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Right-handed vector fields & the Lorenz attractor*

Étienne Ghys

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Abstract. The main purpose of this paper is to introduce a class of vector fields on the 3-sphere that we call "right-handed". Roughly speaking, they are characterized by the fact that any two orbits link positively. We give various natural examples and provide some kind of homological characterization. We then describe some of the main dynamical properties of these flows.

Keywords and phrases: dynamical systems, Lorenz attractor, positive braids

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1. Motivation

We discuss the dynamics of some vector fields in the 3-dimensional sphere, seen as topological objects. It has been known for quite a long time that a recurrent point for such a vector field can be considered as some kind of "diffuse" knot. Even though the trajectory may not be closed and therefore does not define a knot in a strict sense, one gets very long pieces of trajectories whose endpoints "almost" match so that one is tempted to use knot theoretical methods in order to understand the topology of these "infinitely long knots".

In a seminal paper, J. Birman and R. Williams [3] proposed a topological analysis of the famous Lorenz attractor generated by the following ordinary

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É. Ghys
Unité de Mathématiques, Pures et Appliquées, de l'École Normale Supérieure de Lyon, U.M.R. 5669 du Centre national de la recherche scientifique, 46, Allée d'Italie, 69364 Lyon Cedex 07, France
(e-mail: ghys@umpa.ens-lyon.fr)

differential equation in 3-space (Fig. 1):

$$\frac{dx}{dt} = 10(y-x); \quad \frac{dy}{dt} = 28x - y - xz; \quad \frac{dz}{dt} = xy - \frac{8}{3}z.$$

The orbits accumulate on some fractal compact invariant set containing a dense countable collection of periodic orbits, each one defining a knot called a *Lorenz knot* by these authors. Fig. 2, extracted from [3], shows some of these periodic orbits. Birman and Williams prove many remarkable properties of these knots. For instance, *Lorenz knots are fibered*. Recall that a knot in the 3-sphere is fibered if its complement fibers over the circle, the fibers behaving in the neighborhood of the knot as a pencil of planes containing a straight line. Even more, Lorenz *links* are fibered: any finite collection of periodic orbits defines a fibered link. However, all these fibrations—one for each choice of a finite collection of periodic orbits—are not clearly related. For instance the genera of their fibers are unbounded as the periods go to infinity. As a final open question in their paper, Birman and Williams suggest that there may be some concept of "fibration" of the complement of the attractor, obtained as a "limit" of these fibrations but that they have been unable to pursue this idea. The purpose of these talks is to propose some global point of view, enabling us to understand all these fibrations "at the same time". As a bonus, one gets some additional information concerning non-periodic orbits: do they define in some way some "fibered diffuse knot"?

In order to get this description, we shall introduce a concept of *right-handed* vector fields, somehow the analogue of *positive braids*, and we shall describe their dynamics. In particular, we shall show that their periodic orbits define fibered links. Of course, the Lorenz attractor will be our primary example of right-handed vector field, but we shall also supply many other natural examples and we believe that this category of dynamical systems does deserve attention.

2. Positivity in topology and dynamics

2.1. Positive braids

The key to Birman and Williams' proof of their fibration theorem is that *positive braids are fibered*.

Recall that a braid is positive if all crossings are positive: looking at the braid as going downward, at each crossing, the left strand goes over the right one. It turns out that the closure of a positive braid is a fibered link. This fact has an unclear history but it seems that it was known to Murasugi and Stallings. Birman and Williams give a detailed proof in their paper depending on *Stallings' fibration theorem*, that we now recall.

Given an oriented link λ in the 3-sphere $\mathbf{S}^3$, any loop γ in the complement $\mathbf{S}^3 \setminus \lambda$ has a linking number $lk(\lambda, \gamma) \in \mathbf{Z}$ with λ. This defines a homomorphism

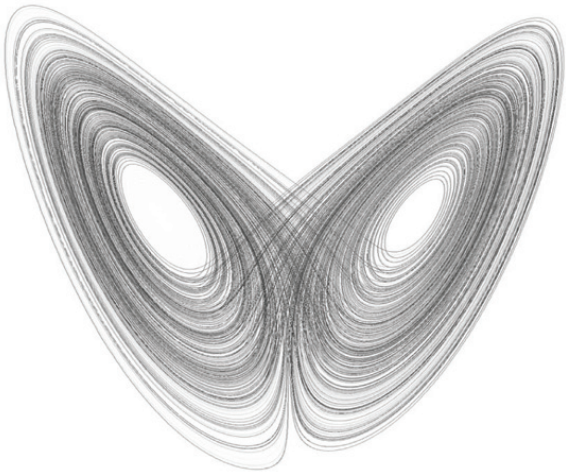

Fig. 1. The Lorenz attractor

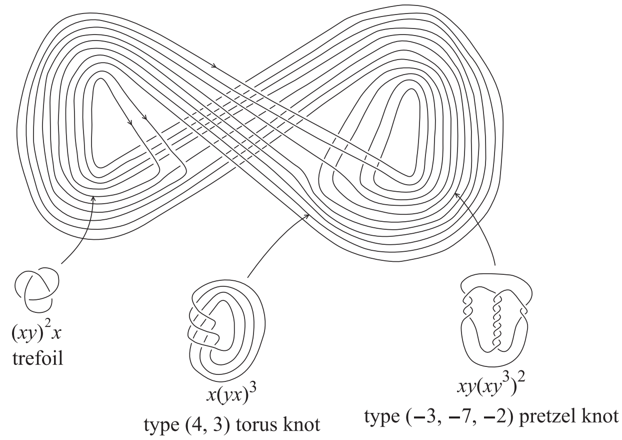

Fig. 2. Some periodic orbits [3]

$lk : \pi_1(\mathbf{S}^3 \setminus \lambda) \to \mathbf{Z}$. Clearly, if the link is fibered, the kernel of this homomorphism is finitely generated since it coincides with the fundamental group of a fiber. Stallings' theorem states that conversely, if the kernel of *lk* is finitely generated, the link is indeed fibered.

The fact that positive braids define fibered links follows by induction on the number of strands. The key point is that a braid determines canonically a *Seifert surface*, i.e., an embedded surface whose boundary is the closure of the braid. The linking homomorphism is nothing but the intersection number with this surface. By induction, one can delete the last strand in a positive braid and describe the Seifert surface as obtained by "band connecting" with a Seifert surface of a positive braid with one strand less. This enables an understanding of the kernel of *lk* by induction and yields the result readily.

2.2. Suspensions

Consider now a non-singular vector field X on a compact manifold M. One says that X is a *suspension* if there is a fibration π of M onto the circle $\mathbf{S}^1$ which is transversal to X. This situation is the paradise for dynamicists since in this case any orbit intersects infinitely many times any fiber F of π. One therefore gets a *Poincaré first return map* $\phi : F \to F$ which reduces the continuous dynamics of X to the discrete dynamics of the diffeomorphism ϕ. Conversely, given the diffeomorphism ϕ of F, one can reconstruct the vector field X (up to reparametrization) by the *suspension construction*: the manifold M is obtained by identifying the two boundary components of $F \times [0,1]$ using ϕ, and X corresponds to (a multiple of) $\partial/\partial t$.

The question of giving efficient criteria implying that a vector field is a suspension has been central in topological dynamics. Following works of Schwarzman, Fried and Sullivan, the situation is now clear and beautiful, and we now describe a summary of the main result [5,8].

The first main object to consider is the non-empty compact convex set $\mathscr{P}_X$ of probability measures on M which are invariant under (the flow generated by) X. Among these invariant measures, the one concentrated on periodic orbits are especially clear, and in the spirit of what has been said above, one should think of invariant probability measures as generalized periodic orbits. Indeed, it is not difficult to associate a 1*-cycle* c_μ in M to an invariant measure μ. For this purpose, the integral $c_\mu(\omega) = \int_M \omega(X)\, d\mu$ defines a linear functional on the space of 1-forms ω, hence a de Rham current. Clearly c_μ vanishes on exact 1-forms so that it is a 1-cycle: the *Schwarzman cycle* associated to the invariant measure μ. These cycles have a homology class in $H_1(M;\mathbf{R})$, so that we finally get a linear Schwarzman map: $S : \mathscr{P}_X \to H_1(M;\mathbf{R})$ whose image is a closed convex set in the homology $\mathscr{S}_X \subset H_1(M;\mathbf{R})$.

Of course, if X is a suspension, the fibration $\pi : M \to \mathbf{S}^1$ which is transverse to X defines—by pullback of $d\theta$ on the circle—a closed non-singular 1-form ω, which is positive when evaluated on X. This defines a linear map on $H_1(M;\mathbf{R})$ which is positive on $\mathscr{S}_X$. In other words, *when X is a suspension, the convex set $\mathscr{S}_X$ lies in some open half space in homology.*

The main result of Schwarzman–Fried–Sullivan's theory can be stated as follows:

Theorem. *Let X be a non-singular vector field on a compact manifold M and denote by $\mathscr{S}_X \subset H_1(M;\mathbf{R})$ the Schwarzman convex set defined above. The following conditions are equivalent:*

- *X is a suspension.*
- *There is a closed* 1*-form which is positive on X.*
- *The cone $\mathscr{S}_X$ lies in some open half space.*

Several proofs are available and will be sketched later.

2.3. Birkhoff sections

Most manifolds don't fiber over the circle... . Hence, most vector fields are not suspensions... . In the first half of the 20th century, Birkhoff developped a useful concept which is especially powerful in dimension 3. Let X be a non-singular vector field on a closed 3-manifold M and suppose λ is the link in M defined by a finite collection of periodic orbits. One says that X admits a *Birkhoff section with binding* λ if there is an embedded oriented surface $F \subset M$ whose oriented boundary is λ and which is transversal to X away from λ. One requires furthermore that every orbit of X intersects F infinitely many times. This situation is almost as paradisiacal as the case of suspension since we still have a first return map ϕ which is now defined in the interior of F. Collapsing the boundary components of F to single points, we produce in this way a boundaryless surface $\overline{F}$ and a first return map $\overline{\phi}$ which is now a homeomorphism of $\overline{F}$. Again, one can reconstruct the dynamics of the vector field from the information contained in the pair (F,ϕ). Taking the suspension of the homeomorphism $\overline{\phi}$ one gets a 3-manifold (fibering over the circle) with a topological flow exhibiting a finite number of periodic orbits, associated to the boundary components which have been collapsed. One recover the manifold M and the vector field X by some Dehn surgery along these periodic orbits.

The first significant example is given by the Hopf vector field on the 3-sphere (which certainly does not fiber over the circle). All its orbits are of course periodic and any finite union of them can be used as a binding. Indeed, consider the product $P : \mathbf{C}^2 \to \mathbf{C}$ of a finite number of linear maps. The argument $\pi : \mathbf{C}^2 \setminus \{P = 0\} \to P/|P| \in \mathbf{S}^1$ is defined in the complement of a finite number of lines and its restriction to the unit sphere is a (Milnor) fibration of the complement of a finite number of Hopf circles onto the circle. Obviously, the standard Hopf vector field is transversal to π. As a matter of fact this construction comes back to Poincaré in his study of the restricted 3-body problem.

An example which is much more interesting dynamically has been described by Birkhoff: the *geodesic flow on a closed surface* Σ *with negative curvature*, say of genus 2. Cutting Σ along the union of six closed geodesics decomposes the surface in four hyperbolic hexagons. One can construct explicitly a Birkhoff section in the unit tangent bundle of Σ whose binding consists of the corresponding twelve periodic orbits (six times two orientations) and which projects in Σ on the union of two of these four hexagons in a two to one way. This construction is so explicit that one can describe the topology of the Birkhoff section and the corresponding first return map. As it turns out, in this case, F is homeomorphic to a 2-torus minus twelve disks and the first return map $\overline{\phi}$ is conjugate to the action of an explicit 2×2 integral matrix having twelve fixed points.

This remarkable construction has been generalized by Fried who showed that any transitive Anosov flow on a closed 3-manifold admits many Birkhoff sections [6].

3. Right-handed vector fields

3.1. The quadratic linking form

For simplicity, consider a non-singular vector field X on the 3-sphere (even though it would be easy to generalize to any homology sphere). Any two periodic orbits can be considered as two knots and hence one can compute their linking number. Again, with the idea of considering invariant measures as cycles, one can study linking numbers between the Schwarzman cycles. This was initiated by Arnold [1] (even though he only discussed the case of divergence free vector fields, but it is not so hard to generalize).

Let μ_1 and μ_2 be two invariant ergodic probability measures for the flow X^t generated by X. Choose p_1 and p_2 two points which are generic respectively for μ_1 and μ_2 and two (big) times t_1 and t_2. The arc of trajectory from p_1 to $X^{t_1}(p_1)$ is not closed in general but one can connect the endpoints by the shortest geodesic arc connecting them (assuming that they are not antipodal, which one can assume generically). This produces a closed loop $k(t_1, p_1)$ in the 3-sphere. Adapting the proof of Arnold, it is not so difficult to prove that *if μ_1 and μ_2 are not the same periodic orbit*, the limit of linking numbers

$$lk(\mu_1,\mu_2) = \lim_{t_1,t_1\to\infty} \frac{1}{t_1 t_2} lk(k(t_1,p_1),k(t_2,p_2))$$

exists for $\mu_1 \times \mu_2$ almost all pairs of (p_1, p_2) and is independent of (p_1, p_2).

A problem arises when μ_1 and μ_2 are distributed on the same periodic orbit γ since in that case, the two knots $k(t_1, p_1)$ and $k(t_1, p_1)$ are the same. One cannot define the self linking number of a knot in space unless one has a preferred trivialization of the normal bundle. In our situation, one can define some kind of self linking number of a periodic orbit going through a point p. Fix times t_1 and t_2 and consider two sequences of points p_1^n, p_2^n on different orbits, converging to p. If n is large enough, the linking number $lk(k(t_1, p_1^n), k(t_2, p_2^n))$ is essentially independent of n, i.e., is well-defined up to a bounded error. Dividing this number by t_1t_2 and letting t_1 and t_2 go to infinity, one gets a *well-defined notion of self linking number for a periodic orbit of a non-singular vector field on the* 3*-sphere.*

When μ_1 and μ_2 are non-ergodic, one can use their ergodic decomposition and define $lk(\mu_1, \mu_2)$ as a bilinear form, using the previous definitions when the measures are ergodic and not the same periodic orbit, and when they are both the same periodic orbit, respectively. One has to prove that this bilinear extension is indeed possible in a continuous way. As a result, one finally gets the fundamental *linking quadratic form* on the compact convex set of invariant probability measures:

$$lk : \mathscr{P}_X \times \mathscr{P}_X \to \mathbf{R}.$$

Definition. *A vector field X on the* 3*-sphere is* right-handed *if the quadratic linking form is positive on the convex set of invariant probability measures.*

As a *first example*, the Hopf vector field is right-handed. Its ergodic invariant measures are concentrated on Hopf fibers and any two Hopf fibers are linked exactly once. The same is therefore true for any pair of invariant probability measures so that the linking form is indeed positive.

In order to get more interesting dynamics, we shall see later that *the set of right-handed vector fields is open in the C^1-topology* so that small perturbations of the Hopf vector field are still right-handed.

A very tractable family of examples is provided by some kind of suspensions of diffeomorphisms of the unit disc $\mathbf{D}^2$. Let ϕ be some diffeomorphism of the disc which is the identity in the neighborhood of the boundary. By the suspension construction, one can construct a vector field X_ϕ on the product solid torus $\mathbf{D}^2 \times \mathbf{S}^1$ such that the return map is precisely ϕ. One can embed this solid torus in the 3-sphere as the preimage of the northern hemisphere by Hopf fibration, so that X_ϕ coincides with the Hopf vector field in the neighborhood of the boundary. Extending outside the solid torus by the Hopf vector field, we get in this way a vector field $\overline{X}_\phi$ on the 3-sphere. Basically, the dynamics of $\overline{X}_\phi$ reduces to the dynamics of ϕ. Ergodic invariant measures of $\overline{X}_\phi$ correspond to ergodic invariant measures of ϕ, and measures concentrated on Hopf fibers outside of the solid torus.

In order to compute the linking form for these examples, consider some isotopy $(\phi_t)_{t\in[0,1]}$ connecting the identity ϕ_0 and $\phi = \phi_1$ in the (contractible) space of diffeomorphisms of the disc which are the identity near the boundary. Given any two distinct points p,q in the disc, one can look at the variation of the argument of $\phi_t(p) - \phi_t(q)$ (seen as a complex number) as t runs from 0 to 1. This variation, denoted by $Ang_\phi(p,q)$ is easily seen to be independent of the choice of the isotopy $(\phi_t)_{t\in[0,1]}$ and is a bounded continuous function on the complement of the diagonal of $\mathbf{D}^2 \times \mathbf{D}^2$. Given two ϕ invariant probability measures μ_1 and μ_2 on the disc, the associated linking number between the corresponding $\overline{X}_\phi$ invariant measures on the sphere is

$$\int_{\mathbf{D}^2 \times \mathbf{D}^2} Ang_\phi(p,q)\, d\mu_1(p)\, d\mu_2(q) + 1.$$

Indeed, one has to take into consideration the fact that the "trivial" circles $\{\star\} \times \mathbf{S}^1$ are Hopf fibers and any two of them are therefore linked once. When

μ_1 and μ_2 coincide with some ϕ periodic point, one has to define appropriately the corresponding value of *Ang* but the reader will easily provide the right rotation number of the differential of ϕ at this periodic point. As a corollary of this preliminary discussion aiming at finding sufficiently many examples of right-handed vector fields, we conclude that any vector field constructed by this procedure for which

$$Ang_{\phi}(p,q) > -1$$

on $\mathbf{D}^2 \times \mathbf{D}^2$ (minus the diagonal) provides an example of a right-handed vector field. The reader will notice that these examples are close in spirit to *positive braids*.

One more example is provided by the Lorenz attractor. Since this may not be true strictly speaking, we shall come back to this example in due time. However, we recommend that the reader looks at Figs. 1 and 2, where it "clearly" appears that all crossings in the projected orbits are positive.

As should be clear from the introduction, the concept of right-handed vector field has been introduced in particular because we aimed at the following result.

Theorem. *Let X be a right-handed vector field in the* 3*-sphere. Then any finite collection of periodic orbits is a fibered link. More precisely, any finite collection of periodic orbits is the binding of some Birkhoff section.*

We now need to introduce more concepts that will actually give more detailed information.

3.2. Linking $(1,1)$*-forms*

Recall first the nature of the well known *Gauss formula* for the computation of the linking number between two disjoint loops $\gamma_1, \gamma_2 : \mathbf{S}^1 \to \mathbf{R}^3$

$$lk(\gamma_1,\gamma_2) = \iint_{\mathbf{S}^1\times\mathbf{S}^1} \frac{\det(\gamma_1(t_1)-\gamma_2(t_2), d\gamma_1/dt_1, d\gamma_2/dt_2)}{\|\gamma_1(t_1)-\gamma_2(t_2)\|^3}\, dt_1\, dt_2.$$

It is amazing that one had to wait a long time before one could write an *explicit* similar formula for the computation of the linking number between two disjoint loops *in the* 3*-sphere*, if one requires this formula to be invariant under the symmetries of the 3-sphere (i.e., left and right rotations when one considers $\mathbf{S}^3$ as a compact Lie group). Such a formula appears in a preprint form in a paper of Deturck and Gluck [4].

A *Gauss linking form* on the 3-sphere is a $(1,1)$-form Ω on the complement of the diagonal of $\mathbf{S}^3 \times \mathbf{S}^3$ which computes the linking number. More precisely, given two distinct points p_1, p_2 in the 3-sphere, and two tangent vectors v_1, v_2 at these two points, $\Omega_{p_1,p_2}(v_1,v_2)$ is a real number depending bilinearly in the two variables v_1 and v_2. To say that Ω computes the linking number means that

$$lk(\gamma_1,\gamma_2) = \iint_{\mathbf{S}^1\times\mathbf{S}^1} \Omega_{\gamma_1(t_1),\gamma_2(t_2)}\left(\frac{d\gamma_1}{dt_1},\frac{d\gamma_2}{dt_2}\right) dt_1\, dt_2.$$

One can find a discussion about these linking forms in the book by Arnold and Khesin [2]. The differential forms on $\mathbf{S}^3\times\mathbf{S}^3$ can of course be bi-graded, using both coordinates, and we shall restrict our Gauss linking forms to those which are $(1,1)$ currents on the product of two copies of the sphere; this can be achieved by specifying the kind of pole that is allowed along the diagonal in $\mathbf{S}^3\times\mathbf{S}^3$. The condition that Ω computes the linking number means that its boundary, as a current, is the sum of the integration cycle over the diagonal $\Delta\subset\mathbf{S}^3\times\mathbf{S}^3$, a $(3,0)$ form and a $(0,3)$ form.

We can now state some theorem which is analogous to the Schwarzman–Fried–Sullivan theorem.

Theorem. *Let X be a non-singular vector field on the 3-sphere, generating a flow X^t. Choose some Gauss linking form Ω. The following conditions are equivalent:*

- *X is right-handed, i.e., the quadratic linking form is positive on the convex set $\mathscr{P}_X$.*
- *There is some $T>0$ such that for every pair of points p_1,p_2 on different orbits, the integral $\int_0^T\int_0^T \Omega_{X^{t_1}(p_1),X^{t_2}(p_2)}(X_{X^{t_1}(p_1)},X_{X^{t_2}(p_2)})\,dt_1\,dt_2$ is positive.*
- *There is some Gauss linking form $\overline{\Omega}$ which is pointwise positive on X: for every distinct points p_1,p_2, one has $\overline{\Omega}_{p_1,p_2}(X_{p_1},X_{p_2})>0$.*

Let us state some corollaries.

Let X be right-handed and choose a Gauss linking form $\overline{\Omega}$ as in the third item of the previous theorem. If μ is an invariant measure one can define in a natural way a 1-form *in the complement of its support* by integration with respect to the second variable. This yields a 1-form ω_μ:

$$\omega_p(v) = \int_{\mathbf{S}^3} \overline{\Omega}_{p,q}(v,X_q)\, d\mu(q).$$

The facts that $\overline{\Omega}$ is a Gauss linking form and that we integrate over the 1-cycle defined by μ means that this form ω_μ is closed. It follows from the third item that ω_μ is a *closed non-singular form* in the complement of the support of μ.

To be more precise, suppose that one has an embedded loop γ in a small transversal disc and that γ is away from the support of the measure μ. Then the integral of ω_μ along γ is nothing more than the μ (transversal) measure of the small disc bounded by γ in the transversal. One could of course express that in terms of the boundary of ω_μ as a current.

In the special case where μ is a barycenter (say with equal weights) of a finite collection of invariant measures concentrated on periodic orbits, we get a closed form ω_μ on the complement of the corresponding link. In a small neighborhood of a point of this link, on a small transversal disc, the form can be written as (some multiple of) $d\theta$ in some polar coordinates. The codimension 1 foliation defined by the closed form ω_μ is therefore a fibration of the complement of the link over the circle, which is transversal to X away from the link. The local description of ω_μ in the neighborhood of the link guarantees that we do get in this way a Birkhoff section for X with binding the given link. This is the proof of the fact that we stated above.

For more complicated invariant measures, it is more difficult to call this ω_μ a "fibration" but in any case, this provides a non-singular codimension 1 foliation on the complement of the support. The local behavior of this foliation in the neighborhood of the support is not difficult to describe.

In any case, it seems to us that this positive Gauss linking form $\overline{\Omega}$ *is the global single object we were looking for, which incarnates as the collection of the fibrations of the complements of all the finite links of periodic orbits.*

4. The case of the Lorenz attractor

We now come back to our initial problem of understanding those fibrations in the context of the Lorenz attractor.

4.1. The template

Recall that the basic idea of Birman and Williams, in order to study the topology of Lorenz knots, is to use a *template*. Fig. 3 is also extracted from [3]. It represents a surface T embedded in 3-space branched along some horizontal interval. It is equipped with an obvious semi-flow $(\phi^t)_{t\geq 0}$. Indeed, one can see orbits running positively on the template but when one tries to track an orbit backward, one hits from time to time the horizontal interval from which two orbits bifurcate, so that the flow is indeed not defined for negative times. To transform it in a flow defined for all times, one can use the standard inverse limit procedure. One considers the space $\hat{T}$ of "orbits", i.e., of curves $c : \mathbf{R} \to T$ which satisfy $\phi^t(c(s)) = c(s+t)$ for all $t \geq 0$ and $s \in \mathbf{R}$. This space is naturally equipped with a *flow* $\hat{\phi}^t$ acting by $\mathbf{R}$-translation on the source, and a projection $\pi : c \in \hat{T} \mapsto c(0) \in T$ which conjugates the flow $\hat{\phi}^t$ and the semi-flow ϕ^t ($t \geq 0$): one has $\pi \circ \hat{\phi}^t = \phi^t \circ \pi$. In the neighborhood of $T \subset \mathbf{R}^3$ one can find some compact set homeomorphic to $\hat{T}$ and a vector field L on $\mathbf{R}^3$ preserving this compact set and inducing a flow conjugate to $\hat{\phi}^t$. One can even arrange things in such a way that $\hat{T}$ appears as an attractor: there is an open neighborhood of $\hat{T} \subset \mathbf{R}^3$ such that every orbit starting in this open set accumulates inside $\hat{T}$.

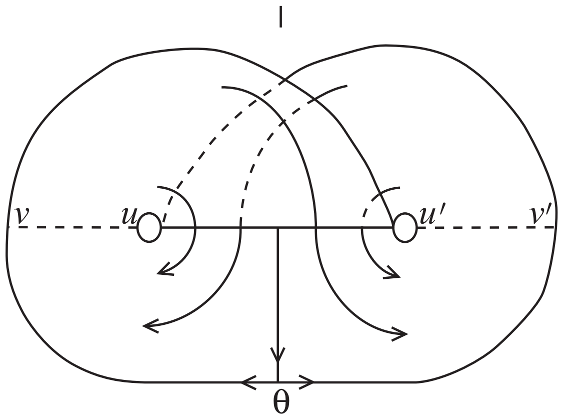

Fig. 3. The template

The first return map on the branching interval has not been specified but we assume that it is simply the doubling map ($x \in [0,1] \mapsto 2x \bmod 1 \in [0,1]$).

This model is called the *geometrical Lorenz attractor* and this is the object of study of Birman and Williams. The exact relation with the actual Lorenz dynamical system was unsettled at the time but has been clarified since then by Tucker [9]. It suffices to say that the actual attractors appearing in Lorenz equations (for various values of the parameters) are conjugate to invariant subsets of this model. Hence, the study of the periodic orbits in the geometrical model contains all cases.

4.2. The Lorenz flow is "almost" right-handed

There are several reasons that prevent the vector field L from being right-handed. The first is that the ambient manifold is not the 3-sphere... . Moreover, there is a singular point. And last, but not least, the two obvious periodic orbits, on each side of the template, are not linked at all!!! However, one readily sees that these two periodic orbits are the only two which are not linked. All the other orbits travel along the template and intersect the branch locus several times, arriving there sometimes from the upper sheet and sometimes from the lower one. Note also that all the crossings of (the projections of) orbits drawn on the template are clearly positive. It follows that any two periodic orbits which are different from the trivial ones are positively linked.

If one chooses some compact set $K \subset [0,1]$ which is positively invariant under the doubling map, this produces an invariant subset $\hat{K} \subset \hat{T}$ which will be fixed point free if $1/2 \notin K$ and which will not contain the trivial periodic orbits if $0,1 \notin K$. For instance, for each $\varepsilon > 0$, one can consider the closed set K_ε consisting of all x's whose positive orbit under the doubling map stays at distance at least ε from $0, 1/2$ and 1. In this way, we get subsets $\hat{K}_\varepsilon \subset \hat{K}$ which are equipped with a fixed point free flow and in which any two orbits are positively linked. Notice that any finite collection of periodic orbits, different from the trivial ones, is contained in some $\hat{K}_\varepsilon$.

Still, $\hat{K}_\varepsilon$ is not a manifold, and one cannot yet declare this flow to be right-handed. However, it is not difficult to fix the problem. Folding the left side of the template over the right side, one can draw the whole picture in some solid torus, in such a way that the flow on $\hat{K}_\varepsilon$ is the suspension of some homeomorphism of some compact set in the disc. One can even construct a flow on the solid torus, transversal to the boundary, such that $\hat{K}_\varepsilon$ appears as an attractor. Every positive orbit accumulates in $\hat{K}_\varepsilon$. Finally, if one wants to complete this construction as a non-singular flow on the 3-sphere, it is enough to glue some "outside" solid torus equipped with a standard flow, say with one repelling periodic orbit in its core and such that all other orbits go to the boundary and hence finally accumulate in $\hat{K}_\varepsilon$. In other words, there is a standard way to think of $\hat{K}_\varepsilon$ as a compact attractor for some non-singular vector field L_ε in the 3-sphere.

We claim that one can construct these vector fields L_ε so that they are right-handed. Indeed it is not difficult to describe all invariant probability measures. One has first of all the invariant measures supported in $\hat{K}_\varepsilon$ and second of all the measure supported on the repelling orbit that we added on the outside solid torus. As we have seen, the linking form is positive on measures supported in $\hat{K}_\varepsilon$. The additional measure, concentrated on the core of the outside torus can be chosen also positively linked with all the other ones, since the cores of the two solid tori are linked and one just has to choose the right orientation. Finally, one has to make sure that the repelling orbit is such that its self linking number, as defined above, is positive.

As a summary, the Lorenz equation cannot be right-handed because it contains a singular point and two trivial unlinked periodic orbits, but it contains arbitrarily large closed invariant subsets which are also invariant subsets of right-handed vector fields in the 3-sphere.

4.3. *Modular knots and links*

We recall here a result from [7] that can be relevant in this context. Consider the space M of lattices of area 1 in $\mathbf{R}^2$. As a homogeneous space under the action of $\mathrm{SL}(2,\mathbf{R})$, it is identified with $\mathrm{SL}(2,\mathbf{R})/\mathrm{SL}(2,\mathbf{Z})$. As a 3-manifold, it is homeomorphic to the complement in the 3-sphere of the trefoil knot. One can think of the points of this knot as the space of subgroups of $\mathbf{R}^2$ isomorphic to $\mathbf{R}$ (making a real projective line, and hence a circle).

The *modular flow* on M consists of the left action of the diagonal group. As a flow, it can be identified with the geodesic flow of the modular surface. It can be continuously extended to the 3-sphere, on which it has two fixed points. Its periodic orbits are in bijection with conjugacy classes of hyperbolic elements in $\mathrm{PSL}(2,\mathbf{Z})$. Every such periodic orbit is therefore a knot, or more precisely a knot in the complement of the trefoil knot k. We proved that these *modular knots* are isotopic to the *Lorenz knots*. The same proof even shows that modular *links* and Lorenz *links* are isotopic as links. As a simple corollary, which does not appear obvious to the author, *any two periodic orbits of the modular flow are positively linked in the 3-sphere*. Note that not all modular knots are positively linked with the trefoil.

For the same reason as in the case of the Lorenz flow, the modular flow—extended on the 3-sphere—cannot be right-handed since it has fixed points. However, one can use the same kind of procedure. One can consider the set of geodesics in the modular surface which do not come too close to the cusp, say for which no representative in upper half plane climbs to some altitude higher than $1/\varepsilon$ for some $\varepsilon > 0$. This provides closed invariant subsets M_ε which are analogous to the sets $\hat{T}_\varepsilon$ that we already described. Extending slightly [7], one can show that for every $\varepsilon > 0$ there is some $\eta > 0$ and some isotopy sending $M_\varepsilon \subset \mathbf{S}^3$ into $\hat{T}_\eta \subset \mathbf{S}^3$. Therefore we can say that the modular flow is almost right-handed, just as we said it for the Lorenz flow. By this, we mean that the linking form is positive on the convex set of invariant measures concentrated on any of these invariant sets M_ε.

5. About the proofs

We shall only sketch the main ideas. We first recall the key step in the proof of Schwarzman–Fried–Sullivan's fibration theorem, as given by Sullivan [8]. Let X be a non-singular vector field on a closed manifold M and denote by $\mathscr{P}_X$ the compact convex set of invariant probability measures. We defined a Schwarzman map from $\mathscr{P}_X$ to $H_1(M,\mathbf{R})$ and we assume that its image $\mathscr{S}_X$ lies in some open half space. The goal is to produce a fibration of M onto the circle which is transversal to X.

Consider the topological vector space C_1 of de Rham 1-currents on M and the convex cone $\mathscr{C} \subset C_1$ generated by the Dirac currents c_p at points p in M. By definition, c_p evaluated on a 1-form is the value of the form on the vector X_p. The elements of $\mathscr{C}$ are called *foliations currents*. An element of the dual of C_1 which is positive on $\mathscr{C}$ is nothing more than a differential form which is positive on X.

Consider moreover the subspace $B \subset C_1$ consisting of 1-currents which are boundaries. An element of the dual of C_1 vanishing on B is a 1-form which vanishes when evaluated on boundaries, so that it defines a closed 1-form on M.

We are looking for a closed 1-form which is positive on X. By Hahn–Banach theorem, such a form exists if and only if no non-trivial element of $\mathscr{C}$ is in B. In other words, this is equivalent to the condition that no foliation cycle is homologous to zero. Then Sullivan interprets foliation cycles as being precisely the cycles c_μ that we associated to invariant measures μ. As a consequence, if the image $\mathscr{S}_X$ of the Schwarzman map in $H_1(M,\mathbf{R})$ is in some open half space, no one of these cycles c_μ is homologous to zero and one can indeed find a closed 1-form which is positive on X. Approximating by a form with rational periods, à la Tischler, and multiplying by some integer, one finally gets a non-singular closed form with integral periods, which produces the required fibration over the circle.

Several other proofs are available. On the topological side, Fried provides a cut and paste construction of a surface of section for X. Starting from an immersed surface of section, one simplifies it progressively to suppress all multiple points (see [5]). Another approach, that the author learned from Tischler, consists in an averaging process. Starting from a 1-form ω on M whose cohomology class is positive on all cycles c_μ, one averages it along the flow X^t during some period of time:

$$\omega_T = \frac{1}{T}\int_0^T (X^t)^\star \omega \, dt$$

and one shows that ω_T is positive on X for a sufficiently large T.

We now come to right-handed vector fields. The main point is to show that there exists some Gauss linking form $\overline{\Omega}$ which is positive when evaluated on the vector field X at distinct points. The general strategy is the same but one runs against several difficulties. The first is that Gauss forms are not smooth on $\mathbf{S}^3 \times \mathbf{S}^3$ since they have poles along the diagonal. The second problem is that a Gauss form is not necessarily closed: its boundary, as a de Rham current, is the sum of the integration current on the diagonal and two currents of bidegrees $(0,3)$ and $(3,0)$. There are technical difficulties but one proceeds exactly as Sullivan. The space of Gauss linking forms is not a vector space but rather an affine space: one can add to any Gauss linking form a $(1,1)$-form written as $d_x d_y f$ where f is a function on $\mathbf{S}^3 \times \mathbf{S}^3$ and d_x, d_y are the partial derivatives with respect to the two variables. The condition that a Gauss linking form is positive on X is also a convex condition and one can use Hahn–Banach theorem. The main problem is to identify the dual obstructions to positivity as being given by the existence of two invariant probability measures whose linking number is zero. Moreover, one has to look carefully at the behavior in the neighborhood of the diagonal. The final output is that if the linking quadratic form is positive on $\mathscr{P}_X$, one can indeed find a Gauss linking form which is positive on X.

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