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Déformations de flots d'Anosov et de groupes fuchsiens. (French. English summary)
[Deformations of Anosov flows and Fuchsian groups]
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A flow φ^t generated by a vector field X on a closed oriented 3-manifold V is an Anosov flow if TV is written as the Whitney sum of line bundles $E^{ss}\oplus E^{uu}\oplus\mathbf{R}\cdot X$ invariant under the flow such that, for some positive constants C and λ, the inequalities $\|d\varphi^t(v_{ss})\|\le C\exp(-\lambda t)\|v_{ss}\|$ for $v_{ss}\in E^{ss}$ and $\|d\varphi^t(v_{uu})\|\ge C^{-1}\exp(\lambda t)\|v_{uu}\|$ for $v_{uu}\in E^{uu}$ hold. (A Riemannian metric on V is fixed.)

There are two families of classical examples of Anosov flows. One family consists of suspensions of the hyperbolic toral automorphisms. The other consists of the geodesic flows of surfaces of curvature -1. The algebraic Anosov flows are those obtained from these examples by taking finite coverings and finite quotients. The algebraic Anosov flows are volume preserving and the invariant distributions E^{ss} and E^{uu} are of class C^∞.

S. E. Hurder and A. Katok showed [Inst. Hautes Études Sci. Publ. Math. No. 72 (1990), 5–61 (1991); MR1087392 (92b:58179)] that if an Anosov flow is volume preserving with E^{ss} and E^{uu} being of class C^2, then E^{ss} and E^{uu} are in fact of class C^∞. The author previously showed that if E^{ss} and E^{uu} are of class C^∞, then the flow is C^∞ conjugate to an algebraic flow after a time change [Ann. Sci. École Norm. Sup. (4) **20** (1987), no. 2, 251–270; MR0911758 (89h:58153)].

In the present paper the author first shows that if an Anosov flow is volume preserving with 2-dimensional distributions $E^{ss}\oplus\mathbf{R}\cdot X$ and $E^{uu}\oplus\mathbf{R}\cdot X$ of class C^2, then the flow is C^∞ conjugate to an algebraic flow after a time change. This settles a problem raised by Hurder and Katok [op. cit.].

The author then constructs nonalgebraic examples of Anosov flows in the unit tangent bundle of a surface S of genus greater than 1 with the 2-dimensional distributions $E^{ss}\oplus\mathbf{R}\cdot X$ and $E^{uu}\oplus\mathbf{R}\cdot X$ being of class C^∞. Of course these are neither volume preserving nor are both the line bundles E^{ss} and E^{uu} of class C^2. Up to time changes the examples are parametrized by a pair of distinct points in the Teichmüller space of the surface S. Hence they are called quasi-Fuchsian.

The author conjectures that if an Anosov flow is not volume preserving but with the 2-dimensional distributions $E^{ss}\oplus\mathbf{R}\cdot X$ and $E^{uu}\oplus\mathbf{R}\cdot X$ of class C^∞, then it is quasi-Fuchsian up to finite coverings. He proves that the conjecture is true under the pinching condition $\lambda_{ss}(x,t)/\alpha\le\lambda_{uu}(x,t)\le\alpha\lambda_{ss}(x,t)$ for $\alpha<2$, where $\lambda_{ss}(x,t)=|\log\|d\varphi^t|E^{ss}_x\||$ and $\lambda_{uu}(x,t)=|\log\|d\varphi^t|E^{uu}_x\||$.

Finally, the author shows the differentiable rigidity of Fuchsian representations. The Riemannian metrics of curvature -1 on a closed oriented surface S define the Fuchsian representations of $\pi_1(S)$ in the group $\mathrm{Diff}^\infty_+(S^1)$ of diffeomorphisms of the circle at infinity of the universal covering space of S. These are in fact representations in $\mathrm{PSL}(2,\mathbf{R})$. The rigidity shown by the author is that any representation sufficiently C^1 near to a Fuchsian representation is differentiably conjugate to some Fuchsian representation. This improves his previous result in another paper [Invent. Math. **82** (1985), no. 3, 479–526; MR0811548 (87f:58084)].

These results come out of the study on the transverse projective structure of the foliation defined by the 2-dimensional distributions $E^{ss} \oplus \mathbf{R} \cdot X$ and $E^{uu} \oplus \mathbf{R} \cdot X$. In a recent paper ["Rigidité différentiable des groupes fuchsiens", Preprint, École Normale Supérieure Lyon, Lyon; per revr.], the author develops this study and proves that any smooth (C^3) representation topologically conjugate to a Fuchsian representation is differentiably conjugate to some Fuchsian representation, as well as that the above conjecture on the quasi-Fuchsian Anosov flows is true (without pinching condition).

Reviewed by *Takashi Tsuboi*