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Knots and dynamics. (English summary)
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This article gives a beautiful and fairly wide-ranging survey of some topological phenomena in certain dynamical systems on low-dimensional manifolds. The paper starts by recalling some older results which are both interesting in their own right and serve as motivation for what follows, beginning with the persistence of vortex rings in perfect incompressible fluids (which follows from a result of Helmholtz and Kelvin), followed by a discussion of the existence or non-existence of periodic orbits of various vector fields on S^3 and of the knot types that they can represent. The Schwartzman asymptotic cycle of a flow on a manifold is then introduced, and a result of Arnold identifying the helicity of a flow on S^3 with an averaged asymptotic linking number in the spirit of Schwartzman's work is discussed.

The paper then turns to some more recent results concerning groups of area-preserving diffeomorphisms of surfaces. A basic ingredient here is the Calabi homomorphism from the group of area-preserving diffeomorphisms of the closed unit disc D which restrict as the identity on ∂D to $\mathbb{R}$; to such a diffeomorphism f this homomorphism associates the integral of the primitive H of the 1-form $f^*\alpha-\alpha$ which vanishes on ∂D, where α is a primitive for the area form on the disc. This invariant has an alternate interpretation due to Fathi: Given distinct points $x_1,x_2\in D$ and an isotopy f_t from id to f, one defines the angle $\mathrm{Angle}(f;x_1,x_2)$ through which the vector $f_t(x_1)-f_t(x_0)$ rotates as t varies from 0 to 1 (this is independent of the isotopy f_t), and then the Calabi invariant is

$$\int_{D\times D}\mathrm{Angle}(f;x_1,x_2)dx_1dx_2.$$

In particular, this alternate characterization shows that the Calabi invariant is unchanged under conjugation by (not necessarily differentiable) homeomorphisms. This serves as a jumping-off point for the discussion of quasi-morphisms from certain groups G of diffeomorphisms to $\mathbb{R}$ (which is to say, maps $F\colon G\to\mathbb{R}$ satisfying $|F(gh)-F(g)-F(h)|\le C$ for some constant C; the groups G considered here often turn out to be simple, making the existence of nontrivial quasi-morphisms significant). A prototype for this is the Poincaré rotation number of a path of homeomorphisms of S^1, which gives a quasi-morphism of $\widetilde{\mathrm{Homeo}}(S^1)$. Moving up a dimension, the author recalls a construction from [D. P. Ruelle, Ann. Inst. H. Poincaré Phys. Théor. **42** (1985), no. 1, 109–115; MR0794367 (86k:58075)] which is somewhat similar to Fathi's account of the Calabi invariant and which induces a quasi-morphism on the group of area-preserving diffeomorphisms of D restricting to ∂D as the identity; further, it is explained how one can modify this construction to get a quasi-morphism on the identity component of the group of area-preserving diffeomorphisms of any closed oriented surface, as was done in [J.-M. Gambaudo and E. Ghys, Ergodic Theory Dynam. Systems **24** (2004), no. 5, 1591–1617; MR2104597 (2006d:37071)]. Still more quasi-morphisms can be defined (as in [J.-M. Gambaudo and E. Ghys, op. cit.]) on the area-preserving diffeomorphism group of either the disc or the sphere by replacing the rotation of segments as in Fathi's construction of the Calabi homomorphism (which corresponds to the signature of the closure of a two-stranded braid) by the signature of the closure of an n-stranded (standard or spherical) braid. Attention then turns to Calabi quasi-morphisms on the group $\mathrm{Ham}(S)$ of Hamiltonian diffeomorphisms of a symplectic surface S, which is to say, quasi-morphisms whose restriction to the group of Hamiltonian diffeomorphisms which are supported in a given sufficiently small disc coincides with the Calabi homomorphism. Such quasi-morphisms are now known to exist on all symplectic surfaces; a construction of P. Py [Ann. Sci. École Norm. Sup. (4) **39** (2006), no. 1, 177–195; MR2224660 (2007f:53116)] in the case where S is hyperbolic is sketched (this construction is somewhat similar to that of Gambaudo-Ghys' quasi-morphism for hyperbolic surfaces), as is the construction of M. Entov and L. Polterovich [Int. Math. Res. Not. **2003**, no. 30, 1635–1676; MR1979584 (2004e:53131)] in the case where S is the sphere; this latter construction is quite different, being based on Floer homology.

The final section deals with the dynamics of a particular vector field on the manifold $M = \mathrm{PSL}(2,\mathbb{R})/\mathrm{PSL}(2,\mathbb{Z})$; M can be identified with the unit tangent bundle of the quotient $D/\mathrm{PSL}(2,\mathbb{Z})$ (where D is the Poincaré disc), and the vector field v in question is then the geodesic flow. Now M can also be identified, using classical modular functions, with the complement of the trefoil knot in S^3; thus the knots which arise as periodic orbits of v in M have a linking number with the trefoil. But periodic orbits can easily be put into a natural correspondence with conjugacy classes of hyperbolic elements A of $\mathrm{PSL}(2,\mathbb{Z})$. The paper ends by summarizing three different proofs of the remarkable fact that the linking number of the periodic orbit k_A associated to $A \in \mathrm{PSL}(2,\mathbb{Z})$ with the trefoil T is equal to $\mathfrak{R}(A)$, where the $\mathfrak{R}\colon \mathrm{PSL}(2,\mathbb{Z}) \to \mathbb{Z}$ denotes the Rademacher function, a function which is related to the logarithm of the Dedekind η function and was extensively studied in [M. F. Atiyah, Math. Ann. **278** (1987), no. 1-4, 335–380; MR0909232 (89h:58177)]. The first of these proofs uses the fact that the linking number of k_A with T is the degree of the restriction of a fibration $S^3 \setminus T \to S^1$; for one particular fibration this degree can be computed using an old theorem of Jacobi. The second proof relies on an obstruction-theoretic argument and a result of Atiyah [op. cit.] and J. Barge and Ghys [Math. Ann. **294** (1992), no. 2, 235–265; MR1183404 (95b:55021)]. Finally, an argument is given which involves a deformation of the geodesic flow under consideration to a dynamical system which can be understood in terms of the template theory of J. S. Birman and R. F. Williams [Topology **22** (1983), no. 1, 47–82; MR0682059 (84k:58138)]. This latter approach yields not just the result about linking numbers, but also the result that the knot types realized by periodic orbits of v on $M \subset S^3$ are precisely the same as the knot types realized by the periodic orbits of the Lorenz dynamical system.

{For the entire collection see MR2334180 (2008b:00006)}

Reviewed by *Michael J. Usher*