that $h^{-1}(x)$ is an interval if x is in $\mathcal{O}$ and a point otherwise. The complement of the interior of these intervals is a Cantor set K. Then we construct a homeomorphism f of the circle which preserves K. On K the homeomorphism f is uniquely defined by the fact that h is a semi-conjugacy with the rotation. On the intervals of the complement of K, there is still some freedom in the construction: we choose any homeomorphism f which sends the interval $h^{-1}(x)$ to the interval $h^{-1}(x+\rho)$ for x in the orbit $\mathcal{O}$. The problem with this construction is that it is not clear whether or not we might do it in such a way that the corresponding homeomorphism f is smooth. Poincaré thought that there could exist an example of type 2 for which f is a real analytic diffeomorphism [59]: he was wrong, as shown later by Denjoy ! Again, we refrain from discussing this point here since we decided to restrict these notes to topological problems.

Suppose now that $\rho(f)$ is *rational* so that f has a periodic point. Replacing f by one of its powers f^q, we study the case where f has a fixed point. To understand the dynamics of f, we have first to describe the set *Fix*(f) of fixed points which can be an arbitrary compact set in the circle (so that it could be rather complicated). Then, f induces a homeomorphism of each connected component of the complement of *Fix*(f). On each component f can move points "to the right" or "to the left" and this information is the only dynamical information: it is easy to show that up to orientation preserving conjugacy, there are two kinds of fixed point free homeomorphisms of an open interval, those going to the right and to the left respectively.

Summing up, we have a complete description of conjugacy classes of homeomorphisms of the circle. To give a complete list of invariants is possible but not very pleasant: for instance in the case of vanishing rotation number, we should describe a compact set up to homeomorphism and labels "left" or "right" on each component of the complement.

As a corollary, we get a description of those elements of $\mathrm{Homeo}_+(\mathbf{S}^1)$ which have the form ϕ^1 for some topological flow ϕ^t on the circle. This follows immediately from our description of homeomorphisms and the description that we gave earlier of topological flows.

PROPOSITION 5.10. *An element f of* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *can be included in a topological flow if and only if $\rho(f) = 0$ or f is conjugate to a rotation.*

Note that it is possible to find elements f which are not included in flows arbitrarily close to the identity.

We can now prove an important fact that we have already used in the proof of the simplicity of $\mathrm{Homeo}_+(\mathbf{S}^1)$.

PROPOSITION 5.11. *Every element of* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *can be written as a product of two commutators.*

Proof. Consider a topological flow on the closed interval, *i.e.* a continuous homomorphism $t \in \mathbf{R} \mapsto \phi^t \in \mathrm{Homeo}_+([0,1])$. Assume that for $t > 0$ the homeomorphism ϕ^t satisfies $\phi^t(x) > x$ for $x \in]0,1[$. By the previous discussion, all homeomorphisms ϕ^t with $t > 0$ are conjugate in $\mathrm{Homeo}_+([0,1])$. In particular, there is a homeomorphism l in $\mathrm{Homeo}_+([0,1])$ such that $l\phi^2 l^{-1} = \phi^1$. It follows that $\phi^1 = \phi^2(\phi^1)^{-1} = \phi^2 l(\phi^2)^{-1} l^{-1}$. This shows that ϕ^1 is the commutator of ϕ^2 and l. Since we know that every homeomorphism of $[0,1]$ which fixes only 0 and 1 is conjugate to ϕ^1 or its inverse, it follows that any such homeomorphism can be written as a commutator.

We described the dynamics of homeomorphisms with rotation number $0 \in \mathbf{R}/\mathbf{Z}$: in each connected component of the complement of their non empty fixed point set, they are described by a homeomorphism of the closed interval with no fixed point in the interior. Our discussion therefore implies that every element of $\mathrm{Homeo}_+(\mathbf{S}^1)$ with rotation number 0 can be written as a commutator.

Consider finally an element f of $\mathrm{Homeo}_+(\mathbf{S}^1)$. Clearly, one can choose a rotation r_θ such that fr_θ has a fixed point. In order to prove the proposition, it is therefore enough to show that any rotation can be written as a commutator. We show that this is indeed the case in $\mathrm{PSL}(2,\mathbf{R})$ using some hyperbolic geometry (of course, we could also prove the same thing by direct calculations).

Let a, b, c, d be four points in the Poincaré disc whose hyperbolic distances satisfy $dist(a,b) = dist(c,d)$ and $dist(a,d) = dist(b,c)$. Let A (resp. B) be the orientation preserving isometry of the Poincaré disc such that $A(a) = b$, $A(d) = c$ (resp. $B(a) = d$, $B(b) = c$). The commutator $ABA^{-1}B^{-1}$ fixes the point c: it is therefore a hyperbolic rotation centered at the point c. Figure 5 shows that the angle of this rotation is equal to 2π minus the sum of the angles of the quadrangle $R = abcd$ which is equal to the area of this quadrangle and can take any value between 0 and 2π since it is built out of two hyperbolic triangles. Hence we may realize any rotation as a single commutator. □

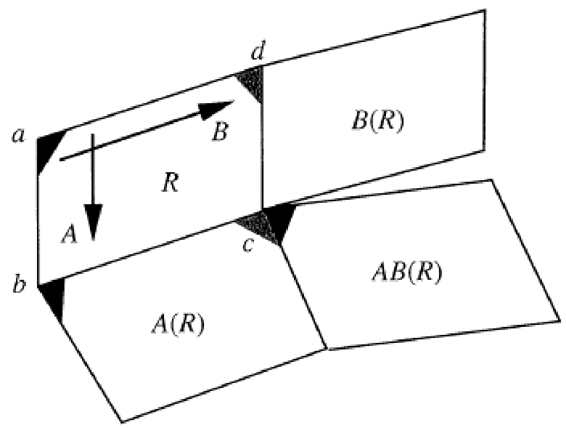

FIGURE 5

It turns out that Proposition 5.11 can be improved: every element of $\mathrm{Homeo}_+(\mathbf{S}^1)$ can be written as a single commutator. This is a special case of a result of [17] that we shall mention later in 6.2.

To conclude, we give some examples. Consider an element A of $\mathrm{PSL}(2,\mathbf{R})$ as a homeomorphism of the circle. The topological dynamics of A are easy to describe. Note that since A is a 2×2 matrix up to sign, the absolute value of the trace of A is well defined. If $|tr(A)| > 2$, then A is called hyperbolic and has two fixed points on the circle. In this case, the rotation number of A is of course 0. If $|tr(A)| = 2$, then A is called parabolic and has only one fixed point; its rotation number is again 0. Finally, if $|tr(A)| < 2$, then A is called elliptic and is conjugate to (the equivalence class of) a rotation matrix by some angle $2\pi\theta$ where $\theta \in \mathbf{R}/\mathbf{Z}$ is such that $2\cos(\theta) = |tr(A)|$. In this case, the rotation number is therefore $\cos^{-1}(tr(A)/2)/2\pi$.

Let us consider a finitely generated fuchsian group $\Gamma \subset \mathrm{PSL}(2,\mathbf{R})$. Since it is a discrete subgroup, any elliptic element in Γ must be of finite order. Assume that Γ is torsion free. (A theorem of Selberg guarantees that any finitely generated subgroup of a matrix group contains a finite index torsion free subgroup.) Then any element of Γ has rotation number equal to 0. However, there are many fuchsian groups exhibiting very rich dynamics, even with dense orbits. These examples show that *the data of all individual rotation numbers of the elements of a group acting on the circle is far from sufficient to describe the dynamics of the group.* In other words, Theorem 5.9 cannot be generalized so easily to groups more complicated than $\mathbf{Z}$. In the next section, we shall define a more subtle invariant suitable for bigger group actions, like fuchsian groups.

Observe that our computations in $\mathrm{PSL}(2,\mathbf{R})$ show that the rotation number is a continuous function on $\mathrm{PSL}(2,\mathbf{R})$ but definitely not a smooth function. On the group $\mathrm{Homeo}_+(\mathbf{S}^1)$, we have the following behaviour:

PROPOSITION 5.12. *The map* $\rho\colon \mathrm{Homeo}_+(\mathbf{S}^1) \to \mathbf{R}/\mathbf{Z}$ *is continuous and the pre-image of* $\mathbf{Q}/\mathbf{Z}$ *contains an open and dense set.*

Proof. The continuity follows immediately from the definitions. Indeed, the continuous function $\widetilde{f} \mapsto \widetilde{f}^n(0)/n$ on $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ differs at most by $1/n$ from $\tau(\widetilde{f})$. This implies the continuity of the translation number.

Suppose that $\widetilde{f}$ in $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ is such that $\widetilde{f}(x)-x$ achieves both positive and negative values. Then $\widetilde{f}$ has at least a fixed point and $\tau(\widetilde{f})=0$. Since this condition is obviously open in the uniform topology, we have found an open set on which the translation number takes the value 0. In the same manner, we construct open sets on which τ takes the value p/q: the set of those $\widetilde{f}$ for which $\widetilde{f}^q(x)-x-p$ takes both positive and negative values.

We leave to the reader the (easy) proof that the set of $\widetilde{f}$ for which $\tau(\widetilde{f})$ is rational is dense in $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$. □

The local structure of the map ρ is quite interesting as was shown by a very nice example due to Arnold [1]. Consider the 2-parameter family of elements of $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ given by

$$\widetilde{f}_{\alpha,\epsilon}(x) = x + \alpha + \epsilon \sin(2\pi x)\,.$$

Here α is a real number and ϵ is a real number which is small enough to guarantee that $\widetilde{f}_{\alpha,\epsilon}$ is a homeomorphism ($|\epsilon| < 1/2\pi$ is enough). We should think of these $\widetilde{f}_{\alpha,\epsilon}$ as a small deformation of the translation by α depending of the small parameter ϵ. Let us look at the behaviour of $\tau(\widetilde{f}_{\alpha,\epsilon})$ as a function of α and ϵ. Of course, we have $\tau(\widetilde{f}_{\alpha,0})=\alpha$. We can check rather easily the following facts. For each ϵ, the function $\alpha \mapsto \tau(\widetilde{f}_{\alpha,\epsilon})$ is continuous and increasing but is not strictly increasing for $\epsilon \neq 0$. The plateau set of this function is the complement of a Cantor set on which τ takes irrational values. The interior of the set of (α,ϵ) on which τ takes the rational value p/q is an "Arnold tongue" which touches the axis $\epsilon=0$ at the point $(p/q,0)$. The bigger the denominator q, the thinner the corresponding tongue.

Another interesting feature of this picture is that the Lebesgue measure of the set of (α,ϵ) for which τ is irrational is not 0. Hence, the translation number takes rational values on an open dense set but takes irrational values on a set of positive Lebesgue measure.

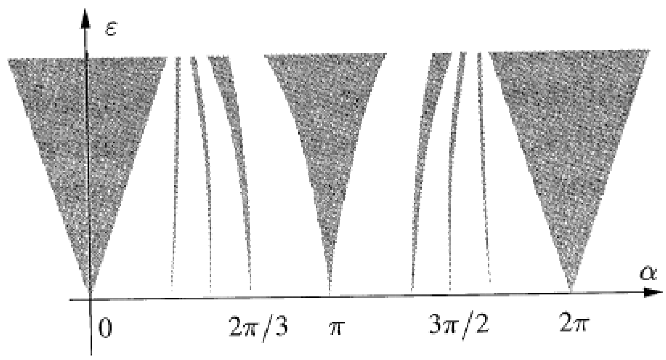

FIGURE 6

As an additional example, consider the case of piecewise linear homeomorphisms of the circle. Since the group $\mathrm{PL}_+(\mathbf{S}^1)$ contains all rotations, it is clear that the rotation number of such a homeomorphism can be arbitrary. However, it is shown in [28] that the rotation number of any element of the Thompson group G is rational and that any rational number is achieved. The proof is very indirect and there is a need for a better proof. We could formulate the problem in the following way.

PROBLEM 5.13. *Consider a rational piecewise linear homeomorphism f of the circle, i.e. such that all its slopes are rational and such that all "break-points" are rational. Is it true that the rotation number of f is rational ?*

We can in fact generalize Thompson's group quite a lot in the following way. Let $\Lambda \subset \mathbf{R}_\star^+$ be a subgroup of the multiplicative group of positive real numbers and let $W \subset \mathbf{R}$ be an additive subgroup invariant under multiplication by Λ. Then we can consider the subgroup $\widetilde{G}_{\Lambda,W}$ of $\widetilde{\mathrm{PL}}_+(\mathbf{S}^1)$ consisting of those elements with slopes in Λ and break-points in W (for instance, Thompson group is the case when Λ consists of powers of 2 and W of dyadic rationals). These groups are quite interesting especially when Λ is finitely generated (see [8, 9, 63]). It would be very useful to understand the nature of translation numbers of elements of $\widetilde{G}_{\Lambda,W}$ for specific Λ and W.

In [34], one can find (among other things !) a very interesting analysis of the rotation numbers of an explicit 1-parameter family of piecewise linear homeomorphisms of the circle.

5.2 TITS' ALTERNATIVE

Recall that J. Tits proved a remarkable alternative for finitely generated subgroups Γ of $\mathrm{GL}(n,\mathbf{C})$ (see [65]): either Γ *contains a non abelian free subgroup or* Γ *contains a subgroup of finite index which is solvable.* Such an alternative does not hold for subgroups of $\mathrm{Homeo}_+(\mathbf{S}^1)$. Indeed, we have seen that the group $\mathrm{PL}_+([0,1])$ can be considered as a subgroup of $\mathrm{Homeo}_+(\mathbf{S}^1)$ and contains no free non abelian subgroup. The subgroup F of $\mathrm{PL}_+([0,1])$ consisting of elements whose slopes are powers of 2 and whose break-points are dyadic rationals, is a finitely presented group and is certainly not virtually solvable (since its first commutator subgroup is a simple group, see [28]). However, answering a question of the author, Margulis recently proved the following theorem [49]:

THEOREM 5.14 (Margulis). *Let* Γ *be a subgroup of* $\mathrm{Homeo}_+(\mathbf{S}^1)$. *At least one of the following properties holds:*

- Γ *contains a non abelian free subgroup.*
- *There is a probability measure on the circle which is* Γ*-invariant.*

COROLLARY 5.15. *Let* Γ *be a subgroup of* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *such that all orbits are dense in the circle. Exactly one of the following properties holds:*

- Γ *contains a non abelian free subgroup.*
- Γ *is abelian and is conjugate to a group of rotations.*

The corollary follows easily from the theorem. Indeed, if all Γ-orbits are dense, any invariant probability must have full support and cannot have any non trivial atom. Any such probability is the image of the Lebesgue measure by some homeomorphism of the circle. Hence, up to some conjugacy, one can assume that Γ preserves the Lebesgue measure, *i.e.* consists of rotations. Note however that the proof which follows will begin with a proof of the corollary...

The proof of Margulis' theorem is very elegant and we cannot refrain from giving an account of it. Our presentation is a variation (or maybe a simplification ?) of Margulis' original ideas. More precise results may be found in the recent preprint [6]. We begin by recalling the "ping-pong" lemma, which is the standard way of constructing free subgroups (see [31]). Suppose a set X contains two disjoint non empty subsets A and A'. Let f, f' be two bijections of X which are such that for every $n \in \mathbf{Z} \setminus \{0\}$, we have $f^n(A) \subset A'$ and $f'^n(A') \subset A$. Then we claim that *f and f' generate a free subgroup of the group of bijections of X*. The proof is easy; consider a word $w(f,f') = f^{m_1} f'^{m'_1} \cdots f^{m_k} f'^{m'_k}$ with non zero exponents m_i, m'_i, except maybe the first one m_1 and the last one m'_k (if $k = 1$, we assume that m_1 and m'_1 are not both zero...). We want to show that $w(f,f')$ represents a non trivial bijection of X. This is clear if $m_1 \neq 0$ and $m'_k = 0$ (resp. $m_1 = 0$ and $m'_k \neq 0$) since in this case we have $w(f,f')(A) \subset A'$ (resp. $w(f,f')(A') \subset A$). In the other cases, one can conjugate $w(f,f')$ by a suitable power of f or f' to get a new word which is in the previous form. This proves the ping-pong lemma.

In the case of the circle, the typical application of the ping-pong lemma is the following. Let I, J, I', J' be four closed intervals in the circle and let f,f' be two orientation preserving homeomorphisms of the circle. Assume the following condition holds:

(PING-PONG) The four intervals I, J, I', J' are disjoint, $f'(I) = \mathbf{S}^1 \setminus interior(J)$ and $f(I') = \mathbf{S}^1 \setminus interior(J')$.

Clearly, if one sets $X = \mathbf{S}^1$, $A = I \cup J$ and $A' = I' \cup J'$, we are in the situation of the ping-pong lemma and one can deduce from (PING-PONG) that f and f' generate a free subgroup of $\mathrm{Homeo}_+(\mathbf{S}^1)$.

In order to find free subgroups inside a given subgroup Γ of $\mathrm{Homeo}_+(\mathbf{S}^1)$, we shall try to locate such ping-pong situations.

Assume now that we are given a group Γ such that the following two properties hold:

(MINIMALITY) All Γ-orbits are dense.

(STRONG EXPANSIVITY) There is a sequence of closed intervals I_n in the circle and a sequence γ_n of elements of Γ such that the length of I_n tends to zero as well as the length of the complementary intervals $J_n = \mathbf{S}^1 \setminus int(\gamma_n(I_n))$.

Of course, using subsequences we can assume in (STRONG EXPANSIVITY) that both endpoints of I_n converge to some point x and that both endpoints of J_n converge to some point y. We can also assume that $x \neq y$, since otherwise we could replace γ_n by $\gamma\gamma_n$ where γ is some element of Γ such that $y = \gamma(x) \neq x$.

Choose some γ in Γ such that $x' = \gamma(x)$ and $y' = \gamma(y)$ are both different from x and y (exercise: show that such an element γ exists!) and consider the sequence $\gamma'_n = \gamma^{-1}\gamma_n\gamma$. Of course, if we let $I'_n = \gamma(I_n)$ (resp. $J'_n = \gamma(J_n)$), the sequence of intervals I'_n (resp. J'_n) shrinks to x' (resp. to y'). Clearly, if n is big enough, the four intervals $I = I_n, J = J_n, I' = I'_n, J' = J'_n$ and the two homeomorphisms $f = \gamma_n, f' = \gamma'_n$ satisfy (PING-PONG) and therefore γ_n

and γ_n' generate a free subgroup of Γ. *In other words, if* (MINIMALITY) *and* (STRONG EXPANSIVITY) *hold, then* Γ *contains a free non abelian subgroup.*

The minimality condition is not so restrictive: we saw earlier that any action without a finite orbit is semi-conjugate to such a minimal action. However, the strong expansivity condition is very restrictive. Let us introduce the following weaker condition.

(EXPANSIVITY) There is a sequence of closed intervals I_n and a sequence of elements γ_n of Γ such that the length of I_n tends to zero and the length of $\gamma_n(I_n)$ is bounded away from zero.

Call a closed interval K in the circle *contractible* if there is a sequence of elements γ_n of Γ such that the length of $\gamma_n(K)$ tends to zero. It follows from (EXPANSIVITY) that there exists a non trivial contractible interval. If moreover the condition (MINIMALITY) is also satisfied, then every point of the circle belongs to the interior of some contractible interval. So let us assume now that (MINIMALITY) and (EXPANSIVITY) are satisfied.

For each point x in the circle, consider the set of points y such that the interval $[x, y]$ is contractible. Denote by $\theta(x)$ the least upper bound of those points y (to be correct, one should lift everything to the universal cover). In this way, we get a map θ from the circle to itself. Note that obviously θ commutes with all elements of Γ. Note also that θ is monotone. We claim that θ is a homeomorphism. Indeed if it were not strictly monotone, the union *Plat*(θ) of the interiors of the intervals in which θ is constant would be a Γ-invariant open set. By (MINIMALITY), this open set is empty unless θ is constant, but this is of course not possible since this constant would be fixed by Γ. In the same way, one shows that θ is continuous, using the union *Jump*(θ) of the interiors of the "jump intervals" like in 3.2.

We now consider the rotation number of θ. If this rotation number is irrational, then θ has to be conjugate to an irrational rotation since otherwise its unique invariant minimal set would be a non trivial Γ-invariant compact set. Since a homeomorphism which commutes with an irrational rotation is itself a rotation, that would imply that Γ is conjugate to a group of rotations. This is in contradiction with (EXPANSIVITY).

Hence the rotation number of θ is rational. The union of periodic points of θ is a non empty closed set which is Γ-invariant. It follows that θ is a periodic homeomorphism.

Consider the quotient $\mathbf{S}^{1'} = \mathbf{S}^1/\theta$ of the circle by the finite cyclic group generated by θ. This is a ("shorter") circle on which we have a natural action of Γ since, once again, Γ commutes with θ.

We observe that this new group of homeomorphisms of a circle satisfies (MINIMALITY) and (STRONG EXPANSIVITY). Minimality is obviously inherited from the same property of Γ on $\mathbf{S}^1$. As for (STRONG EXPANSIVITY), it suffices to observe that any compact interval contained in $[x, \theta(x)[$ is contractible, by definition. This means that any compact interval in $\mathbf{S}^{1'}$ is contractible and this implies (STRONG EXPANSIVITY).

We have now proved that if (MINIMALITY) *and* (EXPANSIVITY) *are both satisfied, then the group* Γ *must contain a free non abelian subgroup.*

Now, let us look more closely at (EXPANSIVITY) and observe that the negation of this property is nothing more than the *equicontinuity* property of the group Γ. If a group Γ acts equicontinuously, then its closure in $\mathrm{Homeo}_+(\mathbf{S}^1)$ is a compact group by Ascoli's theorem. We analyzed compact subgroups of $\mathrm{Homeo}_+(\mathbf{S}^1)$ in 4.1 : they turned out to be abelian and conjugate to groups of rotations.

We have shown that if (MINIMALITY) *holds then* Γ *is either abelian or contains a free non abelian subgroup; in other words, we have proved Corollary 5.15.*

Proving Theorem 5.14 in full generality is now an easy matter. Let Γ be any subgroup of $\mathrm{Homeo}_+(\mathbf{S}^1)$ and let us use the structure theorem 5.6–5.8. If Γ is minimal, we have already proved the theorem. If Γ has a finite orbit, there is a Γ-invariant probability which is a finite sum of Dirac masses. Finally, if there is an exceptional minimal set, the Γ-action is semi-conjugate to a minimal action. Applying our proof to this minimal action, we deduce that Γ contains a non abelian free subgroup unless the restriction of the action of Γ to the exceptional minimal set is abelian and is semi-conjugate to a group of rotations. In this case, one finds a Γ-invariant measure whose support is the exceptional minimal set. This is the end of the proof of Theorem 5.14.

6. BOUNDED EULER CLASS

6.1 GROUP COHOMOLOGY

Let us begin this section with some algebra. Let Γ be any group. Let us consider the (semi)-simplicial set $E\Gamma$ whose vertices are the elements of Γ and for which n-simplices are all $(n+1)$-tuples of elements of Γ. The i^{th} face of the simplex $(\gamma_0, \dots, \gamma_k)$ is $(\gamma_0, \dots, \hat{\gamma}_i \dots \gamma_k)$ where the term γ_i is omitted. Note that the set $E\Gamma$ does not depend on the group structure of Γ.

As a matter of fact, $E\Gamma$ is contractible since it is the full simplex over the set Γ. However, there is a simplicial free action of Γ on $E\Gamma$ induced by left translations of Γ on itself. Hence once could think of the quotient $B\Gamma$ of $E\Gamma$ by this action as a space whose fundamental group is Γ and with vanishing higher homotopy groups. One would like to define the cohomology of the group Γ as the cohomology of this quotient space $B\Gamma$. We should be careful with $B\Gamma$ since it has only one vertex (a group acts transitively on itself !).

However, guided by this idea, it is natural to define a *k-cochain* of Γ with values in some abelian group A as a map $c\colon \Gamma^{k+1} \to A$ which is *homogeneous*, *i.e.* such that $c(\gamma\gamma_0, \gamma\gamma_1, \ldots, \gamma\gamma_k) = c(\gamma_0, \gamma_1, \ldots, \gamma_k)$ identically. The set of these cochains is an abelian group denoted by $C^k(\Gamma, A)$. We have a natural coboundary d_k from $C^k(\Gamma, A)$ to $C^{k+1}(\Gamma, A)$ defined by

$$d_k c(\gamma_0, \ldots, \gamma_{k+1}) = \sum_{i=0}^{k} (-1)^i c(\gamma_0, \ldots, \widehat{\gamma_i}, \ldots, \gamma_k) .$$

Of course, we have $d_{k+1} \circ d_k = 0$ and we define the *cohomology group* $H^k(\Gamma, A)$ as being the quotient of cocycles (*i.e.* the kernel of d_k) by coboundaries (*i.e.* the image of d_{k-1}). If A is moreover a ring, then there is a natural cup product from $H^k(\Gamma, A) \times H^l(\Gamma, A)$ to $H^{k+l}(\Gamma, A)$. We refer to [11] for an excellent account of this theory of group cohomology. Note that for any homomorphism ϕ from a group Γ to another group Γ', there is an induced homomorphism $\phi^\star\colon H^k(\Gamma', A) \to H^k(\Gamma, A)$.

A homogeneous map $c\colon \Gamma^{k+1} \to A$ can be written in a unique way in the form $c(\gamma_0, \ldots, \gamma_k) = \overline{c}(\gamma_0^{-1}\gamma_1, \gamma_1^{-1}\gamma_2, \ldots, \gamma_{k-1}^{-1}\gamma_k)$ for a unique function $\overline{c}\colon \Gamma^k \to A$. Conversely, given a map $\overline{c}$ there is a unique homogeneous map c satisfying this relation. One says that $\overline{c}$ is the inhomogeneous cochain associated to c. In other words, the space $C^k(\Gamma, A)$ is canonically isomorphic to the A-module of all maps $\Gamma^k \to A$.

In degree 1, a cochain is a homogeneous map $c\colon \Gamma^2 \to A$ and the corresponding inhomogeneous cochain is a map $\overline{c}\colon \Gamma \to A$. It is interesting to check that c is a cocycle if and only if $\overline{c}$ is a homomorphism. Moreover 0-cochains are constant maps from Γ to A and their coboundary is therefore 0. It follows that *for any group Γ, the cohomology $H^1(\Gamma, A)$ is identified with the set of homomorphisms from Γ to A.*

In degree 2, the interpretation is quite interesting. Consider a central extension of Γ by A :

$$0 \longrightarrow A \xrightarrow{\ i\ } \widetilde{\Gamma} \xrightarrow{\ p\ } \Gamma \longrightarrow 1 .$$

This means that $\widetilde{\Gamma}$ contains a subgroup isomorphic to A contained in its center and that the quotient by this subgroup is isomorphic to Γ. Suppose that the projection p has a section s which is a homomorphism from Γ to $\widetilde{\Gamma}$ such that $p \circ s = Id_\Gamma$. Then it follows that $\widetilde{\Gamma}$ is isomorphic to the direct product $\Gamma \times A$ by the homomorphism sending (γ, a) to $s(\gamma)i(a)$. Hence, in order to measure the non triviality of an extension we try to find the "obstruction" to finding a section s. This is done in the following way. Choose a set theoretical section s from Γ to $\widetilde{\Gamma}$; this is possible since p is onto. If γ_1 and γ_2 are two elements of Γ, consider $\overline{c}(\gamma_1, \gamma_2) = s(\gamma_1\gamma_2)^{-1}s(\gamma_1)s(\gamma_2)$. This element projects on the identity element of Γ under p since p is a homomorphism; it is therefore an element of the image of i and can be identified with an element of A. This defines a map $\overline{c}\colon \Gamma^2 \to A$. Let $c\colon \Gamma^3 \to A$ be the associated homogeneous cochain. One checks that c is a cocycle. Of course, the section s is not unique but another choice s' has the form $s'(\gamma) = s(\gamma)i(u(\gamma))$ for some function $u\colon \Gamma \to A$. If one computes the cocycle c' associated to this new choice of a section s', one finds that $c' - c$ is the coboundary of the 1-cochain associated to the map u. It follows that the cohomology class of c in $H^2(\Gamma, A)$ is well defined, *i.e.* does not depend on the choice of a section. This cohomology class is called the *Euler class of the extension* under consideration.

It is not difficult to check the following properties of the Euler class.

1) Two central extensions $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$ of A by Γ are isomorphic by some isomorphism which is the identity on the central subgroup A and inducing the identity on the quotient Γ if and only if they have the same Euler class in $H^2(\Gamma, A)$.

2) Any class in $H^2(\Gamma, A)$ corresponds to a central extension.

In short, $H^2(\Gamma, A)$ *parametrizes isomorphism classes of central extensions of A by Γ.*

Before coming back to the dynamics of groups acting on the circle, let us consider a few simple examples.

If $\Gamma = \mathbf{Z}$, it is clear that every extension admits a section which is a homomorphism: it suffices to choose arbitrarily $s(1)$ in $p^{-1}(1)$ and to define $s(n) = s(1)^n$ for $n \in \mathbf{Z}$. Hence, if $\Gamma = \mathbf{Z}$ or more generally if Γ is a free group, we have $H^2(\Gamma, A) = 0$.

Let Γ_g be the fundamental group of a closed oriented surface of genus $g \geq 1$. It has a presentation of the form

$$\Gamma_g = \langle a_1, b_1, \ldots, a_g, b_g \mid a_1b_1a_1^{-1}b_1^{-1} \ldots a_gb_ga_g^{-1}b_g^{-1} = 1 \rangle .$$

Now consider the group $\widetilde{\Gamma}_g$ defined by the presentation

$$\widetilde{\Gamma}_g = \big\langle z, a_1, b_1, \dots, a_g, b_g \;\big|\; a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} = z, \quad z a_i = a_i z, \quad z b_i = b_i z \big\rangle .$$

The central subgroup A generated by z turns out to be infinite cyclic so that $\widetilde{\Gamma}_g$ defines a central extension of Γ_g by $\mathbf{Z}$, hence an Euler class in $H^2(\Gamma_g, \mathbf{Z})$. It is a fact that $H^2(\Gamma_g, \mathbf{Z})$ is isomorphic with $\mathbf{Z}$ and that the element that we have just constructed is a generator of this cohomology group. We shall not prove this here but we note that this is related to the fact that a closed oriented surface of genus $g \geq 1$ has a contractible universal cover and that the cohomology of Γ_g can therefore be identified with the cohomology of the compact oriented surface of genus g (see [11] for more details).

6.2 The Euler class of a group action on the circle

We have already met a central extension related to groups of homeomorphisms

$$0 \longrightarrow \mathbf{Z} \longrightarrow \widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1) \xrightarrow{\;p\;} \mathrm{Homeo}_+(\mathbf{S}^1) \longrightarrow 1 .$$

The cohomology group $H^2(\mathrm{Homeo}_+(\mathbf{S}^1), \mathbf{Z})$ has been computed. It is isomorphic to $\mathbf{Z}$ and a generator is the Euler class of this central extension [50].

Consider now a homomorphism ϕ from some group Γ to $\mathrm{Homeo}_+(\mathbf{S}^1)$. Then, we can pull back the previous extension by ϕ. In other words, we consider the set of $(\gamma, \widetilde{f}) \in \Gamma \times \widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ such that $\phi(\gamma) = p(\widetilde{f})$. This is a group $\widetilde{\Gamma}$ equipped with a canonical projection onto Γ whose kernel is isomorphic to $\mathbf{Z}$, *i.e.* $\widetilde{\Gamma}$ is a central extension of Γ by $\mathbf{Z}$. In case ϕ is injective, $\widetilde{\Gamma}$ is just the pre-image of $\phi(\Gamma)$ under p, which is the group of lifts of $\phi(\Gamma)$. The Euler class of this central extension of Γ is called *the Euler class of the homomorphism* ϕ and denoted by $eu(\phi) \in H^2(\Gamma, \mathbf{Z})$. It is obviously a dynamical invariant in the sense that two conjugate homomorphisms ϕ_1 and ϕ_2 have the same Euler class in $H^2(\Gamma, \mathbf{Z})$. Note that it follows from the definition that $eu(\phi)$ is zero if and only if the homomorphism ϕ lifts to a homomorphism $\widetilde{\phi}\colon \Gamma \to \widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ such that $\phi = p \circ \widetilde{\phi}$.

A few examples are in order. In the case of a single homeomorphism, *i.e.* when $\Gamma = \mathbf{Z}$, we saw that $H^2(\mathbf{Z}, \mathbf{Z}) = 0$. Hence the Euler class vanishes and our new invariant is very poor indeed: in particular, it does not detect the rotation number. A similar phenomenon occurs when Γ is free.

If Γ_g is the fundamental group of a closed oriented surface of genus $g \geq 1$, we know that $H^2(\Gamma_g, \mathbf{Z})$ is isomorphic to $\mathbf{Z}$ so that the Euler class

$eu(\phi)$ in this case is an integer. In [51], Milnor gives an algorithm to compute this number. With the same notation as above, for each $1 \leq i \leq g$, choose lifts $\widetilde{a}_i$ and $\widetilde{b}_i$ of $\phi(a_i)$ and $\phi(b_i)$. Now compute the product of commutators $\widetilde{a}_1\widetilde{b}_1\widetilde{a}_1^{-1}\widetilde{b}_1^{-1}\ldots\widetilde{a}_g\widetilde{b}_g\widetilde{a}_g^{-1}\widetilde{b}_g^{-1}$. Since this homeomorphism is a lift of the identity, it is an integral translation. This amplitude of this translation does not depend on the choices made and is the Euler number $eu(\phi)$.

As an explicit example, also computed by Milnor, recall that any closed orientable surface of genus $g > 1$ can be endowed with a riemannian metric of constant negative curvature. Recall also that the Poincaré upper half space $\mathcal{H}$ can be equipped with a metric of curvature -1 whose group of orientation preserving isometries is precisely $\mathrm{PSL}(2,\mathbf{R})$. Moreover, any complete simply connected riemannian surface of curvature -1 is isometric to $\mathcal{H}$. Hence there are embeddings ϕ of the fundamental group Γ_g of a closed oriented surface of genus $g > 1$ in $\mathrm{PSL}(2,\mathbf{R})$ such that the corresponding action of Γ_g on $\mathcal{H}$ is free, proper and cocompact. Since we know that $\mathrm{PSL}(2,\mathbf{R})$ is a subgroup of $\mathrm{Homeo}_+(\mathbf{S}^1)$, we can compute the corresponding Euler number $eu(\phi)$. The result of the computation is $2g-2$. Note that each element of $\phi(\Gamma_g)$ is hyperbolic since the action is free and cocompact so that the rotation number of every element of $\phi(\Gamma_g)$ is 0. So we are in a situation in which the topological invariant $eu(\phi)$ is not 0 but the rotation number invariants are trivial; a situation different from the case where $\Gamma = \mathbf{Z}$.

6.3 Bounded cohomology and the Milnor-Wood inequality

It was observed very early that the Euler class of a homomorphism $\phi\colon \Gamma \to \mathrm{Homeo}_+(\mathbf{S}^1)$ cannot be arbitrary. Milnor and Wood proved the following [51, 71].

THEOREM 6.1 (Milnor-Wood). *Let Γ_g be the fundamental group of a closed oriented surface of genus $g \geq 1$ and $\phi\colon \Gamma_g \to \mathrm{Homeo}_+(\mathbf{S}^1)$ be any homomorphism. Then the Euler number satisfies $|eu(\phi)| \leq 2g-2$.*

Proof. We shall not give a complete proof since this result will follow from later considerations but we prove a weaker version. Keeping the previous notation, we know that $eu(\phi)$ is the translation number of the homeomorphism $\widetilde{a}_1\widetilde{b}_1\widetilde{a}_1^{-1}\widetilde{b}_1^{-1}\ldots\widetilde{a}_g\widetilde{b}_g\widetilde{a}_g^{-1}\widetilde{b}_g^{-1}$. We also know that the translation number function τ is a quasi-homomorphism, *i.e.* there is some inequality of the form $\left|\tau(\widetilde{f}_1\widetilde{f}_2) - \tau(\widetilde{f}_1) - \tau(\widetilde{f}_2)\right| \leq D$ for some D. We also know that $\tau\big(\widetilde{f}^{-1}\big) = -\tau(\widetilde{f})$. So, if we evaluate τ on this element, we get a bound of the form $|eu(\phi)| \leq (4g-1)D$. This is not quite the bound given in the theorem but this explains the idea of the proof: to get the exact bound, one should be a little bit more clever! □

In [17], Eisenbud, Hirsch and Neumann gave a much more precise result that we would like to mention here. If $\widetilde{f}$ is an element of $\widetilde{\text{Homeo}}_+(\mathbf{S}^1)$, define $\underline{m}(\widetilde{f}) = \inf(\widetilde{f}(x)-x)$ and $\overline{m}(\widetilde{f}) = \sup(\widetilde{f}(x)-x)$. Note that $\underline{m}(\widetilde{f}) \leq \tau(\widetilde{f}) \leq \overline{m}(\widetilde{f})$ and $0 \leq \overline{m}(\widetilde{f}) - \underline{m}(\widetilde{f}) < 1$.

THEOREM 6.2 (Eisenbud, Hirsch, Neumann). *An element $\widetilde{f}$ of the group $\widetilde{\text{Homeo}}_+(\mathbf{S}^1)$ can be written as a product of $g \geq 1$ commutators if and only if $\underline{m}(\widetilde{f}) < 2g-1$ and $1-2g < \overline{m}(\widetilde{f})$.*

Any element of $\text{Homeo}_+(\mathbf{S}^1)$ has at least one lift $\widetilde{f}$ in $\widetilde{\text{Homeo}}_+(\mathbf{S}^1)$ such that $-1 < \underline{m}(\widetilde{f}) \leq \overline{m}(\widetilde{f}) < 1$ so that it can be written as one commutator. It follows that every element of $\text{Homeo}_+(\mathbf{S}^1)$ can be written as a commutator. We mentioned this fact earlier.

In [25], we put these inequalities in the context of bounded cohomology, which was introduced by Gromov (see [30] for many geometrical motivations). Consider again an abstract group Γ and let $A = \mathbf{Z}$ or $\mathbf{R}$. Then define a *bounded* k-cochain as a bounded homogeneous map from Γ^{k+1} to A. This defines a sub A-module of $C^k(\Gamma, A)$ denoted by $C_b^k(\Gamma, A)$. It is clear that the coboundary d_k of a bounded k-cochain is a bounded $(k+1)$-cochain so that we can define the cohomology of this new differential complex, that is called the *bounded cohomology of* Γ *with coefficients in* A and denoted by $H_b^k(\Gamma, A)$. We have obvious maps from $H_b^k(\Gamma, A)$ to $H^k(\Gamma, A)$ obtained by "forgetting" that a cocycle is bounded. In general these maps are neither injective nor surjective. See [35, 36] for a detailed algebraic background on this cohomology.

The degree 1 case is trivial. A cocycle is given by a bounded homomorphism from Γ to A and is therefore trivial. Hence $H_b^1(\Gamma, A) = 0$ for any group Γ.

The degree 2 case is the most interesting for us. Let us look first at $H_b^2(\mathbf{Z}, \mathbf{R})$. Consider a bounded 2-cocycle c on $\mathbf{Z}$ with values in $\mathbf{R}$. Since we know that $H^2(\mathbf{Z}, \mathbf{R}) = 0$, we know that c is the coboundary of a 1-cochain of the form $u(n_1, n_2) = \overline{u}(n_1 - n_2)$ for some function $\overline{u} \colon \mathbf{Z} \to \mathbf{R}$. The fact that c is bounded means precisely that $\overline{u}$ is a quasi-homomorphism from $\mathbf{Z}$ to $\mathbf{R}$. We know that this implies the existence of a real number τ such that $\overline{u}(n) - n\tau$ is bounded. Now, if we define $\overline{v}(n) = \overline{u}(n) - n\tau$, then the coboundary of the *bounded* 1-cochain $v(n_1, n_2) = \overline{v}(n_1 - n_2)$ is c. We have shown that $H_b^2(\mathbf{Z}, \mathbf{R}) = 0$.

For a general group Γ, let us define $QM(\Gamma)$ as being the vector space of quasi-homomorphisms from Γ to $\mathbf{R}$. Say that a quasi-homomorphism is trivial if it differs from some homomorphism by a bounded amount. It follows from the definitions and the previous argument that the kernel of the map from $H_b^2(\Gamma, \mathbf{R})$ to $H^2(\Gamma, \mathbf{R})$ is precisely the quotient of $QM(\Gamma)$ by the subspace of trivial quasi-homomorphisms. This gives some intuition about the group $H_b^2(\Gamma, \mathbf{R})$.

Let us compute now some examples with coefficients in $\mathbf{Z}$. Start with $H_b^2(\mathbf{Z}, \mathbf{Z})$. Let c be a bounded integral 2-cocycle. We know that it is the coboundary of a 1-cochain of the form $u(n_1, n_2) = \overline{u}(n_1 - n_2)$ for some function $\overline{u}\colon \mathbf{Z} \to \mathbf{Z}$. Again, we know that there is a real number τ such that $\overline{u}(n) - n\tau$ is bounded but if we define $\overline{v}(n) = \overline{u}(n) - n\tau$ the 1-cochain v is not integral unless τ is an integer! For each real number τ, define c_τ to be the coboundary of the integral 1-cochain $v_\tau(n_1, n_2) = [(n_1 - n_2)\tau]$ where [ ] denotes the integral part of a real number. It is clear that c_τ is bounded (by 1) and our previous computations show that every bounded integral 2-cocycle in $\mathbf{Z}$ is cohomologous to some c_τ for some τ. Moreover, it is clear that c_{τ_1} and c_{τ_2} define the same element in $H_b^2(\mathbf{Z}, \mathbf{Z})$ if and only if $\tau_1 - \tau_2$ is an integer. Summing up, we showed that $H_b^2(\mathbf{Z}, \mathbf{Z})$ is isomorphic to $\mathbf{R}/\mathbf{Z}$. We hope that the reader will recognize that the rotation number is showing up...

As a matter of fact, the argument that we presented is more general and shows immediately that for any group Γ, the kernel of the map from $H_b^2(\Gamma, \mathbf{Z})$ to $H_b^2(\Gamma, \mathbf{R})$ is precisely the quotient $H^1(\Gamma, \mathbf{R})/H^1(\Gamma, \mathbf{Z})$. (Recall that $H^1(\Gamma, A)$ is the set of homomorphisms from Γ to A.)

We now come to the construction of an invariant of a group action on the circle that combines the rotation numbers and the Euler class. Let us look again at the central extension

$$0 \longrightarrow \mathbf{Z} \longrightarrow \widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1) \longrightarrow \mathrm{Homeo}_+(\mathbf{S}^1) \longrightarrow 1$$

and let us try to find some 2-cocycle representing its Euler class (see also [38]). We know that we should choose a set theoretical section s to p. It turns out that there is a natural choice of such a section. Indeed, let $f \in \mathrm{Homeo}_+(\mathbf{S}^1)$, then among the elements in $p^{-1}(f) \in \widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$, there is only one, denoted by $\sigma(f)$, which is such that $\sigma(f)(0)$ lies in the interval $[0, 1[\subset \mathbf{R}$. This σ will be our preferred section. Let us try to evaluate the associated 2-cocycle c on $\mathrm{Homeo}_+(\mathbf{S}^1)$. By definition the associated inhomogeneous cocycle $\overline{c}$ is:

$$\overline{c}(f_1, f_2) = \sigma(f_1 f_2)^{-1} \sigma(f_1) \sigma(f_2).$$

The main (easy) observation is that the cocycle c is bounded. More precisely:

LEMMA 6.3. *The 2-cocycle c takes only the two values 0 and 1.*

Proof. By definition $\sigma(f_2)(0)$ is in $[0,1[$. It follows that $\sigma(f_1)(\sigma(f_2)(0))$ is in the interval $[\sigma(f_1)(0), \sigma(f_1)(0)+1[$ which is contained in $[0,2[$. We know that $\sigma(f_1f_2)$ and $\sigma(f_1)\sigma(f_2)$ are lifts of the same element f_1f_2 and that $\sigma(f_1f_2)(0)$ is in $[0,1[$. It follows that $\sigma(f_1f_2)^{-1}\sigma(f_1)\sigma(f_2)$ is the translation by 0 or 1. □

Hence, for this choice of section σ, the associated 2-cocycle c is bounded and integral. Thus, we have defined an element of $H_b^2(\mathrm{Homeo}_+(\mathbf{S}^1), \mathbf{Z})$ that we call the *bounded Euler class*. It may seem that the definition depends on the choice of the origin 0 on the line but the reader will easily check that a modification of the origin would change the section σ by a bounded amount so that the bounded integral cohomology class is indeed well defined. If we have a homomorphism ϕ from a group Γ to $\mathrm{Homeo}_+(\mathbf{S}^1)$ we can pull back this bounded Euler class. We get an element in $H_b^2(\Gamma, \mathbf{Z})$ that we still denote by $eu(\phi)$ and that we call the bounded Euler class of the homomorphism ϕ. In case $\Gamma = \mathbf{Z}$, it should now be clear that the corresponding bounded Euler class in $H_b^2(\mathbf{Z}, \mathbf{Z}) = \mathbf{R}/\mathbf{Z}$ is exactly the rotation number of the homeomorphism $\phi(1)$. Hence we have proved the following:

THEOREM 6.4 ([25]). *There is a class eu in $H_b^2(\mathrm{Homeo}_+(\mathbf{S}^1), \mathbf{Z})$ such that:*

1) *For every homomorphism $\phi \colon \Gamma \to \mathrm{Homeo}_+(\mathbf{S}^1)$ the image of $\phi^*(eu) \in H_b^2(\Gamma, \mathbf{Z})$ in $H^2(\Gamma, \mathbf{Z})$ under the canonical map is the Euler class.*

2) *If $\phi \colon \mathbf{Z} \to \mathrm{Homeo}_+(\mathbf{S}^1)$ is a homomorphism then $\phi^*(eu) \in H_b^2(\mathbf{Z}, \mathbf{Z}) = \mathbf{R}/\mathbf{Z}$ is the rotation number of $\phi(1)$.*

3) *$\phi^*(eu)$ is a topological invariant, i.e. if ϕ_1 and ϕ_2 are two homomorphisms from Γ to $\mathrm{Homeo}_+(\mathbf{S}^1)$ which are conjugate by an orientation preserving homeomorphism, then $\phi_1^*(eu) = \phi_2^*(eu)$ in $H_b^2(\Gamma, \mathbf{Z})$.*

In other words, the bounded Euler class is a topological invariant which combines the Euler class and the rotation number.

We now show that this new invariant for a group action is as powerful as the rotation number was for a single homeomorphism. Let us begin by the most interesting case.

THEOREM 6.5 ([25]). *Let ϕ_1, ϕ_2 two homomorphisms from a group Γ to* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *such that all orbits are dense on the circle. Assume that the bounded Euler classes are equal: $\phi_1^*(eu) = \phi_2^*(eu)$. Then ϕ_1 and ϕ_2 are conjugate by an orientation preserving homeomorphism.*

Proof. This is very similar to the corresponding statement for rotation numbers: compare with the proof of 5.9. Since $\phi_1^*(eu) = \phi_2^*(eu)$ then in particular the Euler classes in $H^2(\Gamma, \mathbf{Z})$ are equal, which means that ϕ_1 and ϕ_2 define the same central extension $\widetilde{\Gamma}$. In other words, there is a central extension $0 \to \mathbf{Z} \to \widetilde{\Gamma} \to \Gamma \to 1$ and homomorphisms $\widetilde{\phi}_1$ and $\widetilde{\phi}_2$ from $\widetilde{\Gamma}$ to $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ such that $\widetilde{\phi}_1$ and $\widetilde{\phi}_1$ map the generator 1 of $\mathbf{Z}$ on the translation by 1 and such that the induced homomorphisms from $\widetilde{\Gamma}/\mathbf{Z} \simeq \Gamma$ to $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)/\mathbf{Z} \simeq \mathrm{Homeo}_+(\mathbf{S}^1)$ are ϕ_1 and ϕ_2. The assumption that the bounded classes agree means in fact that we can choose those homomorphisms in such a way that for each x in $\mathbf{R}$, the points $\widetilde{\phi}_1(\widetilde{\gamma})\widetilde{\phi}_2(\widetilde{\gamma})^{-1}(x)$ are bounded independently of $\widetilde{\gamma}$ in $\widetilde{\Gamma}$. We now define $\widetilde{h}(x)$ to be the upper bound of this bounded set. This map $\widetilde{h}$ is increasing, commutes with integral translations, and conjugates $\widetilde{\phi}_1$ and $\widetilde{\phi}_2$. The jump and plateau sets of $\widetilde{h}$ are open sets invariant under $\widetilde{\phi}_1(\widetilde{\Gamma})$ and $\widetilde{\phi}_2(\widetilde{\Gamma})$ respectively. By our assumption these open sets are empty so that $\widetilde{h}$ is a homeomorphism which induces a conjugacy between ϕ_1 and ϕ_2. For more details, see [25]. □

In case the group $\phi(\Gamma)$ does not have all its orbits dense, we saw in 5.6 that there are two possibilities: $\phi(\Gamma)$ can have a finite orbit or $\phi(\Gamma)$ can have an exceptional minimal set. In the second case, we also saw that there is a canonical way of "collapsing" the connected components of the complement of the exceptional minimal set to construct another homomorphism $\overline{\phi}$ which has all its orbits dense: this is the associated "minimal" homomorphism (see 5.8).

Suppose now that $\phi(\Gamma)$ has a finite orbit consisting of k elements. Then, every element of $\phi(\Gamma)$ must permute these k points cyclically so that we get a homomorphism $r\colon \Gamma \to \mathbf{Z}/k\mathbf{Z}$. It is clear that two finite orbits of $\phi(\Gamma)$ have the same number of points and define the same r: we call this r the *cyclic structure of the finite orbits.* Conversely, consider a homomorphism $r\colon \Gamma \to \mathbf{Z}/k\mathbf{Z}$ and the corresponding action on the circle by rotations of order k. The bounded Euler class of this action is an element of $H_b^2(\Gamma, \mathbf{Z})$: we call these elements the *rational elements* in $H_b^2(\Gamma, \mathbf{Z})$. It is not difficult to see that an element in $H_b^2(\Gamma, \mathbf{Z})$ is rational if and only if its pull-back on some finite index subgroup is trivial.

Now, we can state the general result which is the exact analogue of what has been done in 5.9 for the rotation number. We don't give the proof: it can be found in [25] (in a slightly different terminology and with small mistakes...), but the reader should now be in a condition to fill in the missing details by himself.

THEOREM 6.6 ([25]). *Let ϕ_1, ϕ_2 two homomorphisms from a group Γ to* $\mathrm{Homeo}_+(\mathbf{S}^1)$. *Assume that the bounded Euler classes $\phi_1^*(eu) = \phi_2^*(eu)$ are equal to the same class c in $H_b^2(\Gamma, \mathbf{Z})$.*

1) *If c is a rational class, then $\phi_1(\Gamma)$ and $\phi_2(\Gamma)$ have finite orbits with the same cyclic structure.*

2) *If c is not rational, then the associated minimal homomorphisms $\overline{\phi}_1$ and $\overline{\phi}_2$ are conjugate.*

Conversely, if $\phi_1(\Gamma)$ and $\phi_2(\Gamma)$ have finite orbits of the same cyclic structure or if they have no finite orbit and their associated minimal homomorphisms are conjugate (by an orientation preserving homeomorphism), then they have the same bounded Euler class.

Note in particular that the bounded Euler class of an action vanishes if and only if there is a point on the circle which is fixed by all the elements of the group.

6.4 EXPLICIT BOUNDS ON THE EULER CLASS

Since we know that the bounded Euler class of an action contains almost all the topological information, it is very natural to try to determine the part of $H_b^2(\Gamma, \mathbf{Z})$ which corresponds to the bounded Euler classes of all actions of Γ on the circle. In the case $\Gamma = \mathbf{Z}$, we know that $H_b^2(\mathbf{Z}, \dot{\mathbf{Z}}) = \mathbf{R}/\mathbf{Z}$ and that every class corresponds to an action (by rotations). However, in the case where Γ is the fundamental group of a closed oriented surface of genus $g \geq 1$, the Milnor-Wood inequality shows that even the usual Euler class in $H^2(\Gamma, \mathbf{Z}) = \mathbf{Z}$ has to satisfy some inequality.

Given a bounded cochain c in $C_b^k(\Gamma, \mathbf{R})$, we define its norm $\|c\|$ as the supremum of the absolute value of $c(\gamma_0, \dots, \gamma_k)$. Then we define the "norm" of a bounded cohomology class with real coefficients as the infimum of the norms of cocycles that represent it. We should be aware of the fact that this norm is not really a norm but is merely a semi-norm: a non zero class might have zero norm... Consider the case of the bounded Euler class, seen in the real bounded cohomology.

THEOREM 6.7. *The image of the bounded Euler class* eu *in the real bounded cohomology* $H_b^2(\mathrm{Homeo}_+(\mathbf{S}^1), \mathbf{R})$ *has norm* $1/2$.

Proof. This is the abstract version of the Milnor-Wood inequality. Note that a constant 2-cocycle is the coboundary of a constant 1-cochain. We found a representative of the Euler class taking only two values 0 and 1. If we subtract from this cocycle the constant cocycle taking the value $1/2$, we get a cohomologous bounded (real) cocycle taking values $\pm 1/2$. This shows that the norm of the image of eu in $H_b^2(\mathrm{Homeo}_+(\mathbf{S}^1), \mathbf{R})$ is at most $1/2$. The opposite inequality follows from Milnor's computation of the Euler number for an embedding of the fundamental group Γ_g of a closed oriented surface as a discrete cocompact subgroup of $\mathrm{PSL}(2, \mathbf{R})$ that we mentioned in 6.1. If the norm were strictly less than $1/2$, then this number would be strictly less than $2g - 2$. See [25] for more explanations. □

6.5 ACTIONS ON THE REAL LINE AND ORDERINGS

Our main concern is to study actions on the circle but there is a preliminary question which deals with actions on the line. Of course, if a group acts on the line, we can always add a point at infinity to produce an action on the circle (with a common fixed point). In other words studying actions on the line is equivalent to studying actions on the circle with vanishing bounded Euler class. This is the reason why we begin by general remarks on groups acting on the line.

Observe first that the dynamics of a single orientation preserving homeomorphism h of $\mathbf{R}$ are very easy to describe. Let $F = Fix(h)$ be the set of fixed points. Each interval of the complement of F is h-invariant and the action of h on this interval is conjugate to a translation (positive or negative, according to the sign of $h(x) - x$ on this interval).

We say that a group Γ is *left orderable* if there exists a *total* ordering $\preceq$ on Γ which is invariant under left translations (*i.e.* $\gamma_1 \preceq \gamma_2$ implies $\gamma\gamma_1 \preceq \gamma\gamma_2$). We write $\gamma_1 \prec \gamma_2$ if $\gamma_1 \preceq \gamma_2$ and $\gamma_1 \neq \gamma_2$. An obvious necessary condition for a group to be left orderable is that it be torsion free (*i.e.* there is no non trivial element of finite order).

The following theorem is well known but we weren't able to find its origin in the literature.

THEOREM 6.8. *Let Γ be a countable group. Then the following are equivalent:*

1) *Γ acts faithfully on the real line by orientation preserving homeomorphisms.*

2) *Γ is left orderable.*

Proof. Suppose that Γ acts faithfully on the line by orientation preserving homeomorphisms, *i.e.* that there exists an injective homomorphism ϕ from Γ into the group $\mathrm{Homeo}_+(\mathbf{R})$ of orientation preserving homeomorphisms of the real line. Assume first that there is a point x_0 in $\mathbf{R}$ with trivial stabilizer. Then we can define a left invariant total ordering by defining $\gamma_1 \preceq \gamma_2$ if $\phi(\gamma_1)(x_0) \leq \phi(\gamma_2)(x_0)$. If there is no such point x_0, choose a sequence of points $(x_i)_{i\in\mathbf{N}}$ which is dense in the line. Now define $\gamma_1 \preceq \gamma_2$ if $\gamma_1 = \gamma_2$ or if the first i for which $\phi(\gamma_1)(x_i) \neq \phi(\gamma_2)(x_i)$ is such that $\phi(\gamma_1)(x_i) < \phi(\gamma_2)(x_i)$. This defines a left invariant total order on Γ.

Conversely, let $\preceq$ be a left invariant total order on the countable group Γ. Enumerate the elements of Γ, *i.e.*, choose a bijection $i \in \mathbf{N} \mapsto \gamma_i \in \Gamma$. We are going to construct inductively an increasing injection v of $(\Gamma, \preceq)$ in $(\mathbf{R}, \leq)$. Define $v(\gamma_0)$ arbitrarily and suppose that $v(\gamma_0), \ldots, v(\gamma_i)$ have been defined. If γ_{i+1} is smaller (resp. bigger) than all $\gamma_0, \ldots, \gamma_i$ then define $v(\gamma_{i+1})$ as any real number smaller (resp. bigger) than $\min(v(\gamma_0), \ldots, v(\gamma_i)) - 1$ (resp. $\max(v(\gamma_0), \ldots, v(\gamma_i)) + 1$). Otherwise, there is a pair of integers $0 \leq \alpha, \beta \leq i$ such that $\gamma_\alpha \prec \gamma_{i+1} \prec \gamma_\beta$ and such that there is no γ_j $(0 \leq j \leq i)$ between γ_α and γ_β. Then we define $v(\gamma_{i+1})$ as $(v(\gamma_\alpha) + v(\gamma_\beta))/2$. Let $\overline{X} \subset \mathbf{R}$ be the closure of $v(\Gamma)$.

By our construction, it is easy to verify that $\overline{X}$ is unbounded and that any connected component $]a, b[$ of the complement of $\overline{X}$ is such that a and b are in $v(\Gamma)$. The group Γ acts on itself by left translations so that every element γ of Γ induces an increasing bijection $\phi(\gamma)$ of $v(\Gamma)$. We claim that $\phi(\gamma)$ extends continuously to $\overline{X}$. Otherwise, there would exist a point $x = \lim_n v(\gamma_{i_n}) = \lim_m v(\gamma_{i_m})$ for an increasing sequence of elements γ_{i_n} and a decreasing sequence γ_{i_m} and such that $\lim_n v(\gamma\gamma_{i_n}) < \lim_m v(\gamma\gamma_{i_m})$. Then $a = \lim_n v(\gamma\gamma_{i_n})$ and $b = \lim_m v(\gamma\gamma_{i_m})$ would be the endpoints of some connected component of the complement of $\overline{X}$. By our previous observation, a and b would be the image by v of two distinct elements of Γ. On multiplying these two elements on the left by γ^{-1}, this would produce two distinct elements γ_- and γ_+ such that $v(\gamma_{i_n}) \leq v(\gamma_-) < v(\gamma_+) \leq v(\gamma_{i_m})$ and this contradicts the fact that the two sequences have the same limit x.

Therefore we have produced a homeomorphism $\phi(\gamma)$ of $\bar{X}$. We now extend $\phi(\gamma)$ to the whole line $\mathbf{R}$ in such a way that $\phi(\gamma)$ is affine on each interval of the complement of $\bar{X}$. It is now clear that ϕ is an injective homomorphism from Γ to the group of orientation preserving homeomorphisms of the real line. $\square$

Theorem 6.8 produces many examples of actions on the real line. For instance, suppose Γ is a countable group containing a nested sequence of subgroups $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_i \supset \ldots$ (finite or infinite) such that the intersection of this family reduces to the trivial element and that each Γ_i is a normal subgroup in the previous one Γ_{i-1}. Assume that each quotient $Q_i = \Gamma_i/\Gamma_{i-1}$ is left orderable and denote by $\preceq_i$ such a left order on Q_i. Let us construct a left order $\preceq$ on Γ. Consider two distinct elements γ, γ' in Γ and let i be the first integer such that $\gamma\gamma'^{-1}$ is not in Γ_i. Then $\gamma^{-1}\gamma'$ is in Γ_{i-1} and determines an element $[\gamma^{-1}\gamma']$ of Q_i. Then define $\gamma \preceq \gamma'$ if $[\gamma^{-1}\gamma'] \preceq_i 1$. This is a left invariant total order on Γ.

As an example, note that a countable torsion free abelian group A embeds in the tensor product $A \otimes \mathbf{Q}$ which is a $\mathbf{Q}$-vector space whose dimension is at most countable and therefore embeds in $\mathbf{R}$. Hence, countable torsion free abelian groups are orderable. Let us say that a group Γ is solvable (resp. residually solvable) if there is a finite (resp. infinite) decreasing sequence of subgroups as in the previous paragraph such that the quotient groups Q_i are abelian. We have now proved:

PROPOSITION 6.9. *Let Γ be a countable group which is (residually) solvable with torsion free abelian quotients. Then Γ acts faithfully on the real line by orientation preserving homeomorphisms.*

There are many examples of such groups: free groups or fundamental groups of closed orientable surfaces for instance have these properties [46]. Observe that the left orderings that we produced by the previous argument are in fact left *and right* invariant orderings. If we go back to the proof of Theorem 6.8 we can check that for bi-invariant ordered groups, the actions on the line $\phi\colon \Gamma \to \mathrm{Homeo}_+(\mathbf{R})$ produced by the proof are very peculiar: they are such that for every non trivial $\gamma \in \Gamma$, we have either $\phi(\gamma)(x) \leq x$ for all $x \in \mathbf{R}$ or $\phi(\gamma)(x) \geq x$ for all x. In other words the graphs of $\phi(\gamma)$ don't cross the diagonal. However, there will be elements whose graphs touch the diagonals, unless of course the action is free, which is almost never the case because of the following well known theorem of Hölder.

THEOREM 6.10 (Hölder). *If a group acts freely on the real line by homeomorphisms, it is abelian. More precisely, such a group embeds as a subgroup of* $\mathbf{R}$ *and the action is semi-conjugate to a group of translations. In the same way, a group acting freely on the circle is abelian, embeds in* $\mathrm{SO}(2)$, *and is semi-conjugate to a group of rotations.*

Proof. Let $\phi\colon \Gamma \to \mathrm{Homeo}_+(\mathbf{R})$ be a homomorphism such that for all γ different from the identity the homeomorphism $\phi(\gamma)$ has no fixed point. If γ, γ' are elements of Γ, write $\gamma \preceq \gamma'$ if $\phi(\gamma)(0) \leq \phi(\gamma')(0)$ (which implies $\phi(\gamma)(x) \leq \phi(\gamma')(x)$ for all x since the action is free). This defines a left and right invariant ordering $\preceq$ which is archimedean, *i.e.* such that for any pair of non trivial elements γ, γ' for which $id \prec \gamma$ and $id \prec \gamma'$, there is a positive integer n such that $\gamma' \prec \gamma^n$. Indeed, the sequence $\phi(\gamma)^n(0)$ is increasing and has to tend to ∞ since otherwise its limit would be a fixed point of $\phi(\gamma)$; hence for n sufficiently large we have $\phi(\gamma')(0) \leq \phi(\gamma^n)(0)$.

Then we show that any group Γ equipped with a bi-invariant total archimedean ordering embeds in $\mathbf{R}$ and is therefore abelian. Fix a non trivial element γ_0 such that $id \prec \gamma_0$ and for each $\gamma \in \Gamma$, define $\Phi(\gamma)$ as the smallest integer $k \in \mathbf{Z}$ such that $\gamma \preceq \gamma_0^k$. We have

$$\gamma_0^{\Phi(\gamma)-1} \prec \gamma \preceq \gamma_0^{\Phi(\gamma)}.$$

This defines a map $\Phi\colon \Gamma \to \mathbf{Z}$ which satisfies

$$\Phi(\gamma) + \Phi(\gamma') - 1 < \Phi(\gamma\gamma') \leq \Phi(\gamma) + \Phi(\gamma')$$

so that Φ is a quasi-homomorphism. As we have already observed, $\phi(\gamma) = \lim_{n\to\infty} \Phi(\gamma^n)/n$ exists and defines a quasi-homomorphism $\phi\colon \Gamma \to \mathbf{R}$ which is homogeneous (*i.e.* $\phi(\gamma^n) = n\phi(\gamma)$) and which is increasing (*i.e.* $\gamma \preceq \gamma'$ implies $\phi(\gamma) \leq \phi(\gamma')$). Note that $\phi(\gamma_0) = 1$.

We claim that ϕ is a group homomorphism. Indeed, consider two elements γ, γ' in Γ and assume for instance that $\gamma\gamma' \preceq \gamma'\gamma$. It follows easily by induction that for every positive integer n, we have $\gamma^n\gamma'^n \preceq (\gamma\gamma')^n \preceq \gamma'^n\gamma^n$. Evaluating Φ on this inequality, we get

$$\Phi(\gamma^n) + \Phi(\gamma'^n) - 1 \leq \Phi((\gamma\gamma')^n) \leq \Phi(\gamma^n) + \Phi(\gamma'^n).$$

Dividing by n and taking the limit, we obtain

$$\phi(\gamma) + \phi(\gamma') \leq \phi(\gamma\gamma') \leq \phi(\gamma) + \phi(\gamma')$$

so that ϕ is indeed a homomorphism.

We still have to show that ϕ is injective. For any γ such that $id \prec \gamma$ we know, since the ordering is archimedean, that there is some positive integer k

such that $\gamma_0 \preceq \gamma^k$. It follows that $1 \leq k\phi(\gamma)$ so that $\phi(\gamma)$ is non trivial. This proves the injectivity of ϕ.

Observe that the non decreasing embedding ϕ of Γ in $\mathbf{R}$ is unique up to a multiplicative constant. Indeed, if ϕ' is another one, we have by definition $(\Phi(\gamma^n)-1)\phi'(\gamma_0) \leq \phi'(\gamma^n) \leq \Phi(\gamma^n)\phi'(\gamma_0)$. Dividing by n and taking the limit, we get $\phi' = \phi'(\gamma_0).\phi$.

We now show that the action of Γ is semi-conjugate to a group of translations. If Γ is isomorphic to $\mathbf{Z}$, it acts freely and properly on the line so that it is indeed conjugate to the group of integral translations. Otherwise, $\phi(\Gamma)$ is dense in $\mathbf{R}$. Let x be any point in $\mathbf{R}$ and define

$$h(x) = \sup\{\phi(\gamma) \in \mathbf{R} \mid \gamma(0) \leq x\}\,.$$

Clearly, h is non decreasing and satisfies $h(\gamma(x)) = h(x) + \phi(\gamma)$ identically. The continuity of h is easy and follows from the density of the group $\phi(\Gamma)$: if h were not continuous, the interior of $\mathbf{R} \setminus h(\mathbf{R})$ would be a non empty open set invariant by all translations in $\phi(\Gamma)$.

The proof for groups acting on the circle follows easily: if Γ is a group acting freely on the circle, its inverse image in $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ acts freely on the line. □

The following is an elementary corollary of the previous theorem.

PROPOSITION 6.11. *Let Γ be a torsion group (i.e. such that every element in Γ has finite order). Then any homomorphism from Γ to* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *has abelian image.*

Proof. We know the structure of elements of finite order of $\mathrm{Homeo}_+(\mathbf{S}^1)$: they are conjugate to rotations of finite order. It follows that an element having a fixed point and of finite order in $\mathrm{Homeo}_+(\mathbf{S}^1)$ is the identity. In other words, a torsion group acting faithfully on the circle acts freely. The result follows from 6.10. □

There is another very interesting example of a group which admits a left and right invariant total ordering: the group $\mathrm{PL}_+([0,1])$ of orientation preserving piecewise linear homeomorphisms of the interval $[0,1]$. Indeed, let γ, γ' be two distinct elements of $\mathrm{PL}_+([0,1])$ and consider the largest real number $x \in [0,1]$ such that γ and γ' coincide on the interval $[0,x]$. Then for $\epsilon > 0$ small enough, we have either $\gamma(t) < \gamma'(t)$ for $t \in \,]x, x+\epsilon]$ or $\gamma(t) > \gamma'(t)$ for $t \in \,]x, x+\epsilon]$. Say that $\gamma \prec \gamma'$ in the first case and $\gamma' \prec \gamma$ in the second case. This defines a total ordering on $PL_+([0,1])$ and it is clearly left and right invariant. We can induce this ordering on countable subgroups of $PL_+([0,1])$, for instance the subgroup of elements with rational slopes and apply the general construction that we described above. We get an action of this rational group on the line which is very different from the given action of $PL_+([0,1])$ on $]0,1[$: the corresponding graphs don't cross the diagonal.

Remark that an affine bijection of the line $x \mapsto ax+b$ has at most one fixed point (if it is not the identity). Solodov proved that this property essentially characterizes groups of affine transformations.

THEOREM 6.12 (Solodov). *Let Γ be a non abelian subgroup of* $\mathrm{Homeo}_+(\mathbf{R})$ *such that every element (different from the identity) has at most one fixed point. Then Γ is isomorphic to a subgroup of the affine group* $\mathrm{Aff}_+(\mathbf{R})$ *of the real line, and the action of Γ on the line is semi-conjugate to the corresponding affine action.*

Solodov did not publish a proof but mentions his result in [62] and explained it to the author of these notes in 1991. Later T. Barbot needed this theorem for his study of Anosov flows and published a proof in [3]. More recently, N. Kovačević published an independent proof in [43]. See also the recent preprint [20] for a detailed proof.

Proof. Let Γ be a subgroup of $\mathrm{Homeo}_+(\mathbf{R})$ such that every element (different from the identity) has at most one fixed point. If no non trivial element has a fixed point, Hölder's Theorem 6.10 implies that Γ is abelian (and that the action is semi-conjugate to a group of translations). If there is a point x which is fixed by the full group Γ, then one can restrict the action to the two components of $\mathbf{R} \setminus \{x\}$ on which we can use Hölder's theorem again: this would imply that Γ is abelian.

We claim that Γ contains an element γ with a repulsive fixed point x, i.e. such that $\gamma(y) > y$ for every $y > x$ and $\gamma(y) < y$ for every $y < x$. Indeed choose some non trivial γ_0 in Γ fixing some x_0. If x_0 is not repulsive for γ_0 and for γ_0^{-1}, this means that x_0 is a parabolic fixed point, *i.e.* replacing γ_0 by its inverse, we have $\gamma_0(y) > y$ for all $y \neq x_0$. Conjugating γ_0 by some element which does not fix x_0, we get an element γ_1 fixing some x_1 and such that $\gamma_1(y) > y$ for $y \neq x_1$. Assume for instance $x_0 < x_1$ and consider the element $\gamma = \gamma_0\gamma_1^{-1}$. Obviously, one has $\gamma(x_0) < x_0$ and $\gamma(x_1) > x_1$ and since we know that γ has at most one fixed point, γ must have a repulsive fixed point between x_0 and x_1 as we claimed.

Now, we can try to mimic the proof of Hölder's theorem. Consider two elements γ and γ' of Γ. Write $\gamma \preceq \gamma'$ if there is some $x \in \mathbf{R}$ such that $\gamma(y) \leq \gamma'(y)$ for all $y > x$. Clearly, our assumptions imply that this defines a total ordering on Γ which is left and right-invariant. Denote by Γ^+ the subset of elements of $\Gamma \setminus \{id\}$ such that $id \preceq \gamma$.

The next claim is a weak form of the archimedean property. *Fix some γ_0 in Γ^+ with a repulsive fixed point x_0, and let γ be any other element of Γ^+. Then there exists some positive integer k such that $\gamma \preceq \gamma_0^k$.* Indeed, choose some real numbers x_-, x_+ such that $x_- < x_0 < x_+$. For k big enough, one has $\gamma_0^k(x_-) < \gamma(x_-)$ and $\gamma_0^k(x_+) > \gamma(x_+)$ since x_0 is repulsive. It follows that $\gamma^{-1}\gamma_0^k$ has a fixed point in the interval $[x_-, x_+]$ which is therefore the unique fixed point of $\gamma^{-1}\gamma_0^k$. Hence we have $\gamma_0^k(y) > \gamma^{-1}(y)$ for all $y > x_+$ and $\gamma \preceq \gamma_0^k$. This proves our last claim.

Again, we fix some γ_0 in Γ^+ with a repulsive fixed point x_0. For each $\gamma \in \Gamma^+$ we define $\Phi(\gamma) \in \mathbf{N}$ to be the smallest integer k such that $\gamma \preceq \gamma_0^k$. If $\gamma^{-1} \in \Gamma^+$, we let $\Phi(\gamma) = -\Phi(\gamma^{-1})$ and finally we define $\Phi(id) = 0$. This defines a map $\Phi\colon \Gamma \to \mathbf{Z}$. Then we can copy from the proof of Hölder's theorem: Φ is a quasi-homomorphism and the limit $\phi(\gamma) = \lim_{n\to\infty} \Phi(\gamma^n)/n$ exists and defines a group homomorphism $\phi\colon \Gamma \to \mathbf{R}$.

It follows in particular that the first commutator group $[\Gamma, \Gamma]$ is contained in the kernel of ϕ. The final observation is that this kernel acts freely on the line. Otherwise, we saw that $Ker(\phi)$ would contain some element γ with a repulsive fixed point and we have already observed that this implies the existence of some integer k such that $\gamma_0 \preceq \gamma^k$ which in turn implies that $\phi(\gamma) \geq 1/k \neq 0$, a contradiction. Using Hölder's theorem, we conclude that $[\Gamma, \Gamma]$ is abelian.

We know the structure of free actions (of abelian groups) on the line: they are semi-conjugate to translation groups. More precisely, we know that there is a map $h\colon \mathbf{R} \to \mathbf{R}$ and an injective homomorphism $\psi\colon [\Gamma, \Gamma] \to \mathbf{R}$ which are such that for every $\gamma \in [\Gamma, \Gamma]$ and $x \in \mathbf{R}$, one has: $h(\gamma(x)) = h(x) + \psi(\gamma)$. If the image $\psi([\Gamma, \Gamma])$ is non discrete, this map h is unique up to post-composition by an affine map. So assume first that $\psi([\Gamma, \Gamma])$ is non discrete. Note that $[\Gamma, \Gamma]$ is a normal subgroup of Γ. It follows that for every γ in Γ, the map $h \circ \gamma$ coincides with h up to some affine map. This means precisely that h realizes a semi-conjugacy between Γ and some group of affine transformations of $\mathbf{R}$ and shows that Γ is indeed isomorphic to a subgroup of $\mathrm{Aff}(\mathbf{R})$. To finish the proof, we still have to show that $\psi([\Gamma, \Gamma])$ cannot be discrete, *i.e.* isomorphic to $\mathbf{Z}$. In this case, inner conjugacies by an element $\gamma \in \Gamma$ have to preserve the generator 1 of $\mathbf{Z}$ (the unique generator which is bigger that the identity in our ordering). This means that $\mathbf{Z}$ ($\simeq [\Gamma, \Gamma]$) lies in the center of Γ. This is not possible since for every fixed point x of an element γ of Γ, its orbit under $\mathbf{Z}$ would consist of fixed points of γ. $\square$

Hölder's theorem essentially characterizes translation groups as groups acting on the line with no fixed points. Solodov's theorem essentially characterizes groups of affine transformations as groups acting on the line with at most one fixed point. It is very tempting to try to prove a similar characterization of groups of projective transformations as groups acting on the circle with at most two fixed points… Unfortunately, this is not the case! N. Kovačević recently constructed a nice counter-example in [44].

THEOREM 6.13 (Kovačević). *There exists a finitely generated subgroup of* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *such that every element different from the identity has at most two fixed points, such that all orbits are dense, and which is not conjugate to a subgroup of* $\mathrm{PSL}(2,\mathbf{R})$.

Nevertheless, there is a very important characterization of groups which are conjugate to subgroups of $\mathrm{PSL}(2,\mathbf{R})$. This characterization is due to Casson-Jungreis and Gabai [15, 24], following earlier work of Tukia. We would have liked to include a discussion and a proof of this result, but that would be too long and we have to limit ourselves to a statement! Consider a sequence γ_n of elements of $\mathrm{Homeo}_+(\mathbf{S}^1)$. Let us say that γ_n has the *convergence property* if it contains a subsequence γ_{n_k} which satisfies one of the following two properties:

- γ_{n_k} is equicontinuous;
- there exist two points x, y on the circle such that γ_{n_k} (resp. $\gamma_{n_k}^{-1}$) converges to a constant map on each compact interval in $\mathbf{S}^1 \setminus \{x\}$ (resp. in $\mathbf{S}^1 \setminus \{y\}$).

A subgroup Γ of $\mathrm{Homeo}_+(\mathbf{S}^1)$ is called a *convergence group* if every sequence of elements of Γ has the convergence property.

THEOREM 6.14 (Casson-Jungreis, Gabai). *A subgroup of* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *is conjugate to a subgroup of* $\mathrm{PSL}(2,\mathbf{R})$ *if and only if it is a convergence group.*

The reader should at least be able to prove the easy part of the theorem: subgroups of $\mathrm{PSL}(2,\mathbf{R})$ are convergence groups!

We revert now to groups acting on the circle. We state a general criterion which characterizes the bounded classes coming from some action.

THEOREM 6.15 ([25]). *Let* Γ *be a countable group and* c *a class in* $H^2_b(\Gamma, \mathbf{Z})$. *Then there exists a homomorphism* $\phi\colon \Gamma \to \mathrm{Homeo}_+(\mathbf{S}^1)$ *such that* $\phi^\star(eu) = c$ *if and only if* c *can be represented by a cocycle which takes only the values* 0 *and* 1.

Proof. Of course, the necessary condition is clear from 6.3 and the main difficulty will be to construct some action from a cocycle taking two values. Let c be a 2-cocycle on the group Γ taking only the values 0 and 1. We saw that a central extension and a section lead to a 2-cocycle. The process can be reversed and we can construct a central extension $\widetilde{\Gamma}$ in the following way from a 2-cocycle c. As a set, $\widetilde{\Gamma}$ is the product $\mathbf{Z} \times \Gamma$ and we define a multiplication $\bullet$ by:

$$(n_1, \gamma_1) \bullet (n_2, \gamma_2) = (n_1 + n_2 + \bar{c}(\gamma_1, \gamma_2), \gamma_1\gamma_2)$$

where, as usual, $\bar{c}$ denotes the inhomogeneous cocycle associated to c. The fact that $\widetilde{\Gamma}$ is a group is a restatement of the fact that c is a cocycle. The projection $\widetilde{\Gamma} \to \Gamma$ is a group homomorphism.

Assume first that the cocycle c *is non degenerate, i.e.* that $\bar{c}(id, \gamma) = \bar{c}(\gamma, id) = 0$ for every γ in Γ (where id denotes the identity element in Γ). Then the identity element of $\widetilde{\Gamma}$ is $(0, id)$ and the map $n \in \mathbf{Z} \mapsto (n, id) \in \widetilde{\Gamma}$ is also a group homomorphism. Hence, we have a central extension

$$0 \longrightarrow \mathbf{Z} \longrightarrow \widetilde{\Gamma} \longrightarrow \Gamma \longrightarrow 1\,.$$

The fact that c takes non negative values means that the subset P of $\widetilde{\Gamma}$ consisting of elements of the form (n, γ) with $n \geq 0$ is a semi-group, *i.e.* is stable under the product $\bullet$. Moreover, since c takes the values 0 and 1, the inverse of (n, γ) is $(-n, \gamma^{-1})$ or $(-n-1, \gamma^{-1})$. It follows that every element of $\widetilde{\Gamma}$ belongs to P or to its inverse. In other words, if one defines $\widetilde{\gamma}_1 \preceq \widetilde{\gamma}_2$ if $\widetilde{\gamma}_2\widetilde{\gamma}_1^{-1} \in P$ we get a total pre-order on $\widetilde{\Gamma}$ which is left invariant. Denote by $\mathbf{t}$ the element $(1, id)$ in $\widetilde{\Gamma}$. Note that for every $\widetilde{\gamma}$ in $\widetilde{\Gamma}$ we have $\widetilde{\gamma} \preceq \mathbf{t}\widetilde{\gamma}$.

The end of the proof mimics 6.8: One constructs a map $v\colon \widetilde{\Gamma} \to \mathbf{R}$ such that $\widetilde{\gamma}_1 \preceq \widetilde{\gamma}_2$ if and only if $v(\widetilde{\gamma}_1) \leq v(\widetilde{\gamma}_2)$ and such that $v(\widetilde{\gamma}\mathbf{t}) = v(\widetilde{\gamma}) + 1$ for every $\widetilde{\gamma} \in \widetilde{\Gamma}$. We may even choose v in such a way that the action of $\widetilde{\Gamma}$ on itself by left translations defines an action on $v(\widetilde{\Gamma}) \subset \mathbf{R}$ which extends to its closure. Then we extend this action of $\widetilde{\Gamma}$ to $\mathbf{R}$ using affine maps in the connected components of the complement of this closure. Finally, since $\mathbf{t}$ acts on $\mathbf{R}$ by the translation by 1, we get an action of the quotient group Γ on the circle $\mathbf{R}/\mathbf{Z}$. This construction was carried out in such a way that it is clear that the bounded Euler class of this action is precisely the class of the cocycle c.

Finally, we have to deal with the case of degenerate cocycles c. Note that the fact that c is a cocycle can be expressed by the identity:

$$\overline{c}(\gamma_1, \gamma_2) + \overline{c}(\gamma_1\gamma_2, \gamma_3) = \overline{c}(\gamma_2, \gamma_3) + \overline{c}(\gamma_1, \gamma_2\gamma_3).$$

It follows that there exists an integer $\nu = 0$ or 1 such that for every γ in Γ we have $\overline{c}(1, \gamma) = \overline{c}(\gamma, 1) = \nu$. The fact that c is degenerate means that $\nu = 1$. Then we can define $c' = 1 - c$. This is a new cocycle which is non degenerate and takes only the values 0 and 1. By the previous construction, we get an action of Γ on the circle corresponding to the bounded class of c'. Reversing the orientation of the circle, we get finally an action of Γ on the circle whose bounded Euler class is the class of c. □

6.6 Some examples

Recall that a group Γ is called *perfect* if every element is a product of commutators. It is *uniformly perfect* if there is an integer k such that every element is a product of at most k commutators. For such a uniformly perfect group, every quasi-homomorphism from Γ to $\mathbf{R}$ is bounded (since it is bounded on a single commutator) so that the canonical map from $H_b^2(\Gamma, \mathbf{R})$ to $H^2(\Gamma, \mathbf{R})$ is injective. Moreover the map from $H_b^2(\Gamma, \mathbf{Z})$ to $H_b^2(\Gamma, \mathbf{R})$ is also injective since there is no homomorphism from Γ to $\mathbf{R}$. In such a situation, the usual Euler class in $H^2(\Gamma, \mathbf{Z})$ determines the bounded Euler class, and therefore most of the topological dynamics of a group action.

An example of such a group is $\mathrm{SL}(n, \mathbf{Z})$ which is uniformly perfect for $n \geq 3$ and which, moreover is such that $H^2(\mathrm{SL}(n, \mathbf{Z}), \mathbf{Z}) = 0$ (for $n \geq 3$) [52]. As a corollary, we get immediately that *for* $n \geq 3$, *any action of* $\mathrm{SL}(n, \mathbf{Z})$ *on the circle has a fixed point.* This will be strengthened later in 7.1. Some other matrix groups have this property: see for instance [5, 14].

Consider the case of the Thompson group G. We can show that every element in G is a product of two commutators (see [28]) and that $H^2(G, \mathbf{Z})$ is isomorphic to $\mathbf{Z}$. Using the Milnor-Wood inequality we can show that in $H^2(G, \mathbf{Z})$ only the elements $-1, 0, +1$ have a norm less than or equal to $1/2$. Hence we deduce that *any non-trivial action of the Thompson group* G *on the circle is semi-conjugate to the canonical action given by its embedding in* $\mathrm{PL}_+(\mathbf{S}^1)$ or to the reverse embedding obtained by conjugating by an orientation reversing homeomorphism of the circle (see [28] for more details).

Another situation where the bounded cohomology is easy to compute is the case of amenable groups. Let Γ be topological group (which will be frequently a discrete countable group) and denote by $C_b^0(\Gamma)$ the real vector space of bounded continuous functions on Γ with real values. We say that Γ is *amenable* if there is a linear operator $m\colon C_b^0(\Gamma) \to \mathbf{R}$ called a "mean" such that m is non negative on non negative elements, is equal to 1 on the constant function 1 and is invariant under left translations by elements of Γ. See the book [29] for a good description of the theory of these groups. Of course, compact groups are amenable: it suffices to define m as the integral over the Haar measure. Abelian groups are amenable. A closed subgroup of a locally compact amenable group is amenable and an increasing union of amenable groups is amenable. The category of amenable groups is also stable under extensions. In particular, solvable groups are amenable. The following is due to Johnson (see [39]).

THEOREM 6.16 (Johnson). *If Γ is an amenable group then its real bounded cohomology groups $H_b^k(\Gamma, \mathbf{R})$ are trivial for all $k \geq 0$.*

Proof. Strictly speaking, we only defined cohomology and bounded cohomology for discrete groups... but of course we could have done it for a general topological group. Since in any case we don't need this fact for non discrete groups, we assume Γ is a discrete amenable group equipped with a mean m. Let $c\colon \Gamma^{k+1} \to \mathbf{R}$ be a bounded k-cochain. Then we can define $\overline{m}(c)\colon \Gamma^k \to \mathbf{R}$ by taking the mean value with respect to the first variable. This linear operator $\overline{m}\colon C_b^k(\Gamma, \mathbf{R}) \to C_b^{k-1}(\Gamma, \mathbf{R})$ is an algebraic homotopy between the identity and 0, *i.e.* we have $d_{k-1}\overline{m} \pm \overline{m} d_k = id$. It implies immediately that a bounded cocycle is a bounded coboundary. $\square$

Let Γ be an amenable subgroup of $\mathrm{Homeo}_+(\mathbf{S}^1)$ and let $\widetilde{\Gamma}$ be the group of lifts in $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$: this is also an amenable group since it is an extension of the amenable group $\mathbf{Z}$ by the amenable group Γ. The translation number map $\tau\colon \widetilde{\Gamma} \to \mathbf{R}$ is a quasi-homomorphism and is a homomorphism on one generator subgroups; the vanishing of bounded cohomology therefore implies that it is a homomorphism. *The rotation number is a homomorphism when restricted to an amenable group.*

If Γ is an amenable group, the group $H_b^2(\Gamma, \mathbf{Z})$ can easily be determined. Indeed, we know that $H_b^2(\Gamma, \mathbf{R}) = 0$ and that the kernel of the map from $H^2(\Gamma, \mathbf{Z})$ to $H^2(\Gamma, \mathbf{R})$ is the quotient group $H^1(\Gamma, \mathbf{R})/H^1(\Gamma, \mathbf{Z})$. We have therefore proved the following:

PROPOSITION 6.17. *Let Γ be an amenable group and $\phi\colon \Gamma \to \mathrm{Homeo}_+(\mathbf{S}^1)$ a homomorphism. Then the rotation number map $\rho \circ \phi\colon \Gamma \to \mathbf{R}/\mathbf{Z}$ is a homomorphism. If the image of this homomorphism is finite, then $\phi(\Gamma)$ has a finite orbit of the same cyclic structure. Otherwise, ϕ is semi-conjugate to the rotation group $\rho \circ \phi(\Gamma)$.*

Note that there is another approach to the proof of this proposition, using invariant probability measures. Indeed, let Γ be an amenable group acting on the circle by some homomorphism $\phi\colon \Gamma \to \mathrm{Homeo}_+(\mathbf{S}^1)$. If $u\colon \mathbf{S}^1 \to \mathbf{R}$ is a continuous function, we can consider the mean value of the bounded function $\gamma \in \Gamma \mapsto u(\phi(\gamma)(0))$. This gives a linear functional on the space of continuous functions u on the circle, equal to 1 on the function 1, *i.e.* this mean value has the form $\int_{\mathbf{S}^1} u\, d\mu$ for some probability measure μ on the circle. Of course this probability measure is invariant under $\phi(\Gamma)$. Assume now that μ has some non trivial atom, *i.e.* that some point x has some positive mass $\mu(\{x\}) > 0$. Then there is a finite number of atoms of the same mass so that we get a finite orbit for $\phi(\Gamma)$. If there is no atom, then there is a degree 1 map of the circle to itself which sends the measure μ to the Lebesgue measure since in this case the measure of an interval depends continuously on its endpoints. This map collapses each component of the complement of the support of μ to a point. This provides a semi-conjugacy of ϕ with a group of homeomorphisms preserving the Lebesgue measure, *i.e.* a rotation group. This gives another proof of Proposition 6.17.

Invariant probability measures also provide another definition of translation and rotation numbers. Let f be any element of $\mathrm{Homeo}_+(\mathbf{S}^1)$. The qualitative description of the topological dynamics of f that we gave in 5.9 enables us to describe explicitly the probability measures μ on $\mathbf{S}^1$ which are invariant by f.

If the rotation number of f vanishes, the invariant probability measures are characterized by the fact that their support is contained in the fixed point set $Fix(f)$ of f. Indeed we know that the action of f on a connected component of the complement of $Fix(f)$ is conjugate to the translation by 1 on $\mathbf{R}$ and cannot preserve any non trivial finite measure.

If the rotation number is rational, invariant probability measures are concentrated on the set of periodic points.

If the rotation number is irrational and the orbits are dense, we know that f is conjugate to an irrational rotation. In this case, there is a unique invariant probability measure which is the image of the Lebesgue measure by the topological conjugacy (see [41]). If the orbits are not dense, there is an exceptional minimal set $K \subset \mathbf{S}^1$ and the support of any invariant probability has to coincide with K since we know that the connected components of $\mathbf{S}^1 - K$ are wandering intervals. In this case also there is a unique invariant probability μ which is the unique probability which maps to the Lebesgue measure by the degree 1 semi-conjugacy with a rotation.

Let $\widetilde{f}$ be an element of $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ and μ a probability measure on $\mathbf{S}^1$ which is invariant by the corresponding homeomorphism of the circle $f = p(\widetilde{f})$. The function $\widetilde{f}(x) - x$ is $\mathbf{Z}$-periodic and therefore defines a function on $\mathbf{R}/\mathbf{Z}$ that we can integrate with respect to μ. It should be clear to the reader by now that the result is nothing more than the translation number $\tau(\widetilde{f})$. Suppose now that $\widetilde{f}$ and $\widetilde{g}$ are two elements of $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ such that $p(\widetilde{f})$ and $p(\widetilde{g})$ preserve the same measure μ. Note that $\widetilde{f}\widetilde{g}(x) - x = (\widetilde{f}(\widetilde{g}x) - \widetilde{g}(x)) + (\widetilde{g}(x) - x)$ and integrate with respect to μ. We get that $\tau(\widetilde{f}\widetilde{g}) = \tau(\widetilde{f}) + \tau(\widetilde{g})$. So we have proved the following:

PROPOSITION 6.18. *Let μ be a probability measure on the circle. Denote by* $\mathrm{Homeo}_+(\mathbf{S}^1, \mu)$ *the subgroup of* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *consisting of homeomorphisms preserving μ. Then the rotation number* $\rho : \mathrm{Homeo}_+(\mathbf{S}^1, \mu) \to \mathbf{R}/\mathbf{Z}$ *is a homomorphism.*

Of course, in many situations the groups $H_b^2(\Gamma, \mathbf{R})$ can be infinite dimensional. For instance, this is the case of a free non abelian group, of the fundamental group of a closed orientable surface of genus $g > 1$ [4] and more generally of non elementary Gromov hyperbolic groups [19]. This is not a surprise since there are many homomorphisms from a free group for instance to $\mathrm{Homeo}_+(\mathbf{S}^1)$ and their bounded Euler classes are usually distinct.

In some cases, the bounded Euler class of a specific action on the circle might be useful to understand the structure of the group. Suppose for example that a group Γ is such that $H^1(\Gamma, \mathbf{R}) = H^2(\Gamma, \mathbf{R}) = 0$ and that we are given a homomorphism $\phi \colon \Gamma \to \mathrm{Homeo}_+(\mathbf{S}^1)$. Then the image of the bounded Euler class $eu(\phi)$ in $H^2(\Gamma, \mathbf{Z})$ vanishes so that there is a (usually non bounded) quasi-homomorphism $\psi \colon \Gamma \to \mathbf{R}$ such that the bounded Euler cocycle $\phi^\star(c)$ is the coboundary of the 1-cochain $\psi(\gamma_1^{-1}\gamma_0)$. Modifying ψ by a bounded amount, we can assume that ψ is a homomorphism on one generator groups. With this condition, ψ is uniquely defined since we assumed that there is no homomorphism from Γ to $\mathbf{R}$. Of course, for any γ in Γ, the projection of $\psi(\gamma)$ in $\mathbf{R}/\mathbf{Z}$ is nothing more than the rotation number of $\phi(\gamma)$. Summing up, *with these algebraic conditions on the group* Γ, *any action of* Γ *on the circle determines canonically a quasi-homomorphism* $\psi\colon \Gamma \to \mathbf{R}$ *which is a lift of the rotation number map.*

A specific example is the modular group $\mathrm{PSL}(2,\mathbf{Z})$. As a group, it is isomorphic to the free product of two cyclic groups: $\mathrm{PSL}(2,\mathbf{Z}) \simeq \mathbf{Z}/2\mathbf{Z} \star \mathbf{Z}/3\mathbf{Z}$ (see for instance [61]). Of course there is no non-trivial homomorphism from this group to $\mathbf{R}$ since it is generated by two elements of finite order. In the same way, its second real cohomology group is trivial (this follows for instance from the Mayer-Vietoris exact sequence since finite groups have trivial cohomology over the reals). We deduce that every action of $\mathrm{PSL}(2,\mathbf{Z})$ on the circle yields a well defined quasi-homomorphism $\psi\colon \mathrm{PSL}(2,\mathbf{Z}) \to \mathbf{R}$ lifting the rotation number. If we start with the canonical action of $\mathrm{PSL}(2,\mathbf{Z})$ on the circle $\mathbf{RP}^1$, the rotation numbers are not interesting: the only elliptic elements in $\mathrm{PSL}(2,\mathbf{Z})$ have order 2 and 3 so that the rotation number of elements in $\mathrm{PSL}(2,\mathbf{Z})$ are $0, 1/2, 1/3, 2/3 \in \mathbf{R}/\mathbf{Z}$. However the quasi-homomorphism $\Psi\colon \mathrm{PSL}(2,\mathbf{Z}) \to \mathbf{R}$ that we get is very interesting and has been studied in many different contexts: it is called the *Rademacher function*. The explicit formula giving Ψ as a function of the entries of a matrix in $\mathrm{PSL}(2,\mathbf{Z})$ involves the so called Dedekind sums which are important in number theory. We refer to [4] for a description of Ψ and a bibliography on this very nice subject.

7. Higher rank lattices

In this section, we study the problem of determining which lattices in semi-simple groups can act on the circle.

Let G be any Lie group and $\mathfrak{G}$ be its Lie algebra. The *real rank* of G is the maximal dimension of an abelian subalgebra $\mathfrak{A}$ such that for every $a \in \mathfrak{A}$ the linear operator $ad(a)\colon \mathfrak{G} \to \mathfrak{G}$ is diagonalizable over $\mathbf{R}$. For instance, the real rank of $\mathrm{SL}(n,\mathbf{R})$ is $n-1$: its Lie algebra consists of traceless matrices and contains the abelian diagonal traceless matrices. A *lattice* in a Lie group G is a discrete subgroup Γ such that the quotient G/Γ has finite measure with respect to a right invariant Haar measure. A lattice in a *semi-simple group* is called *reducible* if we can find two normal subgroups G_1, G_2 in G, connected and non trivial, which generate G, whose intersection is contained in the (discrete) center of G, and such that $(G_1 \cap \Gamma).(G_2 \cap \Gamma)$ has finite index in Γ. Otherwise, we say that Γ is *irreducible*. Note that lattices in simple Lie groups are obviously irreducible.

The first example of a lattice is $\mathrm{SL}(n,\mathbf{Z})$ in $\mathrm{SL}(n,\mathbf{R})$: the corresponding quotient has finite volume (but is not compact).

Another example to keep in mind is the following. Consider the field $\mathbf{Q}(\sqrt{2})$ and its ring of integers $\mathcal{O}=\mathbf{Z}[\sqrt{2}]$. The field $\mathbf{Q}(\sqrt{2})$ has two embeddings in $\mathbf{R}$ given by $a+b\sqrt{2}\in\mathbf{Q}(\sqrt{2})\mapsto a\pm b\sqrt{2}\in\mathbf{R}$. This gives two embeddings of the group $\mathrm{SL}(2,\mathcal{O})$ in $\mathrm{SL}(2,\mathbf{R})$. The images of these embeddings are dense but the embedding of $\mathrm{SL}(2,\mathcal{O})$ in $\mathrm{SL}(2,\mathbf{R})\times\mathrm{SL}(2,\mathbf{R})$ has a discrete image which is an irreducible lattice in $\mathrm{SL}(2,\mathbf{R})\times\mathrm{SL}(2,\mathbf{R})$ (whose real rank is 2). Of course, we can construct many more examples using this kind of arithmetic construction: Borel showed for instance that any semi-simple Lie group (with no compact factor) contains at least an irreducible lattice (and even a cocompact one).

Note also that if a compact oriented manifold M of dimension n admits a metric with constant negative curvature, its universal cover is identified with the hyperbolic space H^n of dimension n. It follows that the fundamental group Γ of M is a discrete cocompact subgroup of the group of positive isometries of H^n which is the simple Lie group $\mathrm{SO}_0(n,1)$. These examples provide lattices in real rank 1 simple Lie groups.

For the theory of lattices in Lie groups, we refer to [48, 72].

7.1 WITTE'S THEOREM

In [70], Witte proves the following remarkable theorem:

THEOREM 7.1 (Witte). *Let Γ be a finite index subgroup of* $\mathrm{SL}(n,\mathbf{Z})$ *for* $n\geq 3$. *Then any homomorphism* $\phi\colon\Gamma\to\mathrm{Homeo}_+(\mathbf{S}^1)$ *has a finite image.*

The proof will be derived from the following

THEOREM 7.2 (Witte). *A finite index subgroup of* $\mathrm{SL}(n,\mathbf{Z})$ *for* $n\geq 3$ *is not left orderable.*

Proof. It suffices to prove it for a finite index subgroup Γ of $\mathrm{SL}(3,\mathbf{Z})$ since a subgroup of a left ordered group is of course left ordered. Suppose by contradiction that there is a left invariant total order $\preceq$ on Γ. Choose some integer $k\geq 1$ so that the following six elementary matrices belong to Γ:

$$a_1 = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix},$$

$$a_4 = \begin{pmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{pmatrix}, \quad a_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{pmatrix}.$$

It is easy to check the following relations between these matrices. Taking indices modulo 6, for every i the matrices a_i and a_{i+1} commute and the commutator of a_{i-1} and a_{i+1} is $a_i^{\pm k}$. Fix some i and let us analyze the structure of $\preceq$ on the group H_i generated by a_{i-1}, a_i, a_{i+1}. Allowing ourself to replace a_{i-1} or a_{i+1} by their inverses and to permute them, we can define three elements α, β, γ such that $\{\alpha, \beta\} = \{a_{i-1}^{\pm 1}, a_{i+1}^{\pm 1}\}$ and $\gamma = a_i^{\pm k}$ and such that the following conditions are satisfied:

$$\alpha\gamma = \gamma\alpha \quad ; \quad \beta\gamma = \gamma\beta \quad ; \quad \alpha\beta\alpha^{-1}\beta^{-1} = \gamma^{-1}$$

$$1 \prec \alpha \quad ; \quad 1 \prec \beta \quad ; \quad 1 \prec \gamma$$

(1 denotes the identity element). If ξ is an element of Γ, we set $|\xi| = \xi$ if $1 \preceq \xi$ and ξ^{-1} otherwise. If two elements ξ, ζ in Γ are such that $1 \prec \xi$ and $1 \prec \zeta$, we write $\xi \ll \zeta$ if for every integer $n \geq 1$, we have $\xi^n \prec \zeta$. We claim that $\gamma \ll \alpha$ or $\gamma \ll \beta$ (which implies that $|a_i| \ll |a_{i-1}|$ or $|a_i| \ll |a_{i+1}|$). Indeed, suppose that there is some integer $n \geq 1$ such that $\alpha \prec \gamma^n$ and $\beta \prec \gamma^n$ and let us compute

$$\delta_m = \alpha^m \beta^m (\alpha^{-1}\gamma^n)^m (\beta^{-1}\gamma^n)^m .$$

Since δ_m is a product of elements in Γ which are bigger than 1, we have $1 \prec \delta_m$. Now the product defining δ_m can easily be estimated since we know that γ commutes with α and β and that interchanging the order of an α and a β is compensated by the introduction of a γ. We find

$$\delta_m = \gamma^{-m^2+2mn} .$$

Since $1 \prec \gamma$, we know that γ to a negative power is less than 1. For m big enough, we get $\delta_m \prec 1$. This is a contradiction.

Coming back to our six matrices a_i, we find that $|a_i| \ll |a_{i-1}|$ or $|a_i| \ll |a_{i+1}|$. If we assume for instance $|a_1| \ll |a_2|$, we therefore deduce cyclically $|a_1| \ll |a_2| \ll |a_3| \ll |a_4| \ll |a_5| \ll |a_6| \ll |a_1|$, and this is a contradiction. □

Let us now prove Theorem 7.1 using similar ideas. Of course, Theorem 7.2 means that a finite index subgroup of $\mathrm{SL}(n, \mathbf{Z})$ for $n \geq 3$ does not act faithfully on the line (by orientation preserving homeomorphisms).

Consider first a torsion free finite index subgroup Γ of $SL(3,\mathbf{Z})$ and suppose by contradiction that there is an action $\phi\colon \Gamma \to \mathrm{Homeo}_+(\mathbf{S}^1)$ with infinite image. According to an important theorem, due to Margulis, every normal subgroup of a lattice in a simple Lie group of rank at least 2 is either of finite index or is finite (see [48, 64]). It follows that the action ϕ is faithful.

As in the proof of Theorem 7.2, choose an integer k such that the matrices $(a_i)_{i=1\dots 6}$ are in Γ. Note that the group H_i generated by a_{i-1}, a_i, a_{i+1} is nilpotent, hence amenable, so that the rotation number is a homomorphism when restricted to H_i. Since $a_i^{\pm k}$ is a commutator, it follows that the rotation numbers of all $\phi(a_i)$ vanish. Define A_i as being the unique lift of $\phi(a_i)$ whose translation number is 0. We claim that the elements A_i of $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ also satisfy the relations that for every i the homeomorphisms A_i and A_{i+1} commute and the commutator of A_{i-1} and A_{i+1} is $A_i^{\pm k}$. Indeed $A_iA_{i+1}A_i^{-1}A_{i+1}^{-1}$ and $A_{i+1}A_{i-1}A_{i+1}^{-1}A_{i-1}^{-1}A_i^{\mp k}$ project on the identity and have translation number 0 since the inverse image of H_i in $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ is nilpotent and the restriction of τ to this group is a homomorphism. Consider now the (left ordered) group of homeomorphisms of the line generated by the A_i. We can reproduce exactly the same argument that we used in Theorem 7.2 to get a contradiction.

Consider finally the general case of an action $\phi\colon \Gamma \to \mathrm{Homeo}_+(\mathbf{S}^1)$ of a finite index subgroup of $SL(n,\mathbf{Z})$ $(n \geq 3)$. Replacing Γ by a finite index subgroup, we can assume that Γ is torsion free. Of course, $SL(3,\mathbf{Z})$ is the subgroup of $SL(n,\mathbf{Z})$ consisting of matrices preserving $\mathbf{Z}^3 \simeq \mathbf{Z}^3 \times \{0\} \subset \mathbf{Z}^n$ and Γ intersects $SL(3,\mathbf{Z})$ on a subgroup of finite index in $SL(3,\mathbf{Z})$. Since we have already dealt with the case $n=3$, the kernel of ϕ contains a subgroup of finite index in the infinite group $\Gamma \cap SL(3,\mathbf{Z})$. By the theorem of Margulis that we mentioned, the kernel of ϕ is a subgroup of finite index in Γ so that the image of ϕ is a finite group. Theorem 7.1 is proved.

It turns out that the arguments used in this proof can be extended to a family of lattices more general than finite index subgroups of $SL(n,\mathbf{Z})$ for $n \geq 3$. The general situation in which Witte proves his theorem is for *arithmetic* lattices in algebraic semi-simple groups of $\mathbf{Q}$-rank at least 2. We will not define this concept and refer to the original article by Witte. Note however that the method of proof cannot be generalized to an arbitrary lattice since it uses strongly the existence of nilpotent subgroups (which don't exist for example if the lattice is cocompact). However, this strongly suggests the following:

PROBLEM 7.3. *Is it true that no lattice in a simple Lie group of real rank at least* 2 *is left orderable ?*

7.2 Actions of higher rank lattices

We now study actions of the most general higher rank lattices on the circle. Most of this section is an expansion (and a translation) of a small part of [26] to which we refer for more information.

Theorem 7.4 ([26]). *Let Γ be a lattice in a simple Lie group G with real rank greater than or equal to 2. Then any action of Γ on the circle has a finite orbit.*

Of course, in such a situation a subgroup of finite index in Γ acts with a fixed point so that, deleting this fixed point, we get an action of a subgroup of finite index acting on the line. Recall our question 7.3 concerning ordering on lattices; it can be reformulated in the following way:

Problem 7.5. *Let Γ be a lattice in a simple Lie group G with real rank greater than or equal to 2. Is it true that any homomorphism from Γ to* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *has a finite image?*

These notes only deal with actions by homeomorphisms and we decided not to discuss properties connected with smooth diffeomorphisms. However, we mention that the previous question has a positive answer assuming some smoothness.

Theorem 7.6 ([26]). *Let Γ be a lattice in a simple Lie group G with real rank greater than or equal to 2. Then any homomorphism from Γ to the group of C^1-diffeomorphisms of the circle has a finite image.*

This theorem is an immediate consequence of 7.4 and of two important results. The first one, due to Kazhdan, states that a lattice like the one in the theorem is finitely generated and admits no non trivial homomorphism into $\mathbf{R}$ (see [48]). The second, due to Thurston, states that if a finitely generated group Γ has no non trivial homomorphism to $\mathbf{R}$ then any homomorphism from Γ to the group of germs of C^1-diffeomorphisms of $\mathbf{R}$ in the neighbourhood of the fixed point 0 is trivial (see [66]).

If we add more smoothness assumptions (but this is not the goal of this paper…), A. Navas, following earlier ideas of Segal and Reznikov, recently proved a remarkable theorem which applies to groups with Kazhdan's property (T) (see [57]). Note that lattices in higher rank semi-simple Lie groups have this property (see [32]).

THEOREM 7.7 (Navas). *Let* Γ *be a finitely generated subgroup of the group of diffeomorphisms of the circle of class* $C^{1+\alpha}$ *with* $\alpha > 1/2$. *If* Γ *satisfies Kazhdan's property (T), then* Γ *is finite.*

When the Lie group G is not simple but only semi-simple, the situation is more complicated since there are some interesting examples of irreducible higher rank lattices that do act. We have already described some examples of irreducible lattices in $\mathrm{SL}(2,\mathbf{R}) \times \mathrm{SL}(2,\mathbf{R})$ which act on the circle via their projection on the first factor (which is a dense subgroup in $\mathrm{SL}(2,\mathbf{R})$). As a matter of fact, the next result shows that these examples are basically the only ones.

If ϕ_1 and $\phi_2 \colon \Gamma \to \mathrm{Homeo}_+(\mathbf{S}^1)$ are homomorphisms, we say that ϕ_1 is *semi-conjugate to a finite cover of* ϕ_2 if there is a continuous map $h \colon \mathbf{S}^1 \to \mathbf{S}^1$ which is onto and locally monotonous, such that for every $\gamma \in \Gamma$ we have $\phi_2(\gamma)h = h\phi_1(\gamma)$.

THEOREM 7.8 ([26]). *Let* Γ *be an irreducible lattice in a semi-simple Lie group* G *with real rank greater than or equal to 2. Let* ϕ *be a homomorphism from* Γ *to the group of orientation preserving homeomorphisms of the circle. Then either* $\phi(\Gamma)$ *has a finite orbit or* ϕ *is semi-conjugate to a finite cover of a homomorphism which is the composition of:*

i) *the embedding of* Γ *in* G,

ii) *a surjection from* G *to* $\mathrm{PSL}(2,\mathbf{R})$,

iii) *the projective action of* $\mathrm{PSL}(2,\mathbf{R})$ *on the circle.*

These theorems show that higher rank lattices have very few actions on the circle. Hence, according to Section 6.15, the second bounded cohomology groups of lattices should be small. This is indeed what Burger and Monod showed in [12]:

THEOREM 7.9 (Burger, Monod). *Let* Γ *be a cocompact irreducible lattice in a semi-simple Lie group* G *with real rank greater than or equal to 2. Then the second bounded cohomology group* $H_b^2(\Gamma,\mathbf{R})$ *injects in the usual cohomology group* $H^2(\Gamma,\mathbf{R})$.

The assumption that the lattice is cocompact is important in the proof but the theorem probably generalizes to non-cocompact lattices. Note also that for many lattices in semi-simple Lie groups, it turns out that the usual cohomology group $H^2(\Gamma, \mathbf{R})$ vanishes. This is the case for instance for cocompact torsion free lattices in $\mathrm{SL}(n, \mathbf{R})$ for $n \geq 4$ but more generally for cocompact torsion free lattices in the group of isometries of an irreducible symmetric space of non compact type of rank at least 3 which is not hermitian symmetric (see [7]). In these cases, Theorem 7.9 means that $H_b^2(\Gamma, \mathbf{R})$ vanishes. Hence, using 6.6, we deduce that every action Γ on the circle has a finite orbit. In other words, Theorems 7.4 and 7.9 are closely related and, indeed they have been proved simultaneously (and independently). It would be very useful to compare the two proofs.

As we have already noticed, the vanishing of the second bounded cohomology group is closely related to the notion of commutator length. If Γ is any group and γ is in the first commutator subgroup Γ', we denote by $|\gamma|$ the least integer k such that γ can be written as a product of k commutators. We "stabilize" this number and define $\|\gamma\|$ as $\lim_{n\to\infty} |\gamma^n|/n$ (which always exists by sub-additivity). It turns out that for a finitely generated group Γ it is equivalent to say that the second bounded cohomology group $H_b^2(\Gamma, \mathbf{R})$ injects in the usual cohomology group $H^2(\Gamma, \mathbf{R})$, and to say that this "stable commutator norm" $\| \; \|$ vanishes identically [5]. Theorem 7.9 therefore implies that for cocompact higher rank lattices, this stable norm vanishes. The following question is natural:

PROBLEM 7.10. *Let Γ be an irreducible lattice as in Theorem 7.4. Does there exist an integer $k \geq 1$ such that every element of the first commutator subgroup of Γ is a product of k commutators?*

Recall that by a theorem of Kazhdan, there is no non trivial homomorphism from Γ to $\mathbf{R}$; this is equivalent to the fact that the first commutator group of Γ has finite index in Γ. A positive answer to the previous question would be a strengthening of this fact.

7.3 LATTICES IN LINEAR GROUPS

In this section, we prove Theorem 7.4 for lattices in $\mathrm{SL}(n, \mathbf{R})$ $(n \geq 3)$. The general case of a semi-simple Lie group is much harder but the proof that we present here contains the main ideas. As a matter of fact, we shall first concentrate on the case of a lattice Γ in $SL(3,\mathbf{R})$ and we shall easily deduce the general case of $SL(n,\mathbf{R})$ later.

Let us first informally describe the structure of the proof. Let Γ be a lattice in $SL(3,\mathbf{R})$ and consider a homomorphism $\phi\colon \Gamma \to \mathrm{Homeo}_+(\mathbf{S}^1)$.

FIRST STEP. In order to prove the theorem, it is enough to show that there is a probability measure μ on the circle which is invariant under the group $\phi(\Gamma)$.

SECOND STEP (CLASSICAL). A *flag* in $\mathbf{R}^3$ is a pair consisting of a 2-dimensional (vector) subspace E_2 in $\mathbf{R}^3$ and a 1-dimensional (vector) subspace E_1 contained in E_2. Those flags, equipped with the natural topology, define a compact manifold Fl which is a homogeneous space under the action of $SL(3,\mathbf{R})$. Note that in particular, Γ acts on Fl.

Let $Prob(\mathbf{S}^1)$ be the space of all probability measures on the circle. Equipped with the weak topology, this is a compact metrizable space on which the group $\mathrm{Homeo}_+(\mathbf{S}^1)$ acts naturally. The lattice Γ also acts on $Prob(\mathbf{S}^1)$ via the homomorphism ϕ.

Equip Fl with the σ-algebra of Lebesgue measurable sets and $Prob(\mathbf{S}^1)$ with the σ-algebra of Borel sets. In the second step, we construct a measurable map $\Psi\colon Fl \to Prob(\mathbf{S}^1)$ which is equivariant with respect to the actions of Γ on Fl and $Prob(\mathbf{S}^1)$.

In order to prove the theorem, if is enough to show that this map Ψ takes the same value μ almost everywhere with respect to the Lebesgue measure on Fl. Indeed, by equivariance, this measure μ will be invariant by the group $\phi(\Gamma)$.

By way of contradiction, we now assume that Ψ is not constant on a set of full Lebesgue measure.

THIRD STEP. Using ergodic properties of the action of Γ on Fl, we show that there is an integer k and a measurable map Ψ as above such that the image of almost every flag in Fl is the sum of k Dirac masses on the circle (each with weight $1/k$). Let us denote by $\mathbf{S}^1_k$ the set of subsets of $\mathbf{S}^1$ with k elements so that we can now consider Ψ as a map from Fl to $\mathbf{S}^1_k$.

FOURTH STEP. Let X be the space consisting of triples (E_2^1, E_2^2, E_2^3) of distinct planes in $\mathbf{R}^3$ intersecting on the same line E_1. This is again a homogeneous space under the action of $SL(3,\mathbf{R})$. An element of X determines three flags. Therefore the map Ψ enables us to define a measurable map $\Psi^{(3)}: X \to (\mathbf{S}^1_k)^3$. We will get a contradiction between the ergodicity of the action of Γ on X and the non ergodicity of the action of Γ on the set of triples of points: a triple of points on $\mathbf{S}^1$ can be positively or negatively ordered on the circle and this is invariant under $\mathrm{Homeo}_+(\mathbf{S}^1)$.

We now give the detailed proof.

FIRST STEP: FINDING AN INVARIANT MEASURE. Suppose that there is a probability measure μ on the circle which is invariant under $\phi(\Gamma)$.

We know that the rotation number mapping ρ: $\mathrm{Homeo}_+(\mathbf{S}^1) \to \mathbf{R}/\mathbf{Z}$ is not a homomorphism. However by 6.18, the restriction to the subgroup consisting of homeomorphisms preserving a given measure μ is a homomorphism. It follows that the map $\gamma \in \Gamma \mapsto \rho(\phi(\gamma)) \in \mathbf{R}/\mathbf{Z}$ is a homomorphism. According to the result of Kazhdan that we mentioned several times already, Γ is finitely generated and every homomorphism from Γ to $\mathbf{R}$ is trivial. It follows that the image of the restriction of ρ to Γ is a finite cyclic subgroup $\mathbf{Z}/k\mathbf{Z}$. Consider the kernel Γ_0 of this homomorphism: this is a subgroup of index k of Γ, hence a lattice in $\mathrm{SL}(3,\mathbf{R})$. We claim that the support of μ is fixed pointwise by Γ_0. This follows from the fact that for every homeomorphism of the circle with zero rotation number, the support of every invariant measure is contained in the set of fixed points. Hence every point in the support of μ has a finite orbit under $\phi(\Gamma)$. This is the conclusion of Theorem 7.4.

SECOND STEP: FURSTENBERG MAP. This step is classical in the study of actions of lattices and is due to Furstenberg [23].

PROPOSITION 7.11. *There is a Lebesgue measurable map $\Psi: Fl \to Prob(\mathbf{S}^1)$ which is equivariant under the actions of Γ on Fl and $Prob(\mathbf{S}^1)$.*

Proof. We observed that Fl is homogeneous under the action of $\mathrm{SL}(3,\mathbf{R})$. The stabilizer of the flag consisting of the line spanned by $(1,0,0)$ and the plane generated by $(1,0,0)$ and $(0,1,0)$ is the group B of upper triangular matrices. Therefore we can identify Fl and the homogeneous space $\mathrm{SL}(3,\mathbf{R})/B$.

Note that the group B is solvable. Hence B is amenable and there is a linear form m on $L^\infty(B,\mathbf{R})$ which is non negative on non negative functions, takes the value 1 on the constant function 1 and is invariant under left translations. It turns out that it is possible to choose m in such a way that it is a measurable function (see [55]). In other words, if $f_\lambda \in L^\infty(B, \mathbf{R})$ depends measurably on a parameter λ in $[0,1]$, the function $\lambda \mapsto m(f_\lambda)$ is Lebesgue measurable.

Coming back to our problem, we first observe that there are measurable maps $\Psi_0 \colon \mathrm{SL}(3,\mathbf{R}) \to Prob(\mathbf{S}^1)$ which are Γ-equivariant. This follows from the fact that the action of Γ on $\mathrm{SL}(3,\mathbf{R})$ by left translations has a fundamental domain; we define Ψ_0 in an arbitrary measurable way on this fundamental domain and we can therefore define it everywhere using the equivariance.

To complete the proof of the proposition, we modify Ψ_0 to make it invariant under right translations under B. Of course, we use the mean m. We define $\Psi \colon \mathrm{SL}(3,\mathbf{R}) \to Prob(\mathbf{S}^1)$ in the following way. If $g \in \mathrm{SL}(3,\mathbf{R})$, the probability $\Psi(g)$ is defined by its value on a continuous function $u \colon \mathbf{S}^1 \to \mathbf{R}$:

$$\int_{\mathbf{S}^1} u\, d\Psi(g) = m(x \in B \mapsto \int_{\mathbf{S}^1} u\, d\Psi_0(gx)) \,.$$

By construction, Ψ is measurable and invariant by right translations by B; this defines another measurable map $\Psi \colon Fl \simeq \mathrm{SL}(3,\mathbf{R})/B \to Prob(\mathbf{S}^1)$ which is Γ-equivariant, as required. □

THIRD STEP: THE MAP Ψ TO DIRAC MASSES. As mentioned above, *we now assume by contradiction that the map Ψ is not constant on a subset of full Lebesgue measure.*

PROPOSITION 7.12. *There exist an integer $k \geq 1$ and a map $\Psi \colon Fl \to \mathbf{S}^1_k$ to the set of subsets of $\mathbf{S}^1$ with k elements which is Lebesgue measurable and Γ-invariant.*

In order to prove the proposition, we first recall an important ergodic theorem due to Moore that we shall use repeatedly (see for instance [72]). Let $Y = G/H$ be a homogeneous space of a semi-simple Lie group G. Assume that G is connected, has a finite center and has no compact factor. Assume moreover that H is non compact. Let Γ be an irreducible lattice in G. Then the action of Γ on Y is ergodic with respect to the Lebesgue measure (class), *i.e.* every measurable function on Y which is Γ-invariant is constant almost everywhere.

For instance, the stabilizer B of a flag is non compact. *The action of Γ on Fl is ergodic.*

As another example, let us consider the space Y of pairs of flags of $\mathbf{R}^3$ which are in general position. For such a pair of flags, there are three non coplanar lines E_1^1, E_1^2, E_1^3 such that the first flag is given by the line E_1^1 and the plane spanned by E_1^1 and E_1^2 and the second flag is given by the line E_1^3 and the plane spanned by E_1^2 and E_1^3. Since $SL(3, \mathbf{R})$ acts transitively on the space of triples of non coplanar lines, it follows that Y is a homogeneous space of $SL(3, \mathbf{R})$. The stabilizer of an element of Y is the stabilizer of a triple of non coplanar lines: it is clearly non compact. Consequently, the action of Γ on Y is ergodic. Since the set of pairs of flags in general position has full Lebesgue measure in the set of pairs of flags, we deduce that Γ acts ergodically on the set of pairs of flags of $\mathbf{R}^3$.

However, the reader will easily check that this cannot be generalized to the set of triples of flags: the action of $SL(3, \mathbf{R})$ is not transitive on the set of triples of flags in general position.

In order to prove Proposition 7.12, we analyze the action of Γ on the space of pairs of probability measures on the circle.

If μ is a probability on the circle, we define $atom(\mu)$ as the sum of the masses of the atoms of μ (*i.e.* those points x such that $\mu(\{x\}) > 0$). This is a measurable function on $Prob(\mathbf{S}^1)$ which is invariant under the action of $\mathrm{Homeo}_+(\mathbf{S}^1)$. The map:

$$d \in Fl \mapsto atom(\Psi(d)) \in [0, 1]$$

is a measurable Γ-invariant function. Using the ergodicity result that we mentioned above, this function is constant almost everywhere.

Assume first that this constant is not zero. This means that the image of almost every flag under Ψ has at least one atom.

Let $\alpha > 0$ be a positive real number. For each probability measure μ on the circle, consider the points x such that $\mu(\{x\}) > \alpha$. Of course, the number of those points x is finite (possibly zero). Denote this number by $N(\mu, \alpha)$. The map $d \in Fl \mapsto N(\Psi(d), \alpha) \in \mathbf{N}$ is measurable and Γ-invariant; it is therefore constant, equal to some integer N_α almost everywhere. Since we assume that for almost every d the probability $\Psi(d)$ has at least one atom, we can choose some α in such a way that N_α is an integer $k \geq 1$. This enables us to construct a map (defined almost everywhere) from Fl to the set of subsets of $\mathbf{S}^1$ with k elements, sending the flag d to the k atoms of $\Psi(d)$ having a mass greater than or equal to α. Changing our notation, we shall call this new map Ψ: it satisfies Proposition 7.12 which is therefore proved, if almost every $\Psi(d)$ has at least one atom.

We now assume that for almost every d, the probability $\Psi(d)$ has no atom.

We shall show that under this assumption, almost all the measures $\Psi(d)$ have the same support.

Let μ_1 and μ_2 be two probability measures on the circle with no atom. Define $D(\mu_1, \mu_2)$ as the maximum of the μ_2-measures of the connected components of the complement of the support of μ_1. If $D(\mu_1, \mu_2) = 0$, the support of μ_1 contains the support of μ_2. The map

$$(d_1, d_2) \in Fl^2 \mapsto D(\Psi(d_1), \Psi(d_2)) \in [0, 1]$$

is defined almost everywhere and is Γ-invariant. Using the same ergodicity result as before, we deduce that it is constant almost everywhere. *We claim that this constant* δ *is* 0.

Suppose on the contrary that $\delta > 0$. Using Fubini's theorem, we can find a measurable part $\Omega \subset Fl$ such that:

- Ω has full Lebesgue measure.
- If $d \in \Omega$, the probability $\Psi(d)$ has no atom.
- If $d \in \Omega$, then $D(d, d') = \delta$ for almost every d' in Fl.
- If $d \in \Omega$, then $\Psi(d)$ belongs to the support of the measure $\Psi_*(Lebesgue)$ on the compact metrizable space $Prob(\mathbf{S}^1)$.

Fix a point $d \in \Omega$. We can find a sequence $d_i \in \Omega$ such that $\Psi(d_i) = \mu_i$ converges towards $\Psi(d) = \mu$. The probability measures μ_i have no atoms and $D(\mu_i, \mu) = \delta$. This means that there is a component I_i of the complement of the support $supp(\mu)$ such that $\mu_i(I_i) = \delta$. If the sequence of lengths of I_i converges to 0, we can assume that the sequence of intervals I_i shrinks to a point p. This implies that the point p is an atom of μ, contradicting our assumption. Therefore we can assume (after taking a subsequence) that the intervals I_i all coincide with some interval I. Since we know that the endpoints of I are not atoms of μ, that the sequence μ_i converges weakly to μ, and that $\mu_i(I) = \delta$, it follows that $\mu(I) = \delta$. This contradicts the fact that I is in the complement of the support of μ.

We showed that $\delta = 0$. This means that for almost every pair of flags (d, d'), we have $D(\Psi(d), \Psi(d')) = 0$. Therefore, for almost every pair of flags (d, d'), the probability measures $\Psi(d)$ and $\Psi(d')$ have the same support. In other words, *there exists a compact set* $K \subset \mathbf{S}^1$ *with no isolated point, such that for almost every flag* d, *the support of* $\Psi(d)$ *is equal to* K.

Each connected component of $\mathbf{S}^1 - K$ is an open interval. Collapsing the closure of these intervals to a point, we get a space homeomorphic to a circle. Therefore, there exists a continuous $\pi \colon \mathbf{S}^1 \to \mathbf{S}^1$ such that each fiber of π is a point or the closure of a component of the complement of K. If μ is a measure with no atom whose support is K, the direct image $\pi_\star(\mu)$ is a measure on the circle with no atom and full support on the circle.

Using $\pi_\star$, we get a map $\overline{\Psi}$ from Fl to the space of probability measures on the circle with no atom and full support which is Γ-equivariant with respect to the minimal action $\overline{\phi}$ associated to ϕ (see 5.8).

The space of probability measures with no atoms and full support on the circle is a homogeneous space under the action of $\mathrm{Homeo}_+(\mathbf{S}^1)$ and the stabilizer of the Lebesgue measure is of course $\mathrm{SO}(2)$. This space can therefore be identified with the quotient $\mathrm{Homeo}_+(\mathbf{S}^1)/\,\mathrm{SO}(2)$. The group $\mathrm{Homeo}_+(\mathbf{S}^1)$, as any metrizable topological group, can be equipped with a left invariant metric, that we can average under the action of SO(2) to produce a left invariant metric *dist* on $\mathrm{Homeo}_+(\mathbf{S}^1)/\,\mathrm{SO}(2)$. In practice, we could simply define $dist(\mu_1,\mu_2)$ as the supremum of $|\mu_1(I)-\mu_2(I)|$ where I runs through the collection of intervals on the circle: it is easy to check that this metric indeed defines the weak topology when restricted to the set of probability measures with no atom and full support.

For almost every pair of flags (d,d') the distance $dist(\overline{\Psi}(d),\overline{\Psi}(d'))$ defines a Γ-invariant function of pairs of flags; it is therefore constant almost everywhere. Using the same argument as above, we see that this constant is 0, which means that the map $\overline{\Psi}$ is constant almost everywhere. Of course, two probability measures with no atom and with support in K which have the same image under $\pi_\star$ have to coincide so that we deduce that Ψ is constant almost everywhere. We have found a probability measure on the circle which is invariant under $\phi(\Gamma)$. This is a contradiction with our initial assumption and proves 7.12.

FOURTH STEP: CYCLIC ORDERING ON TRIPLES OF POINTS ON A CIRCLE. In order to explain the general idea, we assume first that the integer k that we introduced is equal to 1. In other words, we have a Γ-invariant map $\Psi\colon Fl \to \mathbf{S}^1$ defined almost everywhere which is not constant on a set of full Lebesgue measure.

As explained above, let X denote the space of triples (E_2^1, E_2^2, E_2^3) of distinct planes in $\mathbf{R}^3$ intersecting on the same line E_1. This is again a homogeneous space under $\mathrm{SL}(3,\mathbf{R})$ and the stabilizer of a point in X is clearly non compact. We deduce from Moore ergodicity theorem that the action of Γ on X is ergodic. Since a point of X determines three flags, we can define a measurable Γ-equivariant map $\Psi^{(3)}\colon X \to (\mathbf{S}^1)^3$ (defined almost everywhere). Indeed, let us consider the projection $pr\colon Fl \to \mathbf{RP}^2$ from Fl to the real projective plane mapping a flag $E_1 \subset E_2$ to the line $E_1 \subset \mathbf{R}^3$. The space X

is therefore the space of triples of flags having the same projection under pr. It follows from Fubini's theorem that for every subset of full measure in Fl, the set of triples of elements of this set having the same projection under pr has full measure in X: this is exactly what we need to define $\Psi^{(3)}$.

The space $(\mathbf{S}^1)^3$ can be decomposed into disjoint parts, invariant under the action of $\mathrm{Homeo}_+(\mathbf{S}^1)$:

i) Triples of the form (x,x,x).

ii) Triples consisting of two distinct points. In turn, this set can be decomposed into three parts: the spaces of triples of the form (x,x,z), resp. (x,y,x), resp. (x,y,y).

iii) Triples (x,y,z) of distinct elements on the circle whose cyclic ordering is positive, *i.e.* such that the interval positively oriented from x to y does not contain z.

iv) Triples (x,y,z) of distinct elements on the circle whose cyclic ordering is negative.

Inverse images of these six parts under $\Psi^{(3)}$ are measurable and disjoint Γ-invariant sets and therefore have to be either of measure 0 or of full Lebesgue measure. This means that there is a subset $\Omega \subset X$ of full measure whose image is contained in one of the six parts that we described. We claim that this is not possible.

Observe that the symmetric group $\mathfrak{S}_3$ of permutations of three objects acts on X and on $(\mathbf{S}^1)^3$, permuting respectively flags and points. Note that these actions commute with the actions of Γ on X and $(\mathbf{S}^1)^3$. Of course $\Psi^{(3)}$ is equivariant with respect to these action of $\mathfrak{S}_3$.

It follows that the part which contains $\Psi^{(3)}(\Omega)$ has to be invariant under $\mathfrak{S}_3$. Among the 6 parts that we described, only the first one has this property. This means that the map $\Psi\colon Fl \to \mathbf{S}^1$ factors through the projection $pr\colon Fl \to \mathbf{RP}^2$. In other words, almost everywhere, the image of a flag by Ψ depends only on the line associated to the flag and not on its plane.

Exactly in the same way, we could have defined a space X' consisting of triples of flags having the same plane, *i.e.* having the same projection in the dual projective plane. The same proof shows that almost everywhere Ψ depends only on the plane of a flag and not on its line.

This implies that Ψ is constant almost everywhere and gives the contradiction we were looking for when $k=1$.

When $k>1$, we shall use a similar idea.

Recall that we denote by $\mathbf{S}^1_k$ the space of subsets A of the circle with k elements. Given two elements (A_1,A_2,A_3) and (A'_1,A'_2,A'_3) of $(\mathbf{S}^1_k)^3$, we say that they have the same cyclic ordering if there is an orientation preserving homeomorphism h of the circle such that $h(A_1) = A'_1, h(A_2) = A'_2, h(A_3) = A'_3$. This gives a partition of $(\mathbf{S}^1_k)^3$ into finitely many parts invariant under the action of $\mathrm{Homeo}_+(\mathbf{S}^1)$. As before, it follows that there is a subset Ω of full measure in X such that $\Psi(\Omega)$ is contained in one of these subsets. Using again the action of $\mathfrak{S}_3$ we conclude that this subset consists of triples (A_1, A_2, A_3) which have the same cyclic ordering as $(A_{\sigma(1)}, A_{\sigma(2)}, A_{\sigma(3)})$ for every element $\sigma \in \Sigma_3$. Therefore, for every σ, there is an orientation preserving homeomorphism h_σ such that $h_\sigma(A_i) = A_{\sigma(i)}$ for $i = 1, 2, 3$. Let A be the union of A_1, A_2 et A_3 : this is a set with $N \leq 3k$ elements. Orientation preserving homeomorphisms globally preserving A must induce a cyclic permutation of its elements. In particular, the commutator of two elements h_σ must fix each element of A since cyclic permutations commute. As the cyclic permutation $\sigma = (1, 2, 3)$ is a commutator in $\mathfrak{S}_3$, the homeomorphism $h_{(1,2,3)}$ acts trivially on A. Since we know that $h_{(1,2,3)}(A_1) = A_2$, $h_{(1,2,3)}(A_2) = A_3$ and $h_{(1,2,3)}(A_3) = A_1$, we have $A_1 = A_2 = A_3$. We showed that there exists a measurable subset of full measure $\Omega \subset X$ such that the image $\Psi(\Omega)$ consists of triples of the form (A, A, A). Exactly as we did in the case $k = 1$, we conclude that Ψ is constant almost everywhere and this is a contradiction.

This is the end of the proof of Theorem 7.4 for lattices in $\mathrm{SL}(3, \mathbf{R})$.

Remark that the core of the proof is the incompability between two facts. The group $\mathrm{Homeo}_+(\mathbf{S}^1)$ does not act transitively on generic triples of points on the circle but $\mathrm{SL}(3, \mathbf{R})$ does act transitively on X. Note that the existence of an element of $\mathrm{SL}(3, \mathbf{R})$ fixing a line and permuting arbitrarily three planes containing this line, means that the real projective plane is not orientable.

The proof for a lattice Γ in $\mathrm{SL}(n, \mathbf{R})$ $(n \geq 3)$ is very similar. For every sequence of integers, $1 \leq i_1 < i_2 < \cdots < i_l \leq n$, we consider the space $Fl_{i_1,\ldots,i_l}$ of flags of type $(i_1, \ldots, i_l)$, *i.e.* sequences of vector sub-spaces $E_{i_1} \subset E_{i_2} \subset \cdots \subset E_{i_l} \subset \mathbf{R}^n$ with $\dim E_{i_j} = i_j$ $(j = 1, \ldots, l)$. This is a homogeneous space under the action of $\mathrm{SL}(n, \mathbf{R})$. The space of complete flags, *i.e.* $Fl = Fl_{1,2,\ldots,n}$ is equipped with projections pr_j on incomplete flag spaces $Fl_{1,2,\ldots,\hat{j},\ldots,n}$ where the index j does not appear. The space X_j consisting of distinct triples of flags Fl having the same projection under pr_j is again a homogeneous space of $\mathrm{SL}(n, \mathbf{R})$, with non compact stabilizer.

Now, the proof is the same as before. We first construct an equivariant map Ψ from Fl to $Prob(\mathbf{S}^1)$ (same proof). Assuming by contradiction that Ψ is not constant almost everywhere, we get another map, still denoted by Ψ from Fl to $\mathbf{S}^1_k$ (same proof). For each $j = 1, \ldots, n$, we consider the corresponding map $\Psi_j^{(3)}: X_j \to \mathbf{S}_k^1$ and we show, as above, that the image of this map consists almost everywhere of triples of the form (A, A, A). It follows that for each $j = 1, \ldots, n$ and on a subset of full measure, the image of a flag by Ψ depends only on its projection by pr_j. Since this is true for every j, this means that Ψ is constant almost everywhere. This is a contradiction and finishes the proof of Theorem 7.4 for lattices in $\mathrm{SL}(n, \mathbf{R})$.

Of course, these proofs immediately generalize to lattices in complex or quaternionic special linear groups $\mathrm{SL}(n, \mathbf{C})$ and $\mathrm{SL}(3, \mathbf{H})$ (for $n \geq 3$).

7.4 Some groups that do act...

We saw that many higher rank lattices don't act on the circle. To conclude these notes, we give some more examples of "big" groups acting on the circle. Let Σ be a compact oriented surface of genus $g \geq 2$ and $x \in \Sigma$ be some base point. The fundamental group $\pi_1(\Sigma, x)$ is a classical example of a hyperbolic group in the sense of Gromov (see for instance [27]). The boundary of this group is a topological circle: indeed $\pi_1(\Sigma, x)$ acts freely and cocompactly on the Poincaré disc so that $\pi_1(\Sigma, x)$ is quasi-isometric to the Poincaré disc. Consequently, the automorphism group $\mathrm{Aut}(\pi_1(\Sigma, x))$ acts naturally on the circle. This action is very interesting and has been very much studied. See for instance [21]. Note that $\mathrm{Aut}(\pi_1(\Sigma, x))$ contains the group of inner conjugacies and that the quotient $\mathrm{Out}(\pi_1(\Sigma, x))$ is the *mapping class group* of the surface (*i.e.* the group of isotopy classes of homeomorphisms of the surface):

$$1 \longrightarrow \pi_1(\Sigma, x) \longrightarrow \mathrm{Aut}(\pi_1(\Sigma, x)) \longrightarrow \mathrm{Out}(\pi_1(\Sigma, x)) \longrightarrow 1 .$$

Fix an element f of infinite order in this mapping class group and consider the group Γ_f which is the inverse image of the group generated by f in the previous exact sequence. We have an exact sequence:

$$1 \longrightarrow \pi_1(\Sigma, x) \longrightarrow \Gamma_f \longrightarrow \mathbf{Z} \longrightarrow 1 .$$

This group Γ_f is the fundamental group of the 3-manifold which fibers over the circle and whose monodromy is given by the class f. Thurston showed that if f is of pseudo-Anosov type, then this 3-manifold is hyperbolic. In particular, for such a choice of f, the group Γ_f embeds as a discrete cocompact subgroup of the isometry group of the hyperbolic 3-ball, isomorphic to $\mathrm{PSL}(2, \mathbf{C})$. This construction provides many examples of faithful actions of (rank 1) lattices on the circle. In [68] Thurston constructs faithful actions of the fundamental group of many hyperbolic 3-manifolds on the circle.

Suppose now that Σ has one boundary component $\partial\Sigma$. Choose the base point on the boundary and equip Σ with a metric with curvature -1 and totally geodesic boundary. The universal cover $\widetilde{\Sigma}$ of Σ is therefore identified with the complement in the Poincaré disc of a disjoint union of half spaces. On the boundary of the disc, these half spaces define an open dense subset Ω whose complement is a Cantor set K which is the boundary of the hyperbolic group $\pi_1(\Sigma, x)$. The union $\partial\widetilde{\Sigma} \cup K$ is a topological circle and if we collapse each connected component of $\partial\widetilde{\Sigma}$ to a point, this circle collapses to another circle that we denote by C. Choose also a base point $\widetilde{x}$ above x in the universal cover. Consider now the mapping class group Γ of Σ *i.e.* the group of homeomorphisms of Σ modulo isotopy. A homeomorphism f of Σ has a lift $\widetilde{f}$ to $\widetilde{\Sigma}$ which fixes the boundary component containing $\widetilde{x}$. This homeomorphism $\widetilde{f}$ extends continuously to $\partial\widetilde{\Sigma} \cup K$ and defines a homeomorphism $\bar{f}$ of the circle C. Note that if two homeomorphisms are isotopic, the two corresponding extensions agree on the Cantor set K. The connected component of $\partial\widetilde{\Sigma}$ containing $\widetilde{x}$ yields a base point $\bar{x}$ in C which is fixed by all homeomorphisms $\bar{f}$. Hence we can define an action of Γ on a line by letting f act via $\bar{f}$ on the line $C - \{\bar{x}\}$. Hence we proved (following an idea of Thurston) that the mapping class group of (Σ, x) acts (faithfully) on a line and is therefore left orderable.

We could also use the same idea for surfaces with several boundary components, for instance the sphere minus a finite number of discs. The corresponding mapping class groups turn into the so called braid groups. In this way we get interesting faithful actions of braid groups on the line, or equivalently total left orderings. It is interesting to note that these orderings were initially discovered from a completely different point of view by Dehornoy [16].

To conclude this paper, we would like to mention a rich family of group actions on the circle, coming from the theory of Anosov flows on 3-manifolds. Let M be a compact connected 3-manifold with no boundary and X a non singular smooth vector field on M. Denote by ϕ^t the flow generated by X. One says that ϕ^t is an *Anosov flow* if there is a continuous splitting of the tangent bundle TM as a sum of three line bundles $TM = \mathbf{R}X \oplus E^{ss} \oplus E^{uu}$ which are invariant under (the differential of) the flow ϕ^t and such that vectors in E^{uu} are expanded, and vectors in E^{ss} are contracted. More precisely, this means that for any riemannian metric on M, there are constants $C > 0$ and $\lambda > 0$ such that for any time $t > 0$ and vectors $v_{ss} \in E^{ss}$ and $v_{uu} \in E^{uu}$,

$$\|d\phi^t(v_{ss})\| \leq C\exp(\lambda t)\|v_{ss}\|\,,$$

$$\|d\phi^t(v_{uu})\| \geq C\exp(\lambda t)\|v_{uu}\|\,.$$

This kind of flow is rather abundant on 3-manifolds. The main example, which gave birth to the theory, is the geodesic flow of a compact surface with negative curvature, acting on the unit tangent bundle of the surface. We refer to [3, 22] for a general presentation of the theory including a bibliography. Starting from some Anosov flow and selecting a periodic orbit, one can perform a Dehn surgery on this closed curve. It turns out that if the surgery is positive, one can define a flow on the new manifold which is still of Anosov type. Using this construction, one constructs many examples. For instance, one can construct Anosov flows on some hyperbolic 3-manifolds (*i.e.* admitting a metric of constant negative curvature).

One of the main properties of Anosov flows is that they give rise to two codimension one foliations. Indeed, it has been shown by Anosov that there are two codimension one foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ whose leaves are everywhere tangent to $E^{ss} \oplus \mathbf{R}X$ and $E^{ss} \oplus \mathbf{R}X$. Verjovsky showed that if one lifts the flow ϕ^t to the universal cover $\widetilde{M}$ of M, the orbits of the resulting flow $\widetilde{\phi}^t$ are the fibers of a (trivial) fibration of $\widetilde{M}$ over a surface S (diffeomorphic to $\mathbf{R}^2$). Lifting the two foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ to $\widetilde{M}$, we get two foliations which project to two transverse foliations by curves $\widetilde{f}^u$ and $\widetilde{f}^s$ on the surface S. One says that the flow is $\mathbf{R}$-*covered* if the leaves of $\widetilde{f}^u$ are the fibers of a (trivial) fibration $p_u \colon S \to \mathbf{R}_u$ (where $\mathbf{R}_u$ is homeomorphic to $\mathbf{R}$). It follows that the leaves of $\widetilde{f}^s$ are also the fibers of a (trivial) fibration $p_s \colon S \to \mathbf{R}_s$. For instance, the geodesic flow on a negatively curved surface is $\mathbf{R}$-covered. It turns out that a positive surgery on an $\mathbf{R}$-covered Anosov flow is still $\mathbf{R}$-covered so that we get many examples. Consider the map $(p_u, p_s) \colon S \to \mathbf{R}_u \times \mathbf{R}_s$. Barbot and Fenley showed independently that this map is bijective if and only if the Anosov flow is the suspension of some Anosov diffeomorphism of the 2-torus. In all other cases, they showed that the image of (p_u, p_s) is an open strip in $\mathbf{R}_u \times \mathbf{R}_s$ of the form $\{(x,y) \mid h_-(x) < y < h_+(x)\}$ where h_- and h_+ are some homeomorphisms from $\mathbf{R}_u$ to $\mathbf{R}_s$. Now, observe that the fundamental group Γ of the manifold M acts on all these objects so that we get in particular actions of Γ on $\mathbf{R}_u$ and $\mathbf{R}_s$ which are conjugate by h_u and h_s. Denote by τ the composition $h_u h_s^{-1}$: this is a homeomorphism of $\mathbf{R}_u$ which acts freely so that we can define a circle $\mathbf{S}^1_u$ by taking the quotient of $\mathbf{R}_u$ by the action of τ. Note that the action of Γ on $\mathbf{R}_u$ obviously commutes with τ so that we get an action of Γ on $\mathbf{S}^1_u$. In case we start with the geodesic flow of a negatively curved surface Σ, the fundamental group Γ is a central extension of the fundamental group $\pi_1(\Sigma)$ by $\mathbf{Z}$. The action of Γ that we get on $\mathbf{S}^1_u$ is not faithful: the center $\mathbf{Z}$ acts trivially and the induced action of $\pi_1(\Sigma)$ on the circle is of course the familiar projective action. If the $\mathbf{R}$-covered Anosov flow is not the geodesic flow (up to a finite cover), the action of Γ on $\mathbf{S}_u^1$ is faithful. For instance, we get in this way some examples of faithful actions of the fundamental group of some hyperbolic 3-manifolds on the circle.

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