A singular mathematical promenade
by Étienne Ghys
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REVIEWED BY SERGEI TABACHNIKOV

Consider two graphs of real polynomials that pass through the origin. Intersect the graphs with a vertical line slightly left of the y-axis and move the line to the right until it is slightly right of the y-axis. Follow the intersection points of the line with the graphs: you obtain a permutation of two points, either trivial or non-trivial.

It is obvious that both possibilities can be realized by choosing appropriate polynomials, for example, $P_1(x) = 0, P_2(x) = x^2$ for the trivial permutation, and $P_1(x) = -x, P_2(x) = x$ for the non-trivial one.

Now consider the case of three polynomials, see Figure 1.[1] Once again, choosing the polynomials appropriately, all six permutations of three points can be realized.

But for four polynomials, one has a surprise: out of 24 permutations of four points, only 22 are realizable in this way. This amazing observation is due to Maxim Kontsevich, who shared it with Étienne Ghys in 2009 (at a boring administrative meeting), and which is the starting point of the 'promenade' described in this unusual book.

Here is a pictorial table of contents, color-coded according to the difficulty of the topics, similarly to color-coding of downhill ski slopes, Figure 3.

The promenade starts with Ghys's solution to the general problem: to describe the permutations of n objects realizable by graphs of n polynomials passing through the origin (this occupies the first four chapters, see also [1]). These permutations are called *polynomial interchanges*.

[1]All the illustrations are taken from the book under review.

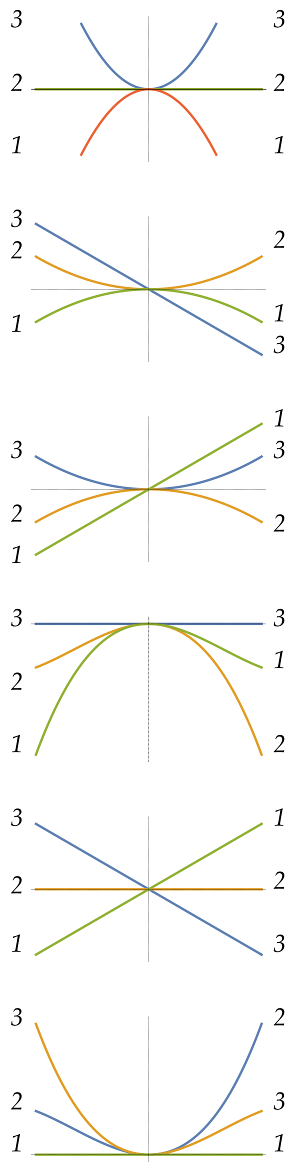

Figure 1: The six permutations of three points.

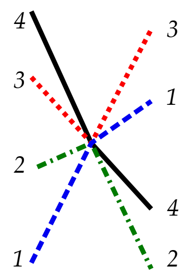

Figure 2: A forbidden permutation (of course, its inverse is also forbidden). Can you prove that this permutation is not realizable by graphs of polynomials?

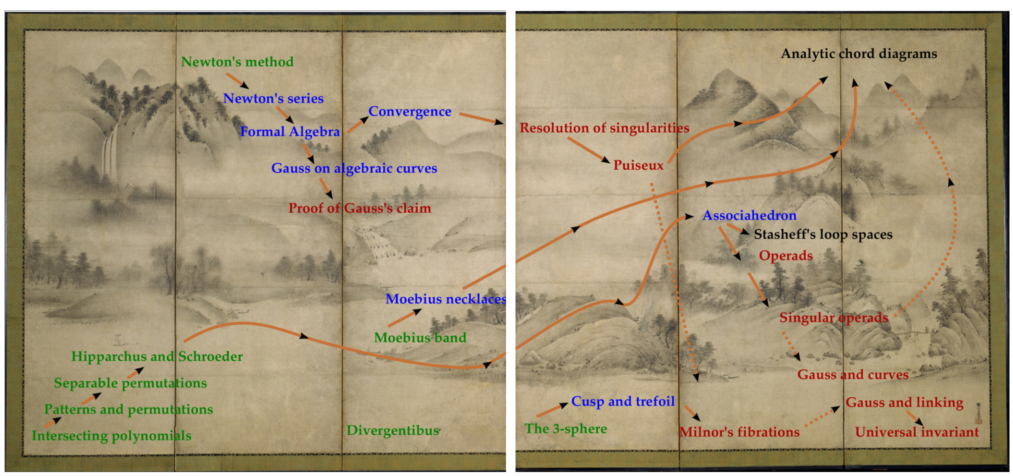

Figure 3: Landscape of the Four Seasons (Eight Views of the Xiao and Xiang Rivers), by Soami, early 16th century.

The first observation is that the only obstruction is the permutation depicted in Figure 2 and its inverse: a permutation π is a polynomial interchange if and only if it does not contain one of the two forbidden permutations, that is, there do not exist four indices $1 \le i_1 < i_2 < i_3 < i_4 \le n$ such that

$$\pi(i_2) < \pi(i_4) < \pi(i_1) < \pi(i_3) \quad \text{or} \quad \pi(i_3) < \pi(i_1) < \pi(i_4) < \pi(i_2).$$

How many polynomial interchanges are there? Here are the first 10 values:

$$1,\ 2,\ 6,\ 22,\ 90,\ 394,\ 1806,\ 8558,\ 41586,\ 206098.$$

This is a well-studied sequence, called the large Schröder numbers; they appear in The Online Encyclopedia of Integer Sequences [10] as A006318. These numbers, except the first one, are even; their halves are called the little Schröder numbers (A001003). One of Ghys's theorems is that the number of polynomial interchanges is equal to the respective large Schröder number.

The nth little Schröder number is the number of ways to insert parentheses in a string of $n+1$ symbols (the parentheses must be balanced, the number of letters inside a pair of parentheses must be at least two, and parentheses enclosing the whole string are ignored). For example, here are all the 11 possibilities for $n = 3$:

$$abcd, (ab)cd, a(bc)d, ab(cd), ((ab)c)d, a(b(cd)), (ab)(cd), (a(bc))d, a((bc)d), (abc)d, a(bcd).$$

A wealth of information about Schröder numbers includes their asymptotic growth and the generating function. The asymptotic is as follows:

$$C\frac{(3+2\sqrt{2})^n}{n\sqrt{n}}, \quad \text{where} \quad C \approx 0.4.$$

Using the Stirling formula for $n!$, one sees that the proportion of the polynomial interchanges among all permutations tends to zero as $n \to \infty$.

Amazingly, the little Schröder number 103,049 appears in the following passage by Plutarkh (about 2,000 years ago):

> Chrysippus says that the number of compound propositions that can be made from only ten simple propositions exceeds a million. (Hipparcus, to be sure, refuted this by showing that on the affirmative side there are 103,049 compound statements, and on the negative side 310,952).

See [9], and also [3] if you are interested in the "negative side" – which appears to be unrelated to polynomial interchanges.

What I have described so far – and a sketchy description it is – is only the beginning. The ultimate goal of the book is to extend the results from graphs of polynomials to real algebraic curves.

In a neighborhood of a point, such a curve consists of a number of branches, each homeomorphic to a segment. Each branch intersects a small circle centered at the point twice, and this yields a chord diagram: an even number of points on a circle, grouped in pairs, see Figure 4.

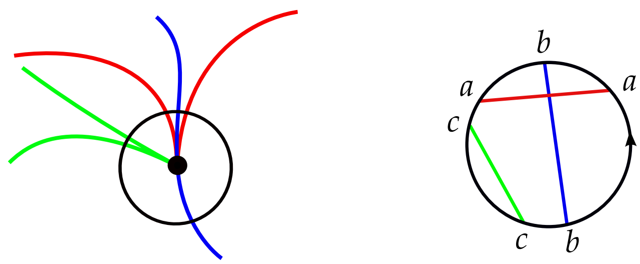

Figure 4: An algebraic curve with three branches and its chord diagram.

The fact that each branch that enters a disc must exit from it is central to Gauss's first proof of the Fundamental Theorem of Algebra; Gauss asserted, but did not prove this fact. Ghys writes:

> The 'proof' is given in a footnote: it is a typical example of a proof by intimidation:
>
> > ... As far as I know, nobody has raised any doubts about this. However, should someone demand it then I will undertake to give a proof that is not subject to any doubt, on some other occasion.
>
> Nobody has raised doubts and he will prove it on some other occasion" ☺ !

A chapter is devoted to a proof of Gauss's claim, and the proof is by no means obvious.

Back to the main problem studied in the book: to describe the chord diagrams that correspond to algebraic curves; the ones that do are called

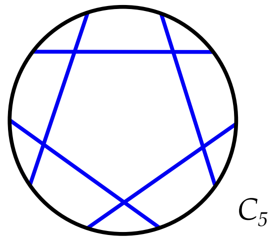

Figure 5: Is this chord diagram analytic?

analytic chord diagrams. For example, what about the chord diagram in Figure 5?

In the words of the author, the following theorem is a highlight in the promenade: *A chord diagram is analytic if and only if it does not contain the sub-chord diagrams depicted in Figure 6.*

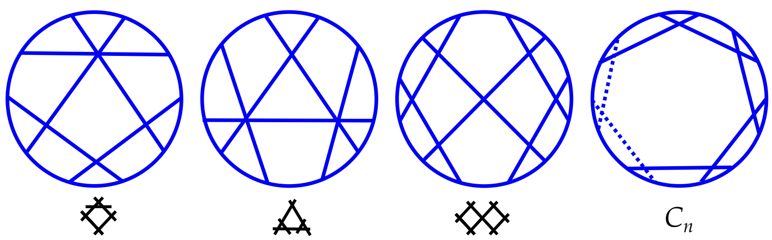

Figure 6: The obstructions to being analytic ($n \geq 5$).

Unlike the case of graphs of polynomials, the number of basic obstructions is infinite (it includes the family of diagrams C_n).

The exposition in the book is very far from being a beeline to the main result[2]: it includes many a detour. Just to mention a few: Newton's "The Method of Fluxions and Infinite Series", Möbius and his band, Hopf fibration, Milnor's theory of singular points of complex hypersurfaces [7], the geometry of associahedron, introduction to operards[3], Gauss coding of planar curves, Gauss theory of linking numbers, the Kontsevich universal knot invariant...

To give a couple of pictorial examples, Figures 7 and 8 are taken from the chapter on the Möbius band and the Hopf fibration.

[2]But it is not a random walk either. To me, as a dog owner, it resembles a trajectory of an off leash dog: one gets to the goal, but not before having thoroughly explored a neighborhood of the path.

[3]"Everything you always wanted to know about operads but were afraid to ask".

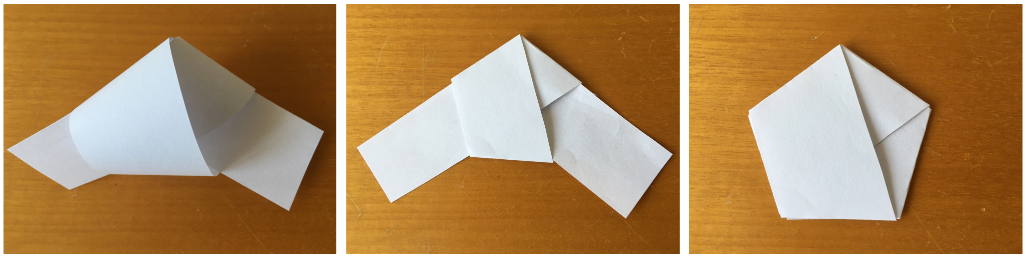

Figure 7: Möbius regular pentagon.

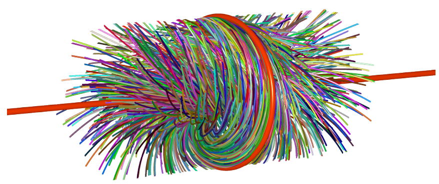

Figure 8: Hopf circles in a neighborhood of one of them, represented by a line.

I hope, the reader's appetite is whetted enough. Let me now address the genre of the book, its unique features, the philosophy and motivation of its author.

There is a huge difference between written and oral presentation of mathematics. This book is probably closer to the latter than the former; I'd describe it as an engaging colloquium talk, spanning a 300 page book.

For example, when giving a colloquium, you will be excused to say that something should be "nice" without saying exactly what "nice" means [6]. In the chapter on operads, Ghys describes this notion and says that operads satisfy ... some axioms.

> "I don't want to write down the formulas expressing these axioms
> since I would be unable myself to read the formulas that I wrote.
> I prefer to give first an example..."

The book is lavishly illustrated, including a full-page picture at the opening of every chapter. Ghys makes an innovative use of the wide margins: they play the role of footnotes for references and comments, but also contain mathematical drawings, photographs and portraits, historical and biographical snippets, etc.

The reader clearly hears the singular voice of the author, who takes the opportunity to make a joke, to share his philosophical, educational, and political views, or just to wink at you. Just a few examples:

- The proof of a proposition that took three pages ends with a TeX symbol for the end of proof, preceded by "Ouf!"
- In the section on linking numbers in astronomy, Ghys talks about the linking of the trajectory of the Earth γ and a planet γ'. "For simplicity, I assume that γ and γ' are disjoint ☺".
- In the chapter *Möbius and his band*, Ghys presents a quotation, in French, from the psychoanalyst Jacques Lacan involving Möbius band and written in a heavy psychoanalytic/literary theory jargon. On the margins, one finds a comment: "I am unable to translate into English (or even understandable French)" accompanied with Figure 9.[4]

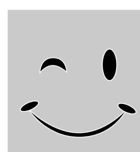

Figure 9: Comment to a quotation from Lacan.

Ghys believes that the mathematical books are too expensive and that they should be essentially free. The copyright page says:

> The text and the illustrations without the symbol $\star$ have been produced by Étienne Ghys who has waived all copyright and related or neighboring rights. You can copy, modify, and distribute them, even for commercial purposes, all without asking permission.

The pdf file of the book can be freely downloaded, for example, from the author's web site `http://perso.ens-lyon.fr/ghys/home/`.

[4]The reader who wishes for more is referred to chapter 2 of [8].

Ghys is one of the main players in popularization of mathematics, and the book under review should be considered in this context. His other work in this direction include the mathematical films “Dimensions” and “Chaos” [4, 5] and the web magazine “Images des Mathématiques” [11].

Regretfully, our “publish or perish” culture is not very conducive to popularization. Ghys makes a strong case for changing this culture in [2]:

> In short, a mathematician answering the traditional question from a colleague “What’s your field?” should not feel anymore ashamed when he or she replies “I work on popularization of mathematics”.

This vision should be attractive to the readers of our magazine as well.

Let me finish by citing from the preface of the book under review:

> I wrote this “petit livre” with one specific reader in mind: myself, when I was an undergraduate... To be very specific, I limited the prerequisites to my own background when I passed the “agrégation” examination, exactly forty years ago ☺! I vividly remember that I had (and I still have) great difficulties reading long mathematical treatises, full of technical details, and that I preferred looking at pictures. I have now learned that precision and details are frequently necessary in mathematics, but I am still very fond of promenades. I did try to imagine what could have been my own reactions faced with this book, as a beginner.

And indeed, after the first draft of the book was finished, its author asked an undergraduate student, Christopher-Lloyd Simon, to become a test reader. Simon did more than refereeing the book, he contributed a new idea that made a complete solution of the problem possible!

This unique book is a trailblazer for a new way of communicating mathematics. Among other things, it projects a strong feeling of unity of mathematics. It is highly recommended to mathematicians of all ages and all sophistication levels.

References

[1] E. Ghys. *Intersecting curves (variation on an observation of Maxim Kontsevich).* Amer. Math. Monthly **120** (2013), 232–242.

[2] E. Ghys. *The internet and the popularization of mathematics*. Proc. ICM Seoul 2014, vol. 4, 1187–2202.

[3] L. Habsieger, M. Kazarian, S. Lando. *On the second number of Plutarch.* Amer. Math. Monthly **105** (1998), 446.

[4] J. Leys, E. Ghys, A. Alvarez. *Dimensions, a walk through mathematics*, 2008. http://www.dimensions-math.org/Dim_E.htm

[5] J. Leys, E. Ghys, A. Alvarez. *Chaos, a mathematical adventure*, 2013. http://www.chaos-math.org/en

[6] J. McCarthy. *How to give a good colloquium.* Canadian Math. Soc. Notes **31** no.5, Sept. 1999, 3–4.

[7] J. Milnor. *Singular points of complex hypersurfaces.* Annals of Mathematics Studies, No. 61 Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo 1968.

[8] A. Sokal, J. Bricmont. *Fashionable nonsense. Postmodern intellectuals' abuse of science.* New York: Picador 1998.

[9] R. Stanley. *Hipparchus, Plutarch, Schröder, and Hough.* Amer. Math. Monthly **104** (1997), 344–350.

[10] The Online Encyclopedia of Integer Sequences https://oeis.org

[11] Images des Mathématiques http://images.math.cnrs.fr/?lang=fr