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Reviews

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REVIEWS

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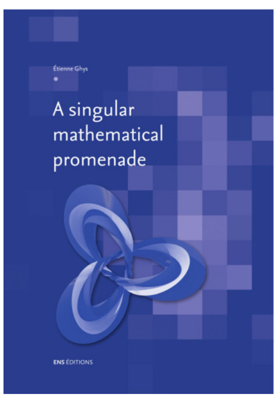
A Singular Mathematical Promenade. By Étienne Ghys, ENS Éditions, Lyon, 2017. 302 pp., ISBN 978-2-84788-939-0, $22.95.

Reviewed by **Alex Kontorovich**

Let us get straight to the point: *A Singular Mathematical Promenade* is a delightful book by a gifted expositor that beautifully showcases both the unity of mathematics, and the realm of possibility which is opened when one retires the austere Bourbaki-style adherence to cycles of Definition-Theorem-Proof. The Bourbaki group of course had good reason for this style, with so many 19th century foundational crises in geometry, analysis, set theory, etc. Hilbert himself famously declared that *Grundlagen der Geometrie*, his 1899 update to Euclid's *Elements*, must be so devoid of potentially flawed human intuition that it should be possible to replace "points," "lines," and "planes" with "chairs," "tables," and "beer bottles" without losing logical validity. However, most mathematicians know that to really learn something, one does not stay on this strict linear path. Instead, we must talk to our friends and colleagues, ask questions, argue, perhaps take a leisurely walk.

The book under review is written in this spirit. It is equal parts deep mathematics, storytelling, and historical vignettes, and Ghys always opts for the scenic route. The author treats the reader as an old friend, writing in a relaxed, leisurely, informal manner with lots of exclamation marks, smiley faces, and figures/drawings/paintings galore (as a highly accomplished researcher, Ghys need not fear that this style may be seen as "unprofessional"). This informality allows his enthusiasm about mathematics to be infectious, and makes the book easy for one to get carried away reading.



Ghys is a prominent researcher, broadly in geometry and dynamics, and for some time now he has been a great advocate and role model for novel ways of communicating mathematical ideas. He does this both at the highest levels (see his movies with Jos

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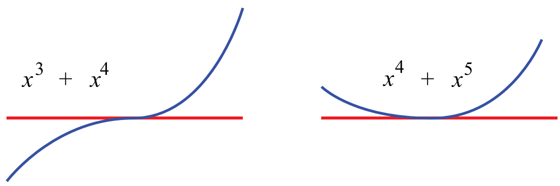
Leys from his 2006 Plenary ICM lecture [1]) and with mathematics aimed at the lay person (see his discussion of popularizing mathematics in the 2014 ICM [3]); in fact, he was awarded the Clay Award for Dissemination of Mathematics in 2015. The book under review, or “petite livre” as he calls it, falls somewhere in between, with most if it at the advanced undergraduate/early graduate level, but also containing some material that is significantly more advanced. In particular, Ghys’s stated intended audience is himself, 40 years ago.

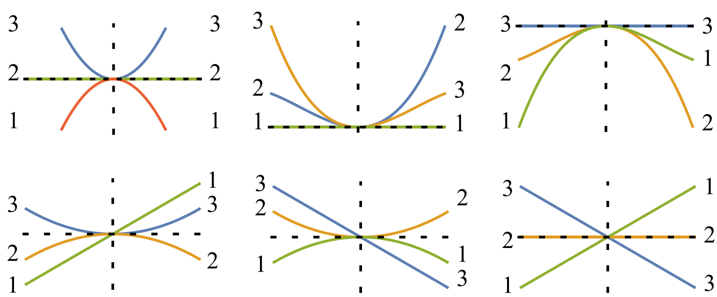
The moral of the book is to regularly attend boring administrative meetings. You never know when you might sit next to Maxim Kontsevich, who may show you a fascinating new fact about polynomials that will send you off on a delightful promenade. Or at least this was Ghys’s experience at one such meeting about a decade ago. The aforementioned fact is as simple as it is striking: there are certain restrictions for how polynomials can cross when they meet!

To explain this result, consider first pairs of polynomials that meet at a single point:

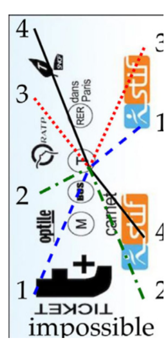
$f(x) = 0,\ g(x) = x^3 + x^4$ $\qquad$ $f(x) = 0,\ g(x) = x^4 + x^5$



As the two images show, the graphs of two polynomials are free to either cross when they meet or not. What if we consider a triplet of polynomials that all meet at a point?



Again, we see that all six crossing permutations are possible. One might naïvely conjecture that one can always find a set of polynomials passing through a common point so that their order, on crossing, is permuted however one likes. But this is false already for quadruples!



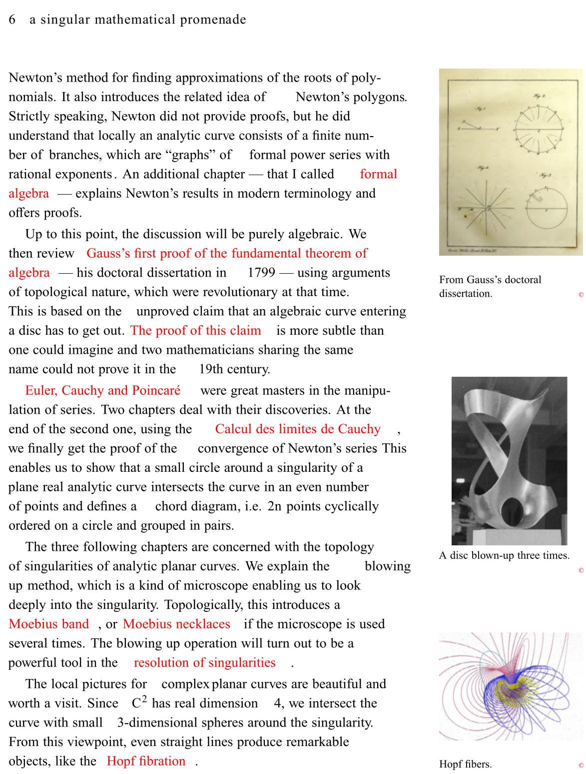
This is the image of a Parisian métro ticket that Kontsevich passed to Ghys during their administrative meeting, to communicate the following result:

Theorem. *There do not exist four polynomials $P_1, \ldots, P_4 \in \mathbb{R}[x]$ with $P_1(x) < P_2(x) < P_3(x) < P_4(x)$ for all small negative x, and $P_2(x) < P_4(x) < P_1(x) < P_3(x)$ for small positive x.*

Amazingly, this result basically characterizes what can or cannot happen for crossings, not only for graphs of arbitrary collections of polynomials, but indeed for all real analytic planar curves! (Note that analyticity is crucial here: there is no crossing obstruction for smooth planar curves, as the reader is encouraged to verify by construction.)

Following this initial curious tip, one finds a vast iceberg, and connections to a dizzying array of ancient and modern mathematics, from Hipparchus (via Schröder numbers, and a key assist from the ability to look up one's favorite integer sequence in Neil Sloane's online encyclopedia [4]), to Newton's method and polygon, to Gauss's flawed first proof of the fundamental theorem of algebra, Hopf and Milnor fibrations, and much, much more. Ghys weaves historical stories in between the combinatorics, complex analysis, and algebraic geometry, and does it all in a very readable way.

As to be expected with this author, the reader is rewarded at all stages of the effort with numerous gorgeous illustrations, diagrams, and pictures—Ghys takes advantage of all of the margin space, and as the reader flips through the pages they are just as likely to be greeted with a series of equations, a quotation from Newton, or a photograph of something such as a railroad station or a palm tree. Here is but a sample of the juicy eye-candy to be found on nearly every page:



6 a singular mathematical promenade

Newton's method for finding approximations of the roots of polynomials. It also introduces the related idea of Newton's polygons. Strictly speaking, Newton did not provide proofs, but he did understand that locally an analytic curve consists of a finite number of branches, which are "graphs" of formal power series with rational exponents. An additional chapter — that I called formal algebra — explains Newton's results in modern terminology and offers proofs.

Up to this point, the discussion will be purely algebraic. We then review Gauss's first proof of the fundamental theorem of algebra — his doctoral dissertation in 1799 — using arguments of topological nature, which were revolutionary at that time. This is based on the unproved claim that an algebraic curve entering a disc has to get out. The proof of this claim is more subtle than one could imagine and two mathematicians sharing the same name could not prove it in the 19th century.

Euler, Cauchy and Poincaré were great masters in the manipulation of series. Two chapters deal with their discoveries. At the end of the second one, using the Calcul des limites de Cauchy, we finally get the proof of the convergence of Newton's series This enables us to show that a small circle around a singularity of a plane real analytic curve intersects the curve in an even number of points and defines a chord diagram, i.e. 2n points cyclically ordered on a circle and grouped in pairs.

The three following chapters are concerned with the topology of singularities of analytic planar curves. We explain the blowing up method, which is a kind of microscope enabling us to look deeply into the singularity. Topologically, this introduces a Moebius band, or Moebius necklaces if the microscope is used several times. The blowing up operation will turn out to be a powerful tool in the resolution of singularities.

The local pictures for complex planar curves are beautiful and worth a visit. Since $\mathbb{C}^2$ has real dimension 4, we intersect the curve with small 3-dimensional spheres around the singularity. From this viewpoint, even straight lines produce remarkable objects, like the Hopf fibration.

From Gauss's doctoral dissertation.

A disc blown-up three times.

Hopf fibers.

For a condensed sampling of this story, see an earlier article by Ghys in this journal [2], though the full book is well worth exploring in detail, and returning to time and again.

One more note: the author of this review did not obtain Professor Ghys's permission (nor that of the publisher, the *École Normale Supérieure de Lyon*) to use any of the pages or figures in this review; indeed, on the copyright page, it is written that: "Étienne Ghys... has waived all copyright and related or neighboring rights. You can copy, modify, and distribute them, even for commercial purposes, all without asking permission... This pdf book is available free of charge on Arxiv, on the author's homepage, and on the website of ENS Éditions." This commitment to open access books should be applauded, and is yet more evidence of Ghys's service towards the dissemination of mathematical ideas.

A Singular Mathematical Promenade is a remarkable achievement in terms of its content, structure, and style. One can only hope that Ghys's manuscript is a shining beacon, lighting the way for a future full of exposition like this, for popular and research mathematics alike. (Let us all be so "unprofessional"!)

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