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Ghys, Étienne (F-LILL)
Sur l'invariance topologique de la classe de Godbillon-Vey. (French. English summary) [On the topological invariance of the Godbillon-Vey class]
Ann. Inst. Fourier (Grenoble) **37** (1987), *no. 4,* 59–76.

A foliation $\mathbf{F}$ of codimension one with transverse differentiability of at least C^2 on a smooth manifold has a much studied invariant in three-dimensional cohomology, the Godbillon-Vey class $\mathrm{GV}(\mathbf{F})$. One of the basic unsolved problems is whether this class is a topological invariant. A result of Raby shows that if there is a C^1-conjugacy between two transversally C^2-foliations, then their Godbillon-Vey classes agree. For Anosov foliations of three-manifolds, the reviewer and A. Katok showed that the class is invariant under homeomorphisms which are bi-absolutely continuous. However, no example is known of two C^2-foliations which are topologically conjugate but have differing invariants. One of the motivations for this problem is due to a theorem of Duminy, that if $\mathrm{GV}(\mathbf{F})$ is nonzero, then $\mathbf{F}$ has a resilient leaf, a leaf which captures itself by a contracting element of its transverse holonomy. This is a topological aspect of the foliation, and the obvious question is what aspects of resiliency contribute to $\mathrm{GV}(\mathbf{F})$. In analogy with the cases of topological and metric entropies for flows, the issue is whether the Godbillon-Vey class is a topological or differentiable invariant.

In the paper under review, the author constructs from surgery on Anosov foliations two foliations which are topologically conjugate, but with different Godbillon-Vey classes! The first foliation can be chosen real analytic, but the key point is that the second foliation is piecewise linear, and the Godbillon-Vey class must be extended to foliations which are transversally of class piecewise C^2. In the paper under review, the author extends the definition by introducing the angle defects at corners. For foliated circle bundles, the extended class is equivalently defined by the Thurston area formula, where the area inside a piecewise-smooth curve in the plane is suitably interpreted by bridging the gaps with straight line segments using the orientation on the source circle. This agrees with the definition of D. B. Fuks, A. M. Gabrièlov and I. M. Gel′fand [Topology **15** (1976), no. 2, 165–188; MR0431199 (55 #4201)]. The author then gives an elegant construction of his examples, based on the technique of choosing an appropriate Birkhoff section for the geodesic flow for a manifold of constant negative curvature, and then performing foliated surgeries. The paper concludes with a section giving speculations on the meaning of the extended Godbillon-Vey class and its relation to the topological invariance problem. Note that the reviewer and Katok showed that there are continuous families of topologically equivalent foliations with continuously varying Godbillon-Vey class, but with the caveat that the foliations are now only C^1 with an r-Hölder estimate on the transverse first derivative, for $r > \frac{1}{2}$, and that the Godbillon-Vey class is also defined for these foliations. Tsuboi has recently observed that both extensions of the author and the reviewer can be put into a common framework, which adds further mystery to the original problem. The evidence suggests that there are two components to the Godbillon-Vey class, a non-C^2-component which is not topologically invariant, and a "smooth" component which is not well understood.

Reviewed by *Steven E. Hurder*