

MR1876932 (2003a:37032) 37C85 (37E10 57S05)**Ghys, Étienne (F-ENSLY-PM)****Groups acting on the circle.***Enseign. Math. (2)* **47** (2001), no. 3-4, 329–407.

This is a well written and informative survey of group actions on the circle. The main topics are rotation numbers, the bounded Euler class and actions of lattices. The focus is on topological dynamics; in particular, differentiability and analyticity considerations are avoided.

The paper has numerous original proofs and interesting discussions and will be of interest to both researchers and students; the reader only needs to have had some exposure to the basics of group theory, measure theory, surface geometry and topology, particularly the notion of covering spaces. Despite requiring remarkably few prerequisites, the paper succeeds in taking the reader through to the boundaries of current research; in particular, several interesting open problems are posed.

The paper has seven parts, starting with a brief introduction, some definitions, and basic examples. Section 4 treats group-theoretic and topological aspects of the group $\text{Homeo}_+(\mathbb{S}^1)$ of orientation-preserving homeomorphisms of the circle. In particular, it is shown that up to conjugacy, the rotation group $\text{SO}(2, \mathbb{R})$ is the only maximal compact subgroup of $\text{Homeo}_+(\mathbb{S}^1)$, the embedding of $\text{SO}(2, \mathbb{R})$ in $\text{Homeo}_+(\mathbb{S}^1)$ is a homotopy equivalence, that $\text{Homeo}_+(\mathbb{S}^1)$ is simple and that a generic (in the sense of Baire) pair of elements of $\text{Homeo}_+(\mathbb{S}^1)$ generates a free group. The classification of faithful group actions of a connected Lie group on the circle is also recalled. Section 5 gives an elegant review of the notion of rotation numbers, and their uses and limitations in describing the dynamics. This is followed by a proof of Margulis' theorem [G. A. Margulis, C. R. Acad. Sci. Paris Sér. I Math. **331** (2000), no. 9, 669–674; [MR1797749 \(2002b:37034\)](#)] (conjectured by the author) that while $\text{Homeo}_+(\mathbb{S}^1)$ doesn't enjoy the Tits alternative, it does satisfy an analogous property: every subgroup either contains a free group on 2 generators, or leaves invariant a probability measure. For the analytic case, see [B. Farb and P. Shalen, *Ergodic Theory Dynam. Systems* **22** (2002), no. 3, 835–844].

Section 6 begins with a succinct review of group cohomology and then recalls the Euler class and the Milnor-Wood inequality. A review is then given of the properties of the bounded Euler class, which was introduced by the author [in *The Lefschetz centennial conference, Part III (Mexico City, 1984)*, 81–106, *Contemp. Math.*, 58, III, Amer. Math. Soc., Providence, RI, 1987; [MR0893858 \(88m:58024\)](#)]; this is a topological invariant which combines both the Euler class and the rotation number. Orderings on groups are examined and a number of results are proven around the theme of Hölder's theorem. Section 7 deals with actions of lattices. A special case of Witte's theorem is proven: For $n \geq 3$, finite index subgroups of $\text{SL}(n, \mathbb{Z})$ can only act on the circle by finite rotations [D. Witte, *Proc. Amer. Math. Soc.* **122** (1994), no. 2, 333–340; [MR1198459 \(95a:22014\)](#)]. Then, after some discussion, a proof is given of the author's result that for any lattice in $\text{SL}(n, \mathbb{Z})$, with $n \geq 3$, all actions on the circle possess a finite orbit. The paper concludes with examples of nontrivial actions of higher rank lattices.

Overall, this is an attractively written survey, and will no doubt encourage interested readers to further examine the author's work on actions of lattices on the circle [Invent. Math. **137** (1999), no. 1, 199–231; [MR1703323 \(2000j:22014\)](#)].

Reviewed by *Grant Cairns*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.