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MR1099877 (92f:57004) 57M07 (20F32 53C23)
Ghys, Étienne (F-ENSLY)
Les groupes hyperboliques. (French) [Hyperbolic groups]
Séminaire Bourbaki, Vol. 1989/90.
Astérisque No. 189-190 (1990), *Exp. No.* 722, 203–238.

This survey of a circle of ideas introduced and publicized by M. Gromov [in *Essays in group theory*, 75–263, Springer, New York, 1987; MR0919829 (89e:20070)] is valuable as a general guide to the theory and its problems. Although few proofs are given, the author's identification of central problems and themes in geometric group theory may be of interest to specialists as well as the general audience.

Two metric spaces (X_1, d_1) and (X_2, d_2) are said to be quasi-isometric if there exist maps $f\colon X_1 \to X_2$ and $g\colon X_2 \to X_1$ such that f and g are uniformly eventually Lipschitz and the composites $f \circ g$ and $g \circ f$ are bounded; f and g need not be continuous. The relevance of this notion to group theory appears in the consideration of finitely generated groups Γ and their Cayley graphs, equipped with the metric determined by word length. Although the structure of such a graph depends upon a choice of generators for the group, any two Cayley graphs for Γ are quasi-isometric through functions rewriting each set of generators in terms of the other. Isometric invariants of Cayley graphs thus depend on Γ together with a choice of generating set, while quasi-isometric invariants of Cayley graphs give invariants of Γ. A crucial observation is that if Γ acts as a group of isometric covering transformations on a space X equipped with a geodesic and proper metric d, and if X/Γ is compact, then (X, d) is also quasi-isometric to any graph of Γ, in its word metric; thus differential geometric properties of X can yield invariants of Γ.

The geometric property emphasized in the work surveyed here is the bound (sectional curvature) $\leq -\delta < 0$, where curvature is understood in the usual sense for a Riemannian manifold and in several synthetic senses for metric spaces and groups; a metric space with this property is said to be hyperbolic. (Toponogov's theorem on triangles in Riemannian geometry motivates the synthetic notion of curvature $\leq -\delta$ presented in the exposé under review.)

If two metric spaces are quasi-isometric and one of them is hyperbolic, then so is the other: for example, the fundamental group of a closed Riemannian manifold of strictly negative sectional curvature is thus seen to be hyperbolic in any of its word metrics. One of the goals of the subject is then the extension of methods and constructions from negatively curved manifolds to more general hyperbolic metric spaces, especially to hyperbolic groups: Gromov constructs compactifications analogous to the compactification of a visibility manifold by the sphere at infinity defined by geodesic rays, proves isoperimetric inequalities, and even constructs an analog of the geodesic flow.

The author discusses Gromov's boundary for a hyperbolic metric space, small cancellation properties for groups which imply hyperbolicity, some other algebraic properties of hyperbolic groups, and algorithmic or automata theoretic results (the last due to Cannon, Epstein, Gromov, Holt, Paterson, and Thurston). The last section of this paper discusses the vexing lack to date of a quasi-isometric property analogous to (sectional curvature) ≤ 0.

The reader seeking details is referred by the author to the notes of several seminars on Gromov's paper, of which those by M. Coornaert, T. Delzant and A. Papadopoulos [*Géométrie et théorie des groupes*, Lecture Notes in Math., 1441, Springer, Berlin, 1990; see the preceding review], and the collection *Sur les groupes hyperboliques d'après Mikhael Gromov* (Bern, 1988) [edited by the author and P. de la Harpe, Progr. Math., 83, Birkhäuser Boston, Boston, MA, 1990] are the most readily available.

{Reviewer's remark: The notes of Coornaert et al. have been published since this exposé was prepared and are cited as a preprint there. Some of the other bibliographic entries also need corrections: the names of S. M. Gersten and M. S. Raghunathan are misspelled, the cited papers of Gersten and H. B. Short have appeared [Invent. Math. **102** (1990), no. 2, 305–334; MR1074477 (92c:20058); Ann. of Math. (2) **134** (1991), no. 1, 125–158], and the work of de la Harpe and A. Valette on Kazhdan's property T has appeared [Astérisque No. 175 (1989); MR1023471 (90m:22001)].}

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