MR930390 (89d:57040) 57R30 (53C12)
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Gauss-Bonnet theorem for 2-dimensional foliations.
J. Funct. Anal. **77** (1988), *no. 1,* 51–59.

Let **F** denote a foliation by two-dimensional leaves of a smooth compact manifold, M. The choice of a Riemannian metric on the tangent bundle to the leaves determines a scalar curvature functional on the ambient manifold M, and it is a very interesting problem to understand the behavior of these scalar curvatures with the addition of some regularizing hypotheses on the foliation. The prototype of course is the Gauss-Bonnet theorem, which asserts that the integral of the scalar curvature over a compact orientable Riemann surface is equal to 2 times the Euler characteristic. A. Connes proved an extension of this theorem to foliations of compact manifolds by oriented surfaces, where the leaves need not be compact. However, Connes assumed that the foliation possessed a transverse, invariant, sigma finite measure. The role of the measure is that, by the construction of D. Ruelle and D. Sullivan [Topology **14** (1975), no. 4, 319–327; MR0415679 (54 #3759)], the measure determines a two-dimensional closed current which represents the fundamental class of the leaves. Connes' theorem then asserts that if the set of spherical leaves for the foliation is negligible with respect to the invariant measure, then the average of the scalar curvature functional with respect to the transverse measure must be nonpositive. In other words, if the average is positive, then the foliation must have some compact spherical leaves. The proof of this result is based on Connes' notion of L^2-Betti numbers for a foliation and is an application of the measured-foliation index theorem. A small technical point is that the average of the scalar curvature is formed by first replacing the transverse measure to **F** with a measure on M, by multiplying the transverse measure with the Riemannian density on leaves.

In the paper under review, the author extends Connes' theorem to include foliations without an invariant transverse measure. By the thesis of L. Garnett, every foliation of a compact manifold admits a harmonic measure, a Borel measure on M which is invariant under the heat flow along leaves. The measure that Connes used to average the scalar curvature in his proof is a harmonic measure. The main result of this paper is that for a harmonic measure with respect to a foliation of a compact manifold by two-dimensional leaves, if the set of spherical leaves is negligible, then the average of the scalar curvature with respect to the harmonic measure will be nonpositive. The proof of this is based on a clever observation by the author that the scalar curvature integral with respect to a harmonic measure is invariant under leafwise conformal change in the metric. The foliated manifold is decomposed into two measurable sets, the leaves whose universal covers are conformally equivalent to a plane, and the leaves equivalent to the disc. The first set is treated via Connes' theorem, for the harmonic measure must be induced from a transverse measure when the leaves are equivalent to planes. The second case is treated by applying the measurable Riemann mapping theorem of Bers, and is the technically novel part of the proof.

Reviewed by *Steven E. Hurder*