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Ghys, Étienne (F-ENSLY-PM)
Groupes aléatoires (d'après Misha Gromov,...). (French. French summary) [Random groups (following Misha Gromov,...)]
Astérisque No. 294 (2004), *viii*, 173–204.

This Bourbaki lecture is a survey of the theory of random groups.

The idea of a random group, as well as almost all its developments, are due to M. L. Gromov who suggested (as early as 1986) in the introduction to his paper [in *Essays in group theory*, 75–263, Springer, New York, 1987; MR0919829 (89e:20070)] which studied presentations from the statistical point of view, that with high probability presentations define hyperbolic groups. This idea was further developed in another paper [in *Geometric group theory, Vol. 2 (Sussex, 1991)*, 1–295, Cambridge Univ. Press, Cambridge, 1993; MR1253544 (95m:20041)], where the same author introduced the idea of the density of a presentation whose relations have the same length, and brought to the fore the astonishing phenomenon of phase transition: with high probability, if the density is $d < 1/2$, then the group is hyperbolic, but if $d > 1/2$, then it is a trivial group. Later, in [Geom. Funct. Anal. **2000**, Special Volume, Part I, 118–161; MR1826251 (2002e:53056)], Gromov introduced the thermodynamical model for a random group and announced the existence of a group whose Cayley graph contains an expander. This fact was established in [M. L. Gromov, Geom. Funct. Anal. **13** (2003), no. 1, 73–146; MR1978492 (2004j:20088a); addendum, L. Silberman, Geom. Funct. Anal. **13** (2003), no. 1, 147–177; MR1978493 (2004j:20088b)].

All these papers stimulated many comments, clarifications and developments, in particular by C. Champetier, A. Y. Ol′shanskiĭ, A. Żuk, Y. Ollivier, and L. Silberman.

The paper under review is a brilliant exposition of the state of our knowledge in 2003. In particular E. Ghys gives a sketch of the proof of all the important results in the theory.

{Reviewer's remark: To complete the sixth section of this article, one should note that two articles by M.-T. Wang [J. Differential Geom. **50** (1998), no. 2, 249–267; MR1684980 (2000e:53051); Comm. Anal. Geom. **8** (2000), no. 3, 545–563; MR1775138 (2001m:58039)] nicely combine with the paper of A. Żuk [Geom. Funct. Anal. **13** (2003), no. 3, 643–670; MR1995802 (2004m:20079)]: with high probability, random groups defined by Żuk not only have Kazhdan property, but also fix a point whenever they act on a complete manifold with pinched nonpositive curvature or on an affine building.}

Reviewed by *Thomas Delzant*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.