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Dynamique des flots unipotents sur les espaces homogènes. (French. French summary)
[Dynamics of unipotent flows on homogeneous spaces]
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This session of the Bourbaki Seminar is concerned with the work of M. Ratner, G. A. Margulis and others on unipotent flows on homogeneous spaces. The main papers under consideration—out of a bibliography with nearly 100 references—are those by Ratner [including Acta Math. **165** (1990), no. 3-4, 229–309; MR1075042 (91m:57031); Duke Math. J. **63** (1991), no. 1, 235–280; MR1106945 (93f:22012)], Margulis [including C. R. Acad. Sci. Paris Sér. I Math. **304** (1987), no. 10, 249–253; MR0882782 (88f:11027)], and S. G. Dani and Margulis [including Math. Ann. **286** (1990), no. 1-3, 101–128; MR1032925 (91k:22026)]. Most of the first section of this report is devoted to the classical example of the horocycle flow, that is, the right action of $\{\left(\begin{smallmatrix}1&t\\0&1\end{smallmatrix}\right)=h^t_+\colon t\in\mathbf{R}\}$ on $\Gamma\backslash\mathrm{SL}(2,\mathbf{R})$ for a lattice Γ, bringing out those aspects of classical (say, before 1985) results and proofs which feature in later generalizations. The geodesic flow is given by the right action of $\{\left(\begin{smallmatrix}e^s&0\\0&e^{-s}\end{smallmatrix}\right)=g^s\colon\ s\in\mathbf{R}\}$. The relationship $g^sh^t_+g^{-s}=h^{e^{2s}t}_+$ has always been important. Hedlund proved that every orbit of the horocycle flow is dense or periodic (and dense if $\Gamma\backslash\mathrm{SL}(2,\mathbf{R})$ is compact). He also proved that the horocycle flow is ergodic with respect to Haar measure, using the relationship above, and Hopf's result that the geodesic flow is ergodic. In fact, Haar measure is the only finite ergodic measure if $\Gamma\backslash\mathrm{SL}(2,\mathbf{R})$ is compact: a number of proofs were given in the 1970s. More generally, if Γ is any lattice, Dani and Smillie proved that every orbit is either periodic or equidistributed with respect to Haar measure.

Ratner's results are for $(\Gamma\backslash G,H)$, where G is a connected Lie group, Γ is a lattice (or sometimes simply discrete) and H is a unipotent subgroup of G, that is, 1 is the only eigenvalue of $\mathrm{Ad}(h)\colon\mathcal{G}\to\mathcal{G}$, for all $h\in H$, where $\mathcal{G}$ is the Lie algebra of G. Her results include both topological and measure-theoretic versions, and one of each is quoted at the end of the first section. The topological result is that, for every $x\in G$, there exists a Lie group $H(x)$ containing H such that $\overline{\Gamma xH}=\Gamma xH(x)$. This solves the so-called Raghunathan conjecture. The measure-theoretic result—which Ratner calls the Raghunathan measure rigidity conjecture—is that every finite H-invariant ergodic measure is a homogeneous measure, that is, a translation of $H(x)$-Haar measure, for $H(x)$ as above.

The Oppenheim conjecture, which is quoted in the introduction, was apparently one of the motivations for the formulation of the Raghunathan conjecture. It says that if Q is a nondegenerate indefinite quadratic form on $\mathbf{R}^n$ for $n\geq 3$, and not a multiple of a rational quadratic form, then for all $\varepsilon>0$ there is $\underline{v}\in\mathbf{Z}^n$ with $0<Q(\underline{v})<\varepsilon$. This was resolved by Margulis [op. cit., 1987] and later improved by Margulis and Dani [op. cit.]. The relationship with the Raghunathan conjecture, explained in the fourth section, is via the right action of $\mathrm{SO}(p,q)$ $(p+q=n)$ on $\mathrm{SL}(n,\mathbf{Z})\backslash\mathrm{SL}(n,\mathbf{R})$, since $\mathrm{SO}(p,q)$ preserves the standard quadratic form of rank n and signature $p-q$. In his second section, the author gives a proof of the results about closures of orbits of the classical horocycle flow, based largely on the methods used in Margulis' work. The section finishes with statements about general $(\Gamma\backslash G, H)$, where $H = \{\varphi^t\colon\ t \in \mathbf{R}\}$ is a 1-parameter unipotent subgroup. The results concern the proportion of time spent by orbits in a compact $K \subset \Gamma\backslash G$, and are due to Dani, Margulis and Ratner.

In the third section, the author gives some of the main ideas of Ratner's work. He discusses Ratner's properties H and R. He describes these as properties of the derivative of h_+^t in the directions of the centraliser and normaliser of H respectively. An example is given of the use of each. Property H is illustrated by a proof of Ratner's result that any nontrivial measurable quotient of the horocycle flow is again a horocycle flow. Property R is used to give Ratner's proof in the classical case that if Γ is discrete (and not necessarily a lattice) then any finite h_+^t-invariant ergodic measure which is not concentrated on a periodic orbit is also g^s-invariant. This is then used to give Ratner's characterisation of all such measures—a result first proved for lattices by Dani—and to give results on ergodic averages. Finally, notes are given on the general case of $\Gamma\backslash G$ (for G solvable and semisimple, both resolved by Ratner).

In the fourth section, applications are discussed: to rigidity of measure-isomorphisms between unipotent flows, to values of quadratic forms at integer vectors, and to nonexistence of nontrivial geodesic laminations of finite volume hyperbolic manifolds.

Reviewed by *M. Rees*