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Laminations par surfaces de Riemann. (French. English, French summaries) [Laminations by Riemann surfaces]
Dynamique et géométrie complexes (*Lyon*, 1997), *ix*, *xi*, 49–95, *Panor. Synthèses*, 8, *Soc. Math. France*, *Paris*, 1999.

The paper under review is an extended written version of a series of talks given at the conference "État de la recherche", Lyon, in January 1997. This very impressive work contains a systematic analysis of the properties of laminations by Riemann surfaces. The intensive use of these "generalized foliations" in the theory of holomorphic dynamical systems during the last few years is sufficient to explain the particular interest of this work.

The reviewer wants to stress that this paper, even if only Section 7 contains really new results, is definitely more than a survey of what is known about laminations. It brings together topics from many areas of mathematics and founds a theory of laminations by Riemann surfaces.

Let us give a precise definition. A lamination by Riemann surfaces $(M, \mathcal{L})$ is a compact metric space M with an atlas $\mathcal{L}$ (in fact an equivalence class of atlases), i.e. a covering of M by open sets U_i homeomorphic to $D \times T_i$ (where D is the unit disk in $\mathbf{C}$ and T_i a topological space) such that the changes of charts have the form $h_{ij} = (f_{ij}(z,t), \gamma_{ij}(t))$ with f_{ij} depending holomorphically on z and continuously on t. Thus a lamination (from now on, we will use the term lamination to mean lamination by Riemann surfaces) is a generalization of a foliation to compact spaces which are not manifolds. In particular, one defines the leaves of the lamination which form a partition of the space M.

The principle which guides this work is the following: a lamination is a generalization of a compact Riemann surface. This is due to the fact that the trivial example of lamination is a single compact Riemann surface (even if generally the leaves of a lamination are non-compact). As a consequence, the theory introduced here is built in parallel to the theory of compact Riemann surfaces.

In all of the paper, the author focuses on the case of minimal laminations, that is to say, laminations with all leaves dense. This is motivated by the problem of the existence of an exceptional minimal set in the complex projective space $\mathbf{P}^2$: take a holomorphic polynomial vector field in $\mathbf{C}^2$ and consider the induced foliation on $\mathbf{P}^2$ (viewed as the compactification of $\mathbf{C}^2$). This foliation has a finite number of singular points and has no compact leaf. It is conjectured that the closure of every leaf contains a singular point. Now, if you can prove that there does not exist any nontrivial minimal lamination embedded in $\mathbf{P}^2$ (by a continuous, holomorphic when restricted to each leaf, injective map), you prove this conjecture, since an exceptional minimal set (the closure of a leaf which does not contain any singular point) would contain such an embedded lamination.

The paper begins with a series of examples, such as the construction by Sullivan of a lamination associated to a C^r expanding map of the circle. Another example associates to a family of polygons a lamination which describes the action of $\mathbf{R}^2$ by translations on the space of tilings of the plane by this family. All of the constructions detailed show how a lamination can arise naturally from very different situations, and how it may be a great help in solving the problems coming from these situations.

In Section 3, differential forms on a lamination $(M, \mathcal{L})$ are constructed as differential forms on each leaf which depend continuously on the transverse parameter. The author describes a linear operator on the space of two-forms (an operator generalizing the integration on a compact surface) and obtains from it a particular harmonic measure on M. By use of ergodic theory applied to this measure, the homeomorphism type of a generic non-compact leaf is proved to belong to a class of only six cases.

Section 4 is devoted to a Riemann-Roch formula for laminations. This is done in the case where there is an invariant transverse measure on the space by applying A. Connes' theory of the foliated index. More precisely one defines a line bundle E over a lamination $(M, \mathcal{L})$ as a line bundle over each leaf whose transition functions depend continuously on the transverse parameter. Then the Chern class of E may be defined and it is a two-form in the previous sense. On each leaf (in fact on the holonomy covering of each leaf), consider the Hilbert space of square integrable holomorphic l-forms with values in E. These Hilbert spaces fit together in a sort of bundle over M. Following Connes, the author recalls how to associate to this object the so-called Murray-von Neumann dimension. It plays the role of the dimension of the l-th cohomology group with values in the sheaf of holomorphic sections of a line bundle for a compact Riemann surface. From this, a Gauss-Bonnet formula and a Riemann-Roch formula are derived.

Sections 5 and 6 deal with the problem of uniformization of a lamination. Each leaf, being a Riemann surface, is, by the uniformization theorem, elliptic, parabolic, or hyperbolic, that is to say, has a Hermitian metric of constant curvature 1, 0 or -1. The question is: is it possible to put on $(M, \mathcal{L})$ a Hermitian metric of constant curvature on each leaf of $\mathcal{L}$ which is continuous in the transverse parameter? By the Reeb stability theorem (applied to laminations), elliptic leaves form an open subset with a structure of local product, and the answer is yes for this subset; so we may consider that there are no elliptic leaves. The author explains the result of A. Candel which states the existence of such a metric in the case where all the leaves are hyperbolic. He then gives a counterexample with all leaves parabolic and dense and proves at the same time an approximation theorem. He then constructs a minimal lamination with all leaves parabolic except for one hyperbolic leaf.

In Section 7, the author proves three theorems on the existence of non-constant meromorphic functions on a lamination. The definition of such a function is not so clear: it is finally defined as a quotient of two holomorphic sections of a line bundle. It is not constant if it is not constant on any leaf. The three theorems state the existence of meromorphic functions which separate the points of M in the three cases where all the leaves have the same conformal type and under some topological hypotheses. For example, in the hyperbolic case, one assumes the existence of a global topological transverse $\mathcal{T}$. One then puts on M a Hermitian metric of constant curvature (by use of Candel's result) and considers a sort of laminated universal cover $\tilde{M}$ of $\mathcal{T}$ whose leaves are hyperbolic disks. Quadratic differential forms may be constructed in $\tilde{M}$ and projected onto M. By considering these quadratic differential forms as sections of a suitable line bundle, and taking the quotient of two of them, one obtains non-constant meromorphic functions.

As a concluding remark, we note that this paper is very well written and that a particular effort is made to give a precise idea of the techniques involved in each stated result. It is thus a reference text on the subject (*the* reference text?) and should be read by anyone interested in the use of laminations.

Reviewed by *Laurent Meersseman*