Intersecting curves

(variation on an observation of Maxim Kontsevich)

by
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Abstract

Consider the graphs of n distinct polynomials of a real variable intersecting at some point. In the neighborhood of this point, the qualitative picture is described by some permutation of $\{1, \ldots, n\}$. We describe the permutations that occur in such a situation.

In this note, we shall examine the relative positions of the graphs of several functions in the neighborhood of a point where they intersect. In order to keep the discussion as elementary as possible, we shall restrict ourselves to polynomials in the real variable x

$$f(x) = a_0 + a_1 x + \ldots + a_d x^d,$$

where the coefficients a_i are real numbers. As usual, one says that the *valuation* of f at 0 is the integer $k \leq d$ such that $a_0 = a_1 = \ldots = a_{k-1} = 0$ and $a_k \neq 0$ (and ∞ if $f = 0$). It is well known that *the function f changes sign in the neighborhood of the origin if and only if its valuation is odd.*

Two curves

Assume now that the graphs of *two* distinct polynomials f_1, f_2 intersect in some point, say the origin $(0,0)$. Then the relative position of the graphs of f_1 and f_2 in the neighborhood of this point is easy to describe. If the valuation of $f_1 - f_2$ is odd, these graphs cross each other. Otherwise they touch without crossing.

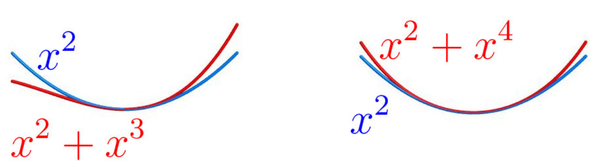

Figure 1: Two intersecting curves

Three curves

Let us now look at the graphs of *three* distinct polynomials f_1, f_2, f_3 vanishing at the origin. The relative positions of these graphs are determined by the valuations of the differences $f_i - f_j$. Looking at small negative values of x, we can rename the polynomials in such a way that $f_1(x) > f_2(x) > f_3(x)$. For small *positive* numbers x, the ordering of the $f_i(x)$ may be different so that the crossing of the three graphs can be described by one of the six permutations of $\{1, 2, 3\}$. It is not difficult to show, by way of examples, that all six permutations occur. Here are six examples:

$$
\begin{array}{ccccc}
(f_1(x) = x^2 & ; & f_2(x) = 0 & ; & f_3(x) = -x^2) \\
(f_1(x) = x^2 & ; & f_2(x) = -x^3 & ; & f_3(x) = 0) \\
(f_1(x) = 0 & ; & f_2(x) = x^3 & ; & f_3(x) = -x^2) \\
(f_1(x) = -x & ; & f_2(x) = x^2 & ; & f_3(x) = -x^2) \\
(f_1(x) = -x & ; & f_2(x) = 0 & ; & f_3(x) = x) \\
(f_1(x) = x^2 & ; & f_2(x) = -x^2 & ; & f_3(x) = x)
\end{array}
$$

leading to the six permutations as illustrated in Figure 2.

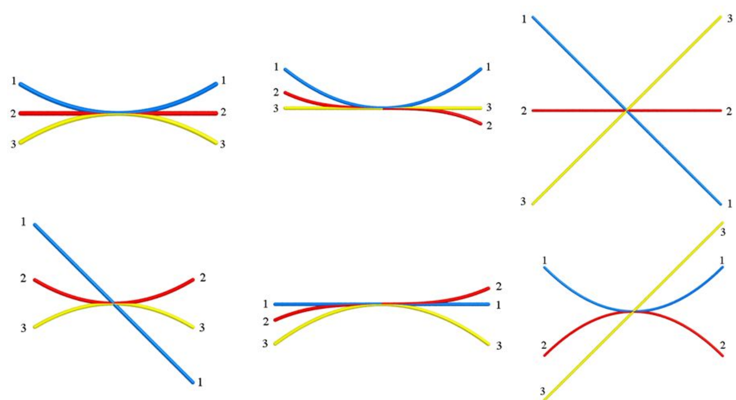

Figure 2: Three intersecting curves

Four curves

Maxim Kontsevich observed that the situation is different when *four* graphs intersect[1]. Not all permutations of $\{1, 2, 3, 4\}$ occur. The following figure 3 shows four piecewise linear functions which intersect in a way which is not possible for polynomials.

Theorem 1. *There do not exist four polynomials f_1, f_2, f_3, f_4 such that:*

1. *they all vanish at the origin,*

[1]Personal communication (2009).

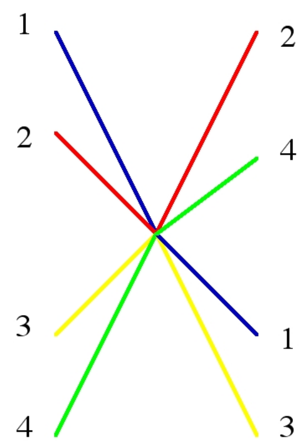

Figure 3: An impossible situation for polynomials

2. $f_1(x) > f_2(x) > f_3(x) > f_4(x)$ *for small negative* x,

3. $f_2(x) > f_4(x) > f_1(x) > f_3(x)$ *for small positive* x.

Proof. The proof is by contradiction.

Replacing the f_i by $f_i - f_4$, we can assume that $f_4 = 0$. Since f_1 and f_3 change sign at the origin, their valuations $val(f_1), val(f_3)$ are odd. Since f_2 does not change sign, its valuation $val(f_2)$ is even. From $f_1(x) > f_2(x) > f_3(x) > 0$ for small negative x, we deduce that $val(f_3) \geq val(f_2) \geq val(f_1)$. Similarly, from $0 > f_1(x) > f_3(x)$ for small positive x, we deduce $val(f_1) \geq val(f_3)$. That would force the three valuations to be equal, but two of them are odd and the third is even! This yields a contradiction. $\boxdot$

Note that the same proof applies to real analytic functions but does not apply to smooth functions. Indeed the reader will easily find four C^∞ functions f_i crossing at the origin according to the "forbidden" permutation $(1, 2, 3, 4) \rightarrow (2, 4, 1, 3)$.

Changing orientations along the x-axis, one can see that the inverse permutation $(1, 2, 3, 4) \rightarrow (3, 1, 4, 2)$ is also forbidden. As an exercise, I recommend showing that the remaining 22 permutations of $\{1, 2, 3, 4\}$ occur for suitable choices of the f_i.

Many curves

Planting trees

Let us now analyze the situation when the graphs of n polynomials intersect at the origin. Let us first coin a name.

Definition. *Let* $n \geq 2$ *be some integer and* $\pi : \{1, 2, ..., n\} \rightarrow \{1, 2, ..., n\}$ *some permutation. We say that* π *is an* interchange *if there exist* n *polynomials* $f_1, ..., f_n$ *vanishing at* 0 *and such that:*

1. $f_1(x) > f_2(x) > \ldots > f_n(x)$ *for small negative* x*, and*

2. $f_{\pi(1)}(x) > f_{\pi(2)}(x) > \ldots > f_{\pi(n)}(x)$ *for small positive* x*.*

Our goal is to give a fairly precise description of interchanges. We start with some estimate of the growth of the number of interchanges.

Theorem 2. *Let* $a(n)$ *denote the number of interchanges of* $\{1, 2, \ldots, n\}$*. Then* $\frac{1}{n} \log a(n)$ *converges to* $\log\left(3 + 2\sqrt{2}\right)$ *when* n *tends to infinity.*

Note in particular that $a(n)$ is much smaller than the total number $n!$ of permutations. Here is a "concrete" way of computing $a(n)$.

Theorem 3. *The number* $a(n)$ *of interchanges is twice the number of possible ways of parenthesizing a word of length* n *in a "correct way". More precisely, one requires that each pair of opening-closing parenthesis contains at least two letters and that there is a pair containing the full word.*

Examples.

1. For $n = 3$, one can parenthesize abc in three different ways: $((ab)c)$, $(a(bc))$, (abc). We do not count "useless" parenthesis, like double (( )), as in $(((ab))c)$ containing a useless pair around ab.

2. For the 4-letters word $abcd$, there are 11 ways: $(abcd)$, $((ab)cd)$, $(a(bc)d)$, $((ab(cd))$, $((ab)(cd))$, $((abc)cd)$, $(a(bcd))$, $(((ab)c)d)$, $((a(bc))d)$, $(a((bc)d))$, $(a(b(cd)))$. Twice eleven gives twenty-two and we do recover the twenty-two allowable permutations of four graphs.

Theorem 4. *One can program a computer to answer the question: "Is a given permutation* π *of* $\{1, 2, \ldots, n\}$ *an interchange?" in such a way that the computing time is bounded by some polynomial in* n*.*

Theorem 5. *A permutation* π *of* $\{1, 2, \ldots, n\}$ *is an interchange if and only if there does not exist four integers* a, b, c, d *which are permuted as in the forbidden permutations, i.e., satisfying* $n \geq a > b > c > d \geq 1$ *and* $\pi(b) > \pi(d) > \pi(a) > \pi(c)$ *or* $\pi(c) > \pi(a) > \pi(d) > \pi(b)$*.*

Actually, we shall not prove these results since they will appear as obvious consequences of the description of interchanges that follows.

Some arboriculture

Let us consider the following eight polynomials, (almost) chosen at random.

$$
\begin{aligned}
&f_1(x) = x^2 - x^3 + x^5 + x^6 + x^8,\\
&f_2(x) = 2x + 2x^2 - x^3 - x^4 + x^5 + x^6,\\
&f_3(x) = -x + x^2 + x^3 + x^4 - x^5 + x^6 + x^7 - x^8,\\
&f_4(x) = 2x + x^2 + x^4 + 2x^5 - x^6,\\
&f_5(x) = -x^2 - x^3 + 2x^4 + x^5,\\
&f_6(x) = -x^2 - x^3 - x^4 + 2x^5 + x^6 - x^7,\\
&f_7(x) = -x + x^2 + x^3 + x^4 + x^5 - x^6, \text{and}\\
&f_8(x) = -x + x^2 + x^3 + x^4 + x^5 - 2x^6.
\end{aligned}
$$

Figure 4 shows their graphs.

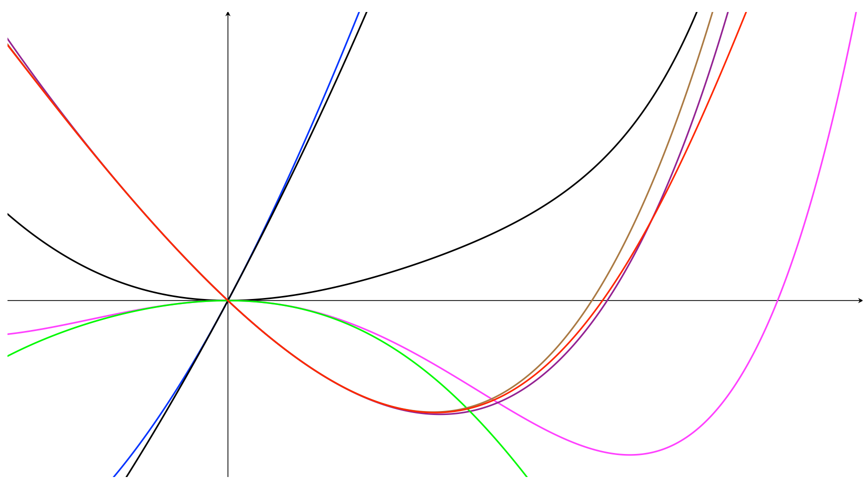

Figure 4: Eight graphs

We are going to analyze their relative positions for small x, positive or negative.

Let us consider first the degree 1 terms. One finds only three possibilities.

- $2x$ (for f_2 and f_4),
- 0 (for f_1, f_5, f_6), or
- $-x$ (for f_3, f_7 and f_8).

This corresponds to the fact that if we zoom on the origin, we essentially only see three graphs. Of course, for $x > 0$, one has $-x < 0 < 2x$, so that we can say that for small positive x, the graph of f_3 for instance is below the graph of f_2. However, the degree 1 terms do not contain enough information to distinguish between f_1 and f_5. We picture the situation in Figure 5.

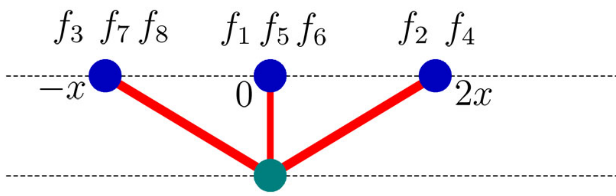

Figure 5: Degree one terms

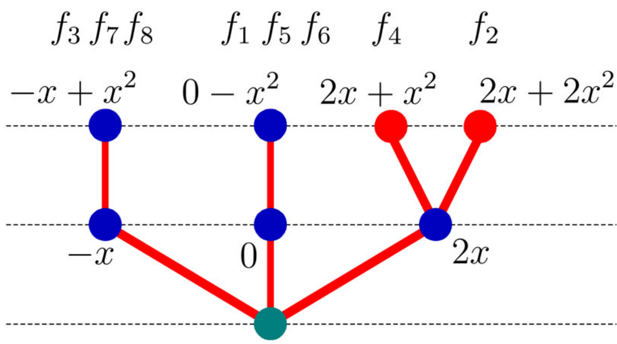

Figure 6: Degree two terms

Let us now consider the degree 2 terms. We have now four possibilities $-x+x^2$, $-x^2$, $2x+x^2$, $2x+2x^2$, pictured in Figure 6.

The second degree terms enabled us to distinguish between f_2 and f_4 and it will be therefore useless to continue analyzing these two polynomials to higher degrees. However, f_3, f_7, f_8 an f_1, f_5, f_6 are still equally placed. We have to proceed to a finer analysis.

Degree 3 is useless, and does not allow us to distinguish between two previously undistinguishable f_i as illustrated in Figure 7. We have to go on.

The final result us illustrated in Figure 8: we have to wait until the fifth degree to separate all these polynomials.

We can therefore encode the situation by a *planar tree*: at each level the nodes are ordered from left to right. In this paper, we only deal with planar trees and for simplicity we skip the word planar.

Let us check that this tree contains enough information to construct the associated interchange. Indeed, here is the recipe.

For small $x > 0$, it should be obvious to the reader, from the tree, that:

$$f_3(x) < f_8(x) < f_7(x) < f_6(x) < f_1(x) < f_5(x) < f_2(x) < f_4(x).$$

How should we order the polynomials for small negative x? We have to evaluate the valuations of the $f_i - f_j$. One should simply determine at which level the two paths

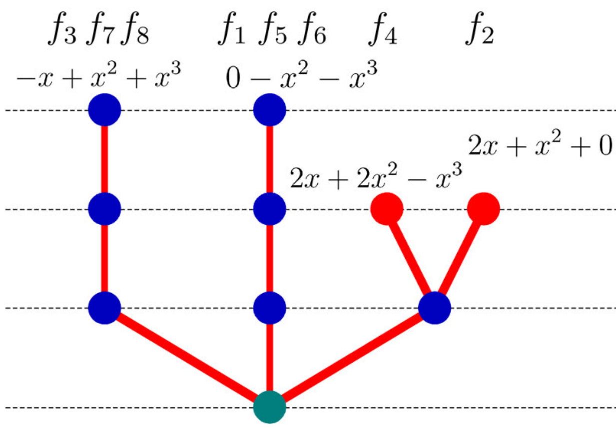

Figure 7: Degree three terms

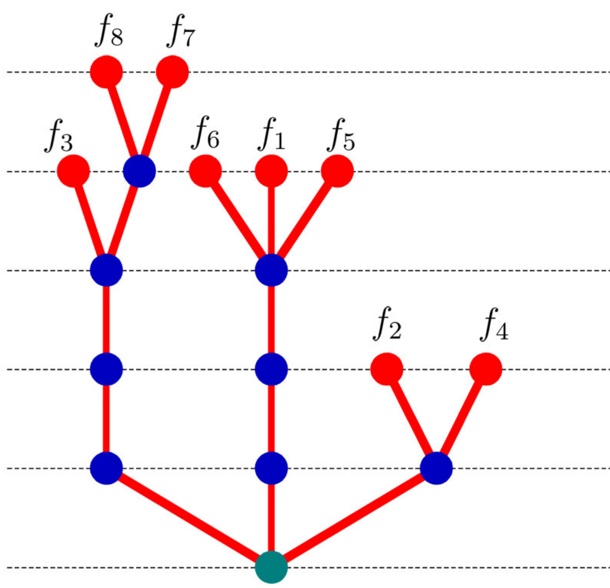

Figure 8: Final picture

starting from the root of the tree and aiming at i and j bifurcate. If this level is odd, the valuation of $f_i - f_j$ is one more, and is hence even: $\pi(i)$ and $\pi(j)$ are in the same order as i and j. If the bifurcation level is even, we obtain the contrary.

For instance, on Figure 8, in order to go in the tree from f_3 to f_8, one has to go down until level 3, so that the valuation of $f_8 - f_3$ is 4. Since $f_8(x) - f_3(x) > 0$ for small positive x, the same is true for small negative x.

In our situation, we find for small negative x that:

$$f_2(x) < f_4(x) < f_6(x) < f_1(x) < f_5(x) < f_3(x) < f_7(x) < f_8(x).$$

In order to determine the corresponding interchange, we have to rename the f_i in such a way that for small negative x the sequence $f_i(x)$ is decreasing and then observe the ordering for small positive x. One finds

$$(1,2,3,4,5,6,7,8) \to (7,8,4,5,6,2,1,3).$$

One could also say that the permutation is obtained by twisting the tree "over" at every even level and watching what happens to the leaves.

Of course, the convention that we used, looking first at the positions of the graphs for small negative x and then for small positive x, is only one of the possible conventions. One could change the orientation along either axis, but this would not change the definition of interchange.

In summary, given the polynomials, on can construct a tree and, given the tree, we can easily order the polynomials for small values of x and deduce the corresponding interchange.

Pruning

Our trees contain too much information and we shall prune their branches. Let us introduce some vocabulary. Our trees have three kinds of vertices: one *root*, some *nodes*, and several *leaves* (terminal vertices). An edge connecting two vertices will be called a *branch*. Each vertex lies at some *level* which is the number of branches connecting it to the root. The two endpoints of a branch are called *child* and *parent*, the parent being at a lower level.

Suppose two vertices x, y are connected in the tree by some path consisting of an even number of branches. Suppose moreover that all the intermediate vertices between x and y are non-ramified, i.e., that have only one child. Let us cut all these paths. The tree now consists of several trees, one of which contains the initial root. Let us identify the endpoints x, y on each of these paths. This produces a new tree. Of course, in this process, the levels of some vertices have changed, but only by an even number.

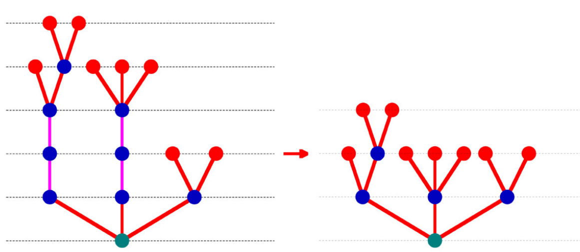

Figure 9: Pruning

Therefore, if one computes the valuation of $f_i - f_j$ in the new tree, the parity did not change and this parity is the only information that matters in order to construct the interchange. Note that the pruned tree has the property that all its nodes have at least two children.

In summary, given n polynomials, one can construct a tree such that:

1. the root can have any number of children,

2. every node has at least two children,

3. there are exactly n leaves, labeled by the n polynomials.

Let us say that a tree is *pruned* if it satisfies these properties. It should be clear that for any pruned tree, one can find n polynomials such that the associated pruned tree is the given one. In particular, the number of interchanges is less than or equal to the number of pruned trees.

We shall now show that these two numbers are actually equal. Two lists of polynomials might theoretically give the same interchange but different pruned trees. The issue is to show that one cannot have two different pruned trees giving the same permutation. In other words, we have to show that the interchange determines uniquely the pruned tree.

From an interchange to a tree

Let T be a pruned tree with $n \geq 2$ leaves and let $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ be the associated interchange. From now on, we always label leaves from 1 to n, from left to right. We want to show that π determines T. On our way, we shall produce an algorithm which determines if a given permutation is associated to some pruned tree, i.e., if it is an interchange.

Let us say that a (non-empty) subset of $\{1, 2, \ldots, n\}$ is an *interval* if it consists of a certain number of consecutive integers. If a is a node, the set $D(a)$ of leaves which are descendants of a is an interval. One easily checks that the interchange π maps each of these intervals $D(a)$ onto some interval. In combinatorics, such permutations are called *separable* (see for instance [3, section 2.2.5] for references).

The root of a pruned tree with $n \geq 2$ leaves can have one, or at least two children.

In the first case, the unique child of the root has at least two children $a_1, \ldots, a_m$. The corresponding consecutive intervals $D(a_1), \ldots, D(a_m)$ are invariant by π. Moreover, this decomposition of $\{1, 2, \ldots, n\}$ is maximal: none of these intervals $D(a_i)$ can be split into two sub-intervals which are invariant by π.

In the second case, the children $a_1, \ldots, a_m$ of the root define consecutive intervals $D(a_1), \ldots, D(a_m)$ which are now invariant by $\bar{\pi}$ where $\bar{\pi}$ denotes the permutation π followed by the flip $i \mapsto n + 1 - i$.

We therefore have the following algorithm constructing the tree T from its interchange π. If it is possible to decompose $\{1, \ldots, n\}$ into at least two intervals which are invariant by π, we are in the first case and we conclude that the root of T has only one child. The children of this unique child are indexed by the unique maximal decomposition of $\{1, \ldots, n\}$ in consecutive intervals invariant by π. By induction, we continue the description of T by looking at the restrictions of π to these invariant intervals.

If it is possible to decompose $\{1, \ldots, n\}$ into at least two intervals which are invariant by $\bar{\pi}$, we are in the second case and we conclude that the root has at least two children. These children are indexed by the unique maximal decomposition of $\{1, \ldots, n\}$ in consecutive intervals invariant by $\bar{\pi}$. By induction, we continue the description of T by looking at the restrictions of $\bar{\pi}$ to these intervals.

Note that these two situations cannot occur simultaneously, since $\pi^{-1}(1) < \pi^{-1}(n)$ in the first case and $\pi^{-1}(1) > \pi^{-1}(n)$ in the second case.

This algorithm can be applied to any permutation, without assuming a priori that it is associated to a pruned tree. If, at some stage of the algorithm, it is not possible to split an interval into sub-intervals which are invariant under the permutation or its composition with the flip, we conclude that the permutation is not an interchange and the algorithm stops.

We therefore have reached the conclusion that the number of interchanges is equal to the number of pruned trees.

Some exercises

Exercise 1: Prove theorem 3.
Hint: The root of a pruned tree can have one child or at least two children. If it has only one child, one can delete the root and the resulting tree has a new root having at least two children. Conversely, given a pruned tree such that the root has at least two children, on can add a new root, one level below, which will be the parent of the previous root.

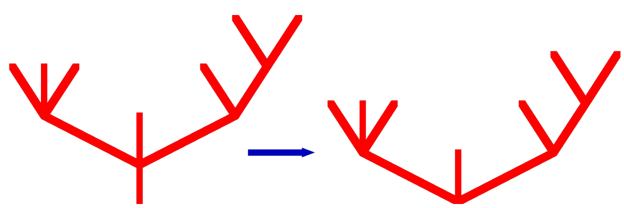

Figure 10: Deleting the root

Hence, the number of pruned trees is twice the number of pruned trees which are such that the root has at least two children. These trees can be interpreted as ways of placing parenthesis on a word. For instance, Figure 11 corresponds to $((ab)c(d(e(fgh))))$.

Exercise 2: Prove theorem 5.
Hint: It is clear that an interchange cannot "contain" the forbidden permutations. The main content of the theorem is the *sufficient condition*: if π does not contain those two permutations, it is possible to construct a pruned tree giving rise to π. The proof is by induction. Starting from some permutation π, one orders first $\pi(1), \pi(2), \ldots, \pi(n-1)$. This produces a permutation of $\{1, 2, \ldots, n-1\}$, for which you apply the induction hypothesis. You therefore get a pruned tree with $n-1$ leaves and you still have to show how to place the last leaf.

Exercise 3: Prove theorem 4. Is is possible to find some algorithm in linear time in n?

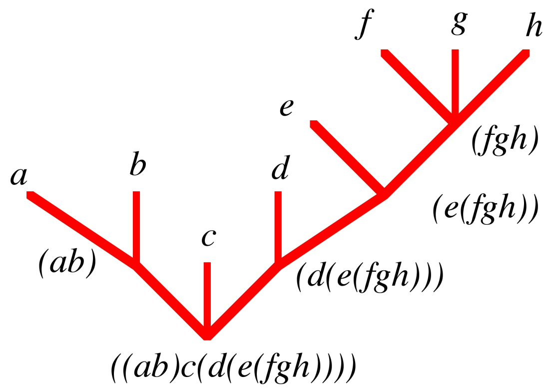

Figure 11: Parenthesis

Let us count!

We are going to count the number $a(n)$ of interchanges.

Let $b(n)$ be the number of pruned trees with n leaves which are such that the root does not have a single child (hence has no child, if $n = 1$, and at least two children if $n \geq 2$). We know that $a(n) = 2b(n)$ for $n \geq 2$. The first values of b are:

1. $b(1) = 1$: a tiny tree whose root is also its unique leaf,
2. $b(2) = 1$: a tiny tree with two branches and two leaves,
3. $b(3) = 3$.

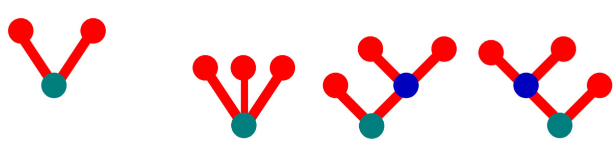

Figure 12: Small trees

It is very tempting to establish an recurrence relation for $b(n)$.

To do this, start with a pruned tree with n leaves such that the root has at least two children. If one deletes the root and the adjacent branches, one gets a certain number of trees, having a total of n leaves. Conversely, if one starts with at least two pruned trees having n leaves in total, one can add a new root and connect it to the previous roots, in order to construct a pruned tree with n leaves.

Therefore, we have the following relation:

$$b(n) = \sum_{k=2}^{n} \sum_{i_1+i_2+\ldots i_k=n} b(i_1)b(i_2)\cdots b(i_k).$$

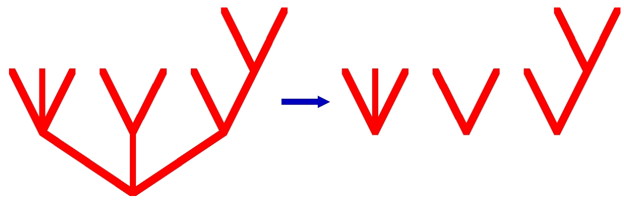

Figure 13: Recurrence

We now use the classical method of generating series (see for instance [1]). Define the formal power series H by:

$$H(t) = \sum_{n=1}^{\infty} b(n)t^n = t + t^2 + 3t^3 + \ldots.$$

Let us square H:

$$H(t)^2 = t^2 + 2t^3 + 4t^4 + \ldots.$$

The coefficient of t^n in this new series is $\sum_{i_1+i_2=n} b(i_1)b(i_2)$, which is equal to the number of pruned trees with n leaves such that the root has exactly two children. Using $H(t)^3$, we would count the number of trees whose root has three children, etc.

The infinite series

$$G(t) = H(t)^2 + H(t)^3 + \ldots$$

counts therefore all trees, except the only one which has a single leaf. Hence, this infinite sum is $H(t) - t$. We have proved that,

$$G(t) = H(t) - t = H(t)^2 + H(t)^3 + \ldots.$$

Summing the geometric series, we get,

$$H(t) - t = \frac{H(t)^2}{1 - H(t)}$$

or

$$2H(t)^2 - (1+t)H(t) + t = 0,$$

which yields,

$$H(t) = \sum_{n=1}^{\infty} b(n)t^n = (1 + t - \sqrt{1 - 6t + t^2})/4.$$

As a function of a complex variable, $(1 + t + \sqrt{1 - 6t + t^2})/4$ is well defined and holomorphic in the disc of center 0 whose radius is the smallest of the two roots of $1 - 6t + t^2 = 0$, i.e., $t = 3 - 2\sqrt{2}$. The radius of convergence of $H(t)$ is therefore $(3 - 2\sqrt{2})$. In other words

$$\limsup_{n\to\infty} \frac{1}{n} \log a(n) = \log(3 + 2\sqrt{2}).$$

This is theorem 2. Not quite..., but the reader will easily show that the lim sup can be replaced by a lim.

Visiting Sloane's website

The "On-Line Encyclopedia of integer sequences" [5], is a very powerful tool. In our example, we know that $a(1) = 1$, $a(2) = 2$, $a(3) = 6$, $a(4) = 22$. Let us type $1, 2, 6, 22$ in the main window of this website, and we get immediately a page fully devoted to the *Schröder sequence*. Indeed the sequence $a(n)$ is not a newcomer in mathematics. It has already appeared in many different contexts, long time ago, and the bibliography on this topic is huge.

In Sloane's site, one finds many recurrence relations, many equivalent definitions of $a(n)$, a numerical table of $a(n)$, and much more.

For instance, one finds a refined description of the asymptotic growth of $a(n)$:

$$a(n) \sim \frac{(3+2\sqrt{2})^n}{\left(n\sqrt{2\pi n}\sqrt{3\sqrt{2}-4}\left(1-\frac{9\sqrt{2}+24}{32n}\right)+\dots\right)}.$$

Or, this recurrence relation, enabling a quick computation:

$$(n+1)a(n+1) - 3(2n-1)a(n) - (n-2)a(n-1) = 0.$$

Prove this relation as an exercise! The trick is to find a second order differential equation satisfied by $H(t)$.

From the bibliography in this website, one can easily trace the history of this sequence. According to Plutarch, it seems that Hipparchus, (second century BC), proved that $b(10) = 103049$. What was his motivation? Which method did he use? See [2, 6] for some interesting historical comments.

However, Ernst Schröder made the first systematic study of $a(n)$ in 1870 [4]. His motivation was not related to the behavior of intersecting curves. If you like counting trees, you will probably enjoy this paper [7].

A slightly different French version of this note appeared online

`http://images.math.cnrs.fr/Quand-beaucoup-de-courbes-se.html`.

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