Appendix E - Riemannian Foliations : Examples and Problems

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The object of this appendix is to present, via some examples, a certain number of open problems which seem to us to be interesting.

1. How to construct Riemannian foliations.

The general theory of Riemannian foliations clearly shows the crucial role played by Lie foliations. For this reason we will concentrate essentially on these Lie foliations.

The following is an extremely general method of construction. Let G and H be two simply connected Lie groups and let $D : H \to G$ be a surjective morphism. Suppose that H contains a uniform discrete subgroup Γ. The foliation of H by the fibers of D is clearly invariant by right translations by the elements of Γ so that the compact manifold H/Γ is equipped with a natural foliation. This is a Lie G-foliation whose developing map is precisely D and for which the holonomy morphism $h : \pi_1(H/\Gamma) \cong \Gamma \to G$ is the restriction of D to Γ.

This type of construction can sometimes be modified. For example, let K be a compact subgroup contained in $D^{-1}(e)$. Then the left action of K on H/Γ preserves the leaves of the foliation that we have constructed. If this action is free, then one obtains a Lie G-foliation in the manifold $K \backslash H/\Gamma$.

The Lie foliations constructed by this type of method will be called "homogeneous foliations".

We now give some concrete examples.

1.1 : "Nilpotent" foliations.

Let G be a simply connected nilpotent Lie group and let $\Gamma \subset G$ be a finitely generated dense subgroup. In this case Malcev's theory [Rag] enables one to construct a simply connected nilpotent Lie group H , an imbedding i of Γ into H and a surjective morphism $D : H \to G$ such that :

1) $i(\Gamma)$ is discrete and uniform in H .
2) $D \circ i = id_\Gamma$.

So one has :

> *Proposition* $[Ha]_4$: Every dense finitely generated subgroup of a simply connected nilpotent Lie group G can be considered as the holonomy group of a Lie G-foliation on a compact manifold.

1.2 : "Solvable" foliations.

We will just give an example, due to A.Haefliger. Consider the Lie group GA , of affine orientation preserving bijections of $\mathbf{R}$, which we identity with the group of matrices $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ $[a > 0]$. This group is a semi-direct product :

$$0 \to \mathbf{R} \to GA \to \mathbf{R}_+^* \to 1$$

It's the simplest example of solvable Lie group that isn't nilpotent.

Let $\mathbf{k}$ be a number field, let A be its ring of integers and let U be the group of units of A . Suppose that $\mathbf{k}$ verifies the following conditions :

i) $\mathbf{k}$ is totally real, i.e., every imbedding of $\mathbf{k}$ in $\mathbf{C}$ has its image contained in $\mathbf{R}$. Let $i : k \to \mathbf{R}$ be one of these imbeddings.

ii) If u is a unit of U such that $i(u) > 0$, then all of the conjugates u' of u verify $i(u') > 0$.

Under these conditions, A , as an additive group, is isomorphic to $\mathbf{Z}^n$, where n is the degree of $\mathbf{k}$ over $\mathbf{Q}$. Moreover, the group of units u such that $i(u) > 0$ is isomorphic to $\mathbf{Z}^{n-1}$ [see, for example [Sam]].

By considering the action of U on A , one can construct a group Γ which is a semi-direct product :

$$0 \to \mathbf{Z}^n \to \Gamma \to \mathbf{Z}^{n-1} \to 0.$$

Conditions i) and ii) enable one to "tensorize" this expression by $\mathbf{R}$, that is, to construct a Lie group H :

$$0 \to \mathbf{R}^n \to H \to \mathbf{R}^{n-1} \to 0.$$

This group H naturally contains Γ as a uniform discrete subgroup. Furthermore, given an imbedding i of k in $\mathbf{R}$, one can imbed Γ in GA. This imbedding can, in its turn, be tensorized by $\mathbf{R}$. One thus obtains a surjective morphism D from H onto GA.

So we have constructed a homogeneous Lie GA-foliation on H/Γ. This manifold H/Γ is a $\mathbf{T}^n$-bundle over the torus $\mathbf{T}^{n-1}$. The leaves are dense as soon as $n \geq 3$. When $n = 2$, one rediscovers the Lie flows described in Appendix A.

1.3 : "Semi-simple" foliations.

Here again, we will just give an example. Consider the group $H = PSL(2, \mathbf{R}) \times PSL(2, \mathbf{R})$. A theorem of A.Borel [see [Bor]] states that H, like any semi-simple group, admits a discrete uniform subgroup Γ. We may assume that Γ is irreducible, that is, that the projections of Γ into each of the two factors of H are dense. By considering the group $G = PSL(2, \mathbf{R})$ and the projection $D : H \to G$ onto the second factor, one obtains a Lie $PSL(2, \mathbf{R})$-foliation with dense leaves on the compact 6-dimensional manifold $PSL(2, \mathbf{R}) \times PSL(2, \mathbf{R}) / \Gamma$.

In this case, one can assume as well that Γ is torsion free, which ensures that the action of Γ on

$$(SO(2) \setminus PSL(2, \mathbf{R})) \times (SO(2) \setminus PSL(2, \mathbf{R}))$$

is proper and has no fixed point. By observing that the homogeneous space $SO(2) \setminus PSL(2, \mathbf{R})$ is the Poincaré disc, one thus obtains examples of Riemannian foliations [which in fact are transversally hyperbolic] on compact 4-manifolds.

1.3 : "Compact" foliations.

The case where G is compact is somewhat different from the general case in that here one has another general method of construction. Let B be an arbitrary compact manifold, let $\tilde{B}$ be its universal covering and let $h : \pi_1(B) \to G$ be an arbitrary homomorphism. One can then "suspend" h, that is, one can consider the quotient of

$\tilde{B} \times G$ by the obvious diagonal action of $\pi_1(B)$. This produces a compact manifold which is a principal G-bundle over B, equipped with a Lie G-foliation transverse to the fibers.

An interesting example of this construction is given by A.Connes [Cn]. If B is a compact orientable surface of genus 2, then it is possible to show that $\pi_1(B)$ injects into $SO(3, \mathbf{R})$. In this way, one obtains a Lie $SO(3, \mathbf{R})$-foliation whose leaves are all simply connected [diffeomorphic to $\mathbf{R}^2$]. The interest of this example, in the theory of A.Connes, comes from the fact that, although the leaves are contractible, one is lead to consider that the "average first Betti number" of the leaves is non-zero.

2. *Towards a classification of Lie foliations ?*

It would seem unreasonable to hope to completely classify Lie foliations with dense leaves. However a "weak" classification is conceivable and that is what we would like to explain in this section.

Let F be a Lie G-foliation with dense leaves on a compact manifold M. If N is a compact manifold and if $f : N \to M$ is a smooth mapping that is transverse to F, then it is clear that f^*F is also a Lie G-foliation. Following A.Haefliger $[\mathrm{Ha}]_4$, we will say that (M, F) is a classifying space for the pair (G, Γ) if, conversely, for every Lie G-foliation F' with holonomy group Γ on a compact manifold N, there exists $f : N \to M$ such that $f^*F = F'$.

The following result characterizes the classifying spaces.

> *Theorem.* $[Ha]_4$: Let F be a Lie G-foliation with holonomy group Γ on a compact manifold M. Then F is a classifying space for the pair (G, Γ) if and only if the leaves of F are all contractible.

We have already observed that compact groups behave differently. For convenience, let us say that a Lie group has no compact factor if the quotient of G by its radical is a semi-simple group having no compact factor.

The following would be an optimistic conjecture for a "weak" classification of Lie foliations :

Conjecture 1 : Let G be a simply connected Lie group having no compact factor. Let F be a Lie G-foliation with holonomy group $\Gamma \subset G$ with dense leaves on a compact manifold N. Is it true that there exists a homogeneous Lie foliation which is a classifying space for the pair (G, Γ) ?

A positive answer to this question would therefore show that the only Lie foliations with dense leaves are the homogeneous foliations and their inverse images by smooth maps.

A certain number of results suggest that the answer to this question may be yes.

Proposition : Conjecture 1 is true if G is nilpotent.

Indeed, for each dense finitely generated subgroup of G, we constructed above a homogeneous foliation with holonomy Γ. It is easy to see that the leaves of this homogeneous foliation are diffeomorphic to a simply connected nilpotent group and are therefore contractible.

Discrete uniform subgroups of Lie groups often have strong arithmetic properties. If conjecture 1 is true, then one could expect that these arithmetic properties would be reflected in the holonomy groups of Lie foliations with dense leaves. There are a few results that run in this direction. The first one treats the case where $G = GA$.

Theorem [Gh]$_4$: Let $\Gamma \subset GA$ be the holonomy group of a Lie GA-foliation on a compact manifold. If Γ is generated by the matrices $\begin{pmatrix} a_i & b_i \\ 0 & 1 \end{pmatrix}$ $[i = 1, 2, ..., N]$, then the $2N$ real numbers $a_1, ..., a_N, b_1, ..., b_N$ are algebraically dependent over the rational numbers.

One can find in [Gh]$_4$ another result related to conjecture 1 which concerns the cohomological dimension of Γ.

But the most encouraging result in favour of conjecture 1 is certainly that of R.Zimmer. It is still only a partial answer since it deals with foliations whose leaves are symmetric spaces, which is obviously a strong restriction. Nevertheless, it is the first general arithmetic result known for holonomy groups. We state it in our own language :

Theorem [Zi]$_2$: Let F be Lie G-foliation with holonomy group Γ on a compact manifold M. Suppose that the leaves are dense and simply connected. As well, suppose that there exists a bundle-like metric on M for which the leaves of F are locally symmetric spaces, with non-positive curvature, whose irreducible factors have rank at least 2. Then G is semi-simple and the pair (G, Γ) admits a classifying space which is a homogeneous foliation.

To finish this section, we would like to stress the fact that knowledge of the existence of a classifying space is an important piece of information which leads, in certain cases, to a complete understanding of the foliation. Here is an example of an application of this method.

Theorem : Let F be Lie G-flow with holonomy group $\Gamma \subset G$ on a compact manifold M. If G is nilpotent then F is conjugated to a homogeneous flow on a homogeneous space H/Γ where H is nilpotent.

Sketch of the Proof : If the leaves of F are compact, then F is a circle bundle over a quotient G/Γ and the theorem is easy to prove.

Otherwise, the leaves of F are all diffeomorphic to $\mathbf{R}$ and are therefore contractible. So (M, F) is a classifying space for the pair (G, Γ). Furthermore, we have seen that in the nilpotent case one can construct a homogeneous foliation F' on H/Γ which is also a classifying space for (G, Γ). So there exists a homotopy equivalence $f : M \to H/\Gamma$ such that $f^*F' = F$. In particular, the dimension of H is equal to that of M and F' also has dimension 1. Parametrize F by a flow ϕ_t and F by a flow ψ_t. Then there exists a function $u : H/\Gamma \times \mathbf{R} \to \mathbf{R}$ such that :

$$f(\phi_t(x)) = \psi_{u(x,t)}(f(x)).$$

The map f may not be a diffeomorphism. This comes from the fact that, x being fixed, the map $t \to u(x, t)$ may not be monotonic. So we modify f along the orbits in order to make it injective by using the averaging technique. Let $\theta : \mathbf{R} \to \mathbf{R}$ be a function with compact support and integral 1. Then set :

$$f_T(x) = \psi_{v(x)}(f(x))$$

where $v(x) = \frac{1}{T}\int_{-\infty}^{+\infty} u(x, t)\theta(Tt)dt$. It is easy to see that, for sufficiently large T, f_T is a diffeomorphism sending F to F'. For more details the reader may consult $[\mathrm{Gh}]_3$ where a similar situation is considered.

In the previous theorem, the fact that F has dimension 1 was only used in order to "smoothen" f along the leaves. It would seem interesting to ask whether such a "smoothing" is possible in higher dimension.

> *Problem* : Let F_1 and F_2 be two Lie G-foliations with the same holonomy group on a compact manifolds. Suppose that the leaves of F_1 and F_2 are diffeomorphic to $\mathbf{R}^n$. Then does there exist a diffeomorphism sending F_1 to F_2 ?

This problem generalizes a well known problem concerning non-singular closed forms on tori [the case where $G = \mathbf{R}^n$]. Let ω_1 and ω_2 be two non-singular closed forms on the torus $\mathbf{T}^{n+1}$ and suppose that they belong to the same cohomology class. Suppose as well that the period groups of the two forms have rank $n + 1$ over $\mathbf{Q}$. Then are the forms ω_1 and ω_2 conjugated ? [See [Sk] for a discussion of this problem].

3. *Deformations of Riemannian foliations.*

Since the structure of Riemannian foliations is now quite well known, it is natural to study the structure of the space of Riemannian foliations on a given manifold. More precisely, the problem that we will consider in this section is the following. Given a Riemannian foliation, is it possible to describe, up to differentiable conjugation, all the Riemannian foliations which are close to it ?

We begin again with the simple case of Lie foliations. Let G

be a Lie group and let F_0 be a Lie G-foliation on the compact manifold M. If x_0 is a base point of M, one can define without ambiguity the holonomy representation $h_0 : \pi_1(M, x_0) \to G$. If F_1 is another Lie G-foliation on M, we will say that F_1 is close to F_0 if there exists an open covering (U_i) of M such that on each U_i the foliations F_0 and F_1 are defined by submersions that are close to one another, from U_i to an open subset of G. It is clear that the holonomy $h_1 : \pi_1(M, x_0) \to G$ of F_1 is then close to h_0, in the following obvious sense : on a finite system of generators $\{\gamma_1,..., \gamma_n\}$ of $\pi_1(M, x_0)$, the images $h_0(\gamma_i)$ and $h_1(\gamma_i)$ are close in G.

The following result generalizes that of Moser [when $G = \mathbf{R}$ see [Mos]]. It is well kown by the experts, though to our knowledge the proof hasn't been published. One has an "intuitive" proof in $[\text{Thu}]_2$.

> *"Moser-type" Theorem* : If $h_1 : \pi_1(M, x_0) \to G$ is a homomorphism close to h_0, then there exists a Lie G-foliation F_1 that is close to F_0 and which has holonomy h_1.
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> If F_1 is a Lie G-foliation close to F_0, then there exists a diffeomorphism close to the identity that conjugates F_0 and F_1 [as Lie foliations] if and only if there exists g in G, close to the identity, such that $h_0 = gh_1g^{-1}$.

Consider the space of homomorphisms from $\pi_1(M, x_0)$ to G. Since $\pi_1(M, x_0)$ is finitely presented, this space is an analytic subset of the cartesian product G^n. Let S be the germ of this analytic set in a neighborhood of h_0. The first part of the above theorem can then be expressed in the following manner : there exists a family of Lie G-foliations F_s, parametrized by $s \in S$, such that the holonomy of F_s is precisely s. As for the second part of the theorem, it can be expressed in the following manner : let H_λ be a family of Lie G-foliations parametrized by $\lambda \in \Lambda$, with $H_0 = F_0$ [Λ is a "reasonable" space of parameters, for example the germ at 0 of an analytic subset of $\mathbf{R}^m$]. Then there exists a map $\phi : \Lambda \to S$ such that H_λ is conjugated to $F_{\phi(\lambda)}$ by a diffeomorphism of M which depends smoothly on λ. We will express this fact by saying that the family $(F_s)_{s \in S}$ is a "versal deformation" of F_0.

In the general case of Riemannian foliations, we are lead to introduce the following definitions. We intentionally formulate them in a vague fashion, some minor changes being certainly possible.

Definitions : Let F be a Riemannian foliation on a compact manifold M and let S be a germ at 0 of an analytic subset of $\mathbf{R}^m$. Then a "deformation" of F indexed by S is a family of Riemannian foliations F_s indexed by $(S, 0)$ which depends smoothly on s and for which $F_0 = F$.

If F_s and H_s are two deformations of $F_0 = H_0$, indexed by $(S, 0)$, then we say that these deformations are "equivalent" if there exists a family of diffeomorphisms $(\phi_s)_{s \in S}$ such that $\phi_0 = id$ and $\phi_s^* F_s = H_s$.

Let F_0 be a deformation of F indexed by $(S, 0)$ and let $(T, 0)$ be another germ of an analytic set. If $u : (T, 0) \to (S, 0)$ is a smooth map, the deformation H_t defined by $H_t = F_{u(t)}$ is the "deformation induced" from F_s by u.

A deformation (F_s) of F indexed by $(S, 0)$ is "versal" if every other deformation indexed by $(T, 0)$ is induced by F_s by some mapping $u : (T, 0) \to (S, 0)$.

Conjecture 2 : Every Riemannian foliation on a compact manifold admits a versal deformation.

In intuitive terms, this conjecture can be expressed in the following manner : the conjugation classes of Riemannian foliations close to a given Riemannian foliation depend only a finite number of parameters.

For example, the classification of Riemannian flows in dimension 3 and 4 [see Appendix A] is so precise that it effectively shows that these flows are described by a finite number of parameters. The proof of conjecture 2 for the case of Riemannian flows in arbitrary dimension shouldn't in fact pose any major problems.

Here are some of the difficulties that one meets while trying to prove this conjecture.

Let H be a Lie group which possesses a discrete uniform subgroup Γ. Let G_α be a Lie subgroup that is normal in H and depends on one parameter α. Then the manifold H/Γ is furnished with a one parameter smooth family of Lie foliations F_α corresponding to the left cosets of G_α in H. The foliation F_α is a Lie H/G_α-foliation. The family of Lie groups H/G_α is a continuous family of Lie groups in the sense that their Lie algebras can be defined by structure constants that depend continuously on α. In certain cases this family may be non-trivial, that is, the groups H/G_α may not be pair-wise isomorphic. So, in short, when a Lie foliation deforms, its structure group may also deform. Here is an example of this situation. The group H will be the nilpotent group whose Lie algebra $\mathfrak{h}$ is generated by X, Y, Z, T with the following relations :

$$\begin{cases} [X, Y] = Z \\ Z \text{ and } T \text{ are in the center of } \mathfrak{h}. \end{cases}$$

The group G_α is the one parameter group generated by $Z + \alpha T$. The reader can easily verify that H/G_0 is Abelian [it is isomorphic to $\mathbf{R}^3$] and if $\alpha \neq 0$, H/G_α isn't Abelian [it is isomorphic to the Heisenberg group of dimension 3].

This suggests that a proof of the conjecture should incorporate two ingredients : the Moser theorem cited above and the theory, now well developed, of deformations of Lie algebras. Of course, such an approach requires a reduction of the families of Riemannian foliations to families of Lie foliations.

Problem : Develop a structure theory for deformations of Riemannian foliations. For example, let $\{F_s\}_{s \in S}$ be a deformation of the transversally parallelizable foliation F_0 through transversally parallelizable foliations parametrized by the germ of an analytic set S. Does there exist a family $\{\phi_s\}_{s \in S}$ of diffeomorphisms and a fibration $\pi : M \to B$ such that :

1) the leaves of $\phi_s^* F_s$ are contained in the fibers of π

2) the restriction of $\phi_s^* F_s$ to a fiber of π is a Lie G_s-foliation where G_s is a deformation of Lie groups.

To finish this section, note that if the ambient manifold is simply connected, then the structure of its Riemannian foliations is extremely simple [see $[\mathrm{Gh}]_2$]. In particular, the structure group is necessarily Abelian [see Chapter 5]. This would suggest that conjecture 2 is probably easier to prove in this case.

4. Cobordism of Riemannian foliations.

Let us recall that A.Haefliger was lead to generalize the notion of a foliation. If X is a topological space, then a Γ-structure of codimension q on X is given by a covering (U_i) of X, continuous maps $f_i : U_i \to \mathbf{R}^q$ and diffeomorphisms ϕ_{ij} of class C^∞, for which $f_j = \phi_{ij} \circ f_i$ and for which the obvious cocycle condition is verified. In the same way, one can define Lie Γ-structures, Riemannian Γ-structures, etc... Two Γ-structures on X are said to be concordant if there exists a Γ-structure on $X \times [0, 1]$ which induces the given Γ-structures on $X \times \{0\}$ and $X \times \{1\}$. A.Haefliger constructed a classifying space $B\Gamma^q$ such that, for all [reasonable] X, the homotopy classes of maps of X into $B\Gamma^q$ are in one to one correspondance with the concordance classes of Γ-structures on X. In the same way, one obtains a Riemannian classifying space $B\Gamma^q_{Rie}$ and, if G is a Lie group, a classifying space for the Lie Γ-structures modelled on G. Let this latter space be denoted by $B\overline{G}$ [this space is different from Milnor's classifying space BG]. For technical reasons, one is lead to consider the classifying spaces $B\overline{\Gamma}^q$ and $B\overline{\Gamma}^q_{Rie}$ of Γ-structures whose normal bundles are trivialized. Notice that a Lie Γ-structure has a normal bundle which is canonically trivialized, so that, by forgetting structure, one has homotopy classes of maps :

$$B\overline{G} \to B\overline{\Gamma}^q_{Rie} \to B\overline{\Gamma}^q .$$

The interest of the spaces $B\Gamma^q$ has become even more obvious because of the work of W.Thurston. If X is a manifold, W.Thurston gives an explicit and concrete condition for a given Γ-structure to be concordant to an actual [non-singular] foliation. The cohomology [and homotopy] of $B\Gamma^q$ is not known ; it is clear that its knowledge would be of fundamental importance for the homotopic classification of foliations. Nevertheless, a small [?] part of the cohomology of $B\Gamma^q$ is known ; this is the continous cohomology. This leads to the theory of secondary characteristic classes of foliations. These classes are constructed by comparing two connections adapted to the foliation : a Bott connection and a Riemannian connection. For example, the secondary characteristic classes of a Riemannian foliation with trivial normal bundle are all zero. For all these notions, the reader may consult [Law], for example.

The problem that we consider here is to explain the vanishing of the characteristic classes in the following way.

Conjecture 3 : The natural map $B\overline{\Gamma}_{Rie} \to B\overline{\Gamma}$ is homotopically trivial.

For example, this conjecture would imply that if F is a Riemannian foliation with trivial normal bundle, then the foliation F is concordant to the trivial Γ-structure [by a non-Riemannian concordance, of course].

As in the general theory, one can begin by considering the case of Lie foliations. The classifying space $B\overline{G}$ can be described in a more direct manner. Let BG^{δ} be the Eilenberg-McLane space $K(G, 1)$. The fundamental group of BG^{δ}, i.e. G, acts on G by translations. So one can consider the associated G-bundle $BG^{\delta} \ltimes G$ over BG^{δ} This space is equipped with a "horizontal Lie G-foliation" transverse to the fibers of the principal G-bundle. It turns out that this total space $BG^{\delta} \ltimes G$, furnished with this "horizontal foliation", is precisely the classifying space $B\overline{G}$ [see $[\mathrm{Ha}]_4$]. When G is contractible, $B\overline{G}$ therefore has the same homotopy type as BG^{δ}. A special case of conjecture 3 is then :

Conjecture 3' : Let G be a Lie group of dimension q. Then the natural map $B\overline{G} \to B\overline{\Gamma}^q$ is homotopically trivial.

This conjecture can indeed be proven in certain cases. If G has dimension q, we will say that G has property (*) if there exists an action of G of class C^{∞} on $\mathbf{R}^q$ having the following properties :

1) the action is trivial on a non-empty open subset of $\mathbf{R}^q$
2) there exists a point of $\mathbf{R}^q$ whose stabilizer is trivial.

For example, $G = \mathbf{R}$ verifies (*). Indeed, it suffices to consider a flow on $\mathbf{R}$ whose fixed point set contains a proper open subset of $\mathbf{R}$. In the same way, $G = \mathbf{R}^q$ verifies (*). It would seem that nilpotent groups verify (*) [by a remark of P.Greenberg].

The interest of this notion comes from the following result which is nothing more than a generalization of a method used by T.Tsuboi in [Ts].

Theorem : Let G be a q-dimensional contractible Lie group verifying (*). Then the natural map $B\overline{G} \to B\overline{\Gamma}^q$ is homotopically trivial.

Sketch of the Proof : Since G acts on $\mathbf{R}^q$, one can construct, by suspension, a $\mathbf{R}^q$-bundle E over BG^δ. This space is equipped with a "horizontal" Γ-structure F. Let s be a section of E. If the image of s is contained in the open set where the action of G is trivial, then s^*F is obviously a trivial Γ-structure on BG^δ. Let $m \in \mathbf{R}^q$ be a point of $\mathbf{R}^q$ whose stabilizer is trivial. Then the orbit of m is diffeomorphic to G and, via this diffeomorphism, the action of G on the orbit of m coincides with the left action of G on itself. So if the image of s is contained in the [open] orbit of m, then s^*F is conjugated to the universal Lie Γ-structure on $BG^\delta \cong B\overline{G}$. Since two arbitrary sections are homotopic, one deduces that this universal Lie structure is concordant to the trivial Γ-structure on $B\overline{G}$.

To finish this section, let us remark once again that the case of Riemannian foliations on simply connected manifolds is probably the simplest case. This follows from the fact the structural group is Abelian and therefore verifies condition (*).

5. "*Qualitative Riemannian foliations*.

Riemannian foliations are those for which two leaves "don't separate too much" one from the other. To formulate this notion, one is lead to introduce bundle-like metrics, which use all the arsenal of differential geometry. In this section, we want to raise the following question : is it necessary to introduce differential geometry in order to develop the theory, or can one instead consider the theory in terms of topological dynamics ?

To formalize this question, we introduce a definition. Let (X, d) be a reasonable metric space [for example, a locally compact locally connected space of finite topological dimension]. Let $\mathbf{H}_1 = \{h_1, \ldots, h_n\}$ be a finite collection of homeomorphisms between open sets of X. We will say that the pseudogroup $\mathbf{H}$ generated by $\mathbf{H}_1$ is equicontinuous if, for all $\epsilon > 0$, there exists $\eta > 0$ such that if x and y are two points of X such that $d(x, y) < \eta$ then $d(h(x), h(y)) < \epsilon$ for all $h \in \mathbf{H}$ whose domain contains x and y.

If E is a compact metric space, then a "foliation" is a covering of E by open sets V_i that are homeomorphic to a product $L_i \times T_i$ such that the "coordinate changes" preserve the "plaques" $L_i \times \{*\}$. Such a foliation obviously possesses a transverse pseudogroup. We will say that the foliation is equicontinuous if its transverse pseudogroup is equicontinuous. A Riemannian foliation is a trivial example of an equicontinuous foliation. A foliation which is topologically conjugated to a Riemannian foliation is also equicontinuous even though it may not be Riemannian. For example, a codimension1 foliation of class C^2 having trivial holonomy is topologically conjugated to a foliation that is defined by a closed form [hence Riemannian] [see [He-Hi]].

> *Problem* : Is it possible to develop a qualitative theory of equicontinuous foliations analogous to that of Riemannian foliations? For example, do the leaf closures also define a partition of the ambient space ?

Here are two examples of analogous situations where such a theory is possible. If ϕ_t is a flow on a compact metric space E, we will say that ϕ_t is equicontinuous if the family of homeomorphisms $\{\phi_t\}_{t \in \mathbf{R}}$ is equicontinuous. This notion is very close to Lyapounov's notion of stability. The following theorem is well known [see [Ne-Sn]]. The reader will notice its analogy with the general structure theorem for Riemannian flows.

Theorem [Ne-Sn] : Let ϕ_t be an equicontinuous flow on a compact metric space. Then the closures of the orbits of ϕ_t define a partition of E. Each of these closures is homeomorphic to a compact Abelian topological group on which ϕ_t acts as a one parameter group.

Sketch of the Proof : Let $x \in E$. On the orbit of x, one has a natural group structure $*$ given by $\phi_{t_1}(x) * \phi_{t_2}(x) = \phi_{t_1+t_2}(x)$. The equicontinuity of ϕ_t implies that this group operation can be extended to the closure of the orbit of x.

This immediately suggests the following question which generalizes the theorem of Caron-Carrière [Appendix A].

Problem : Let ϕ_t be a flow without fixed points on a compact metric space. Suppose that the 1-dimensional foliation generated by ϕ_t is equicontinuous. Then is it true that the closure of an orbit of ϕ_t is homeomorphic to a compact Abelian topological group ?

Another motivation for the general problem of this section is the following remark. In a certain sense, the general structure theorem for Riemannian foliations is a generalization of the well known fact that the group of isometries of a Riemannian manifold is a Lie group. This situation is actually much more general because of the following theorem which follows immediately from the solution of Hilbert's 5^{th} problem.

Theorem : Let (E, d) be a compact connected locally connected finite dimensional metric space. Let Γ be a group of homeomorphisms of E acting equicontinuously on E and having a dense orbit. Then the action of Γ extends to an action of compact Lie group containing Γ.

Sketch of the Proof : By Ascoli's theorem, the closure of Γ in the space of continuous maps from E to itself is a compact topological group G. By considering the metric d' on E defined by

$$d'(x, y) = \operatorname{Sup}_{\gamma} d(\gamma x, \gamma y),$$

one sees that G is a subgroup of the group of isometries of (E, d'). The theorem then follows from the fact that this group of isometries is a compact Lie group [see [Mon-Z]].

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