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Déformations des structures complexes sur les espaces homogènes de $\mathrm{SL}(2, \mathbf{C})$. (French)
[Deformations of complex structures on homogeneous spaces of $\mathrm{SL}(2, \mathbf{C})$]

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In this paper the author studies homogeneous complex 3-folds of the form $M = G/\Gamma$, where G is the complex Lie group $\mathrm{SL}(2, \mathbf{C})$ and $\Gamma \subset G$ is a cocompact lattice. These complex manifolds are not Kähler, and have many interesting properties. When Γ is torsionfree, M is diffeomorphic to $H^3/\Gamma \times S^3$, where H^3/Γ is the hyperbolic 3-manifold with fundamental group Γ.

Much of the paper concerns the deformation theory of M. If $u\colon \Gamma \to G$ is a homomorphism sufficiently close to the trivial homomorphism, then the quotient of G by the action of Γ defined by $\gamma\colon x \mapsto u(\gamma)^{-1} x\gamma$ is a complex manifold $M(u, \Gamma)$ diffeomorphic to M. One of the main results of this paper is that every complex manifold close to M arises from this construction.

Furthermore, two such manifolds $M(u_1, \Gamma_1)$ and $M(u_2, \Gamma_2)$ are biholomorphic if and only if Γ_1 and Γ_2 are essentially conjugate (up to twisting by a character $\Gamma \to \{\pm I\} \subset G$), where the conjugation takes u_1 to u_2. Furthermore every holomorphic surjection $M(u_1, \Gamma_1) \to M(u_2, \Gamma_2)$ is a covering map.

The Kuranishi space of M is computed, as are the various cohomology groups which arise in deformation theory. For example, it is shown that $H^i(M; \mathbf{C}) \cong H^i(M; \mathcal{O})$ for $i = 1, 2$ (where $\mathcal{O}$ is the structure sheaf) and the holomorphic cohomology groups are related to group cohomology of Γ. Many results on holomorphic tensor fields are described (including a discussion of "holomorphic metrics").

The paper contains many other interesting results and remarks. For example, since G doubly covers the bundle of oriented orthonormal frames over hyperbolic 3-space H^3, the complex manifold M identifies with the frame bundle of the hyperbolic 3-manifold H^3/Γ. A general construction of a natural almost complex structure on the frame bundle of any Riemannian 3-manifold is given, and it is shown that this almost complex structure is integrable precisely when W has constant curvature -1.

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