The theorem of the bellows

Étienne Ghys

A ruler, a pencil, cardboard, scissors and glue: one doesn't need more to give a mathematician pleasure, and present interesting problems whose study often turns out to be useful in other areas, in totally unexpected ways.

Let us build a cardboard pyramid... One starts by cutting out the design *SABCDE* in a sheet of cardboard as indicated in figure 1, then one folds along the dotted lines and, finally, one glues the sides *AS* and *ES*.

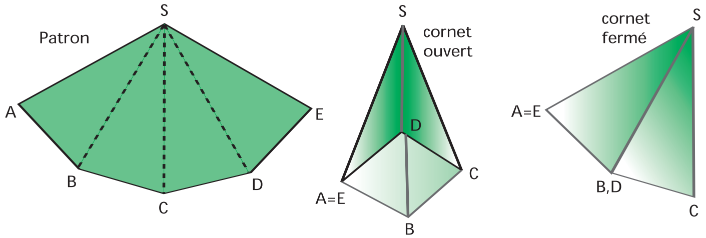

Figure 1. The construction of a pyramid from a piece of cardboard. If the base ABCDA is removed, the object is flexible.

The result is a kind of cone whose vertex is the point and whose base is the quadrilateral ABCD. This object is flexible. If held in the hand, the quadrilateral ABCD can be deformed and opened or closed a little: the construction is not very solid. To complete the pyramid we need to cut out a square from the cardboard and to stick it onto the quadrilateral to form the base. After this operation the pyramid is sturdy and rigid. If one puts it on a table it does not collapse. If one takes it in hand and tries to deform it (softly!), one is unable to do it without deforming the cardboard face.

In the same way, a cardboard cube is rigid, as everyone must have observed at one time. What about a more general polyhedron, having perhaps thousands of faces? Is the *Géode*, a dome at *La Villette* in Paris, rigid? This last question suggests that the subject of rigidity or flexibility is perhaps not only a theoretical one!

A problem going back to Antiquity, but which is still relevant

The problem of rigidity of these type of objects is very old. Euclid probably was aware of it. The great French mathematician Adrien-Marie Legendre became interested in it towards the end of the 18th century and talked to his colleague Joseph-Louis Lagrange about it, who in turn suggested it in to the young Augustin-Louis Cauchy in 1813. It was to be the first major result of baron A.-L. Cauchy, who went on to become one of the greatest mathematicians of his century

Cauchy was interested in convex polyhedra, i.e., polyhedra which do not have any inward-pointing edges. For example, the pyramid that we built or the surface of a football are convex, while the object drawn

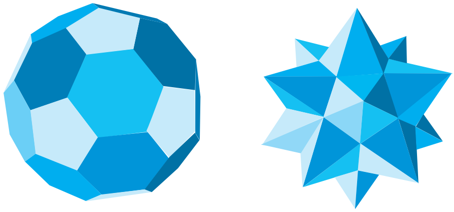

Figure 2. A convex polyhedron and a star-shaped, non-convex, polyhedron.

on the right of figure 2 is not.

Augustin-Louis Cauchy (1789-1857), one of the great mathematicians of his time. (Photo Archives de l'École polytechnique)

The theorem established by Cauchy is the following: any convex polyhedron is rigid. That means that if one builds a convex polyhedron with indeformable polygons (made of metal, for example) adjusted by hinges along their edges, the overall geometry of the object prevents the play of joints. The cone that we built is flexible, but that does not contradict the theorem: a face is missing, and it is the last face which makes the pyramid rigid...

Doing mathematics means proving what one claims! It so happens that Cauchy's proof is superb (even if it was pointed out later that it is incomplete). There is unfortunately no question of giving an idea of this proof in this short article, but I would like to extract from it a ``lemma”, i.e., a step in the proof.

Let us place on the ground a chain made up of some metal bars joined at the ends, as in figure 3. At each angle of this polygonal line, let us move the two bars in order to decrease the corresponding angle. Then the two ends of the chain come closer. Does that seem obvious to you? Try to prove it...

For a long time many mathematicians wondered whether nonconvex polyhedra were also rigid. Can one find a proof of the rigidity which would not use the assumption of convexity? Mathematicians like statements in which all the assumptions are necessary to obtain the conclusion. One had to wait more than 160 years to know the answer in this particular case.

The "géode de la Vilette" in the Cité Des Sciences in Paris, is a convex polyhedron of 1730 triangular faces. The rigidity of this structure of joined polyhedra gives rise to an interesting mathematical problem which was solved only in 1997. (Photo Cosmos/R. Bergerot)

In 1977, the Canadian mathematician Robert Connelly created something surprising. He built a (quite complicated) polyhedron, which is flexible, and, of course, nonconvex not to contradict Cauchy! Since then the construction has been somewhat simplified, in particular by Klaus Steffen. In figure 4, I,ve given a design which will allow the reader to build the ``flexidron" of Steffen. Cut it out and fold along the lines. The solid lines represent edges pointing outward, and the broken lines correspond to edges pointing inward. Stick the free edges in the obvious way. You will obtain a kind of Shadok and you will see that it is indeed flexible (a little...).

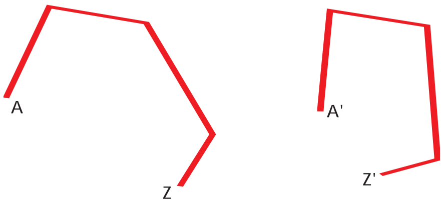

Figure 3. If one decreases the angles which the segments form with each other, the ends of the chain of segments come closer.

Does the volume of a polyhedron change when it is deformed?

At the time, mathematicians were enchanted by this new object. A metal model was built and put in the tea room of the Institut des *Hautes Études Scientifiques*, at Bures-sur-Yvette, near Paris, and one could have fun making this thing move; to tell the truth, it was not very pretty, and squeaked a little. The story goes that Dennis Sullivan had the idea of blowing some cigarette smoke into Connelly's flexidron and he noticed that while the object moved, no smoke came out... So he got the idea that when the flexidron is deformed, its volume does not vary! Is this anecdote true? Whether true or not, Connelly and Sullivan conjectured that when a polyhedron is deformed, its volume remains constant. It is not difficult to check this property in the particular case of the flexidron of Connelly or for that of Steffen (through complicated and uninteresting calculations). But the conjecture in question considers all polyhedra, including those which have never been built in practice! They called this question the

``the bellows conjecture": the bellows at the corner of the fireplace eject air when they are pressed; in other words, their volume decreases (besides, that is what they are meant for). Of course, true bellows do not provide an answer to the problem of Connelly and Sullivan: they are made of leather and their faces become deformed constantly, in contrast to our polyhedra with rigid faces.

In 1997, Connelly and two other mathematicians, I Sabitov and A. Walz, finally succeeded in proving this conjecture. Their proof is impressive, and once more illustrates the interactions between different areas of mathematics. In this eminently geometrical question, the authors have used very refined methods of modern abstract algebra. It is not a proof that Cauchy ``could have found": the mathematical techniques of the time were insufficient. I would like to recall a formula which one used to learn at secondary school at one time. If the sides of a triangle are a, b and $ in length, one can easily calculate the area of the triangle. For that, one calculates first the semi-perimeter $p=(a+b+c)/2$ and then one obtains the area by extracting the square root of $p(p-a)(p-b)(p-c)$. This pretty formula bears the name of the Greek mathematician Hero and its origins are lost in antiquity. Can one calculate, in a similar way, the volume of a polyhedron if the lengths of its edges are given? Our three contemporary mathematicians have shown that one can.

They start from a polyhedron built from a certain design having a certain number of triangles, and they call l_1, l_2, l_3, etc. the lengths of the sides of these triangles (possibly very many). They then find that the volume V of the polyhedron must satisfy an equation of the nth degree, i.e. an equation of the form $a_0 + a_1V + a_2V^2 + ... + a_nV^n = 0$. The degree n depends on the design used, and the coefficients (a_0, a_1, etc.) of the equation depend explicitly on the lengths l_1, l_2, l_3, etc. of the sides. In other words, if the design and the lengths of the sides are known, the equation is known. If the reader remembers that an equation has in general one solution if it is of the first degree, two solutions if it is of the second degree, he will be able to guess that an equation of degree n cannot have more that n solutions. Conclusion: if one knows the design and the lengths, one does not necessarily know the volume, but it is at least known that this volume can take on only a finite number of values. When the flexidron is deformed, its volume cannot vary continuously (otherwise the volume would take on an infinity of successive values); this volume is ``blocked" and the bellows conjecture is established...

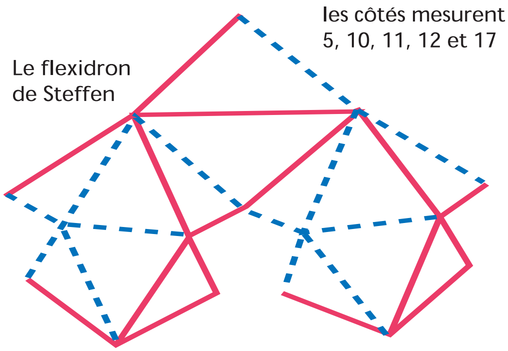

Figure 4. The model for the flexidron of Steffen.

Yes, the bellows problem is worthy of interest!

Is this problem useful, interesting? What is an interesting mathematical problem? That's a difficult question, which, of course, mathematicians have been contemplating for a long time. Here are some partial answers, some indicators of ``quality”. The history of a problem is the first criterion: mathematicians are very sensitive to tradition, to problems stated a long time ago, on which mathematicians of several generations have worked. A good problem must also be stated simply, its solution must lead to surprising developments, if possible connecting very different fields. From these points of view, the problem of rigidity, which we have just discussed, is interesting.

The question as to whether a good problem must have useful practical applications is more subtle. Mathematicians answer it in a variety of different ways. Undoubtedly, ``practical” questions, arising for example from physics, are very often used as a motivation for mathematics. Sometimes it is a question of solving a quite concrete problem, but the relationship is often less direct: the mathematician uses the concrete question only as a source of inspiration and the actual solution of the initial problem is no longer the true motivation. The problem of rigidity belongs to this last category. The physical origin is rather clear: the stability and the rigidity of structures, for example metallic structures. For the moment, Connelly,s examples are of no use to engineers. However, it is clear that this kind of research will not fail, in an indeterminate future, to provide a better overall understanding of the rigidity of vast structures made up of a large number of individual elements (macromolecules, buildings, etc.). It is thus a purely theoretical ``disinterested” kind of research, but which has a good chance one day of being useful ...

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