MR1460126 (98i:57067) 57S20 (57S25)
Cairns, Grant (5-LTRB-SM); **Ghys, Étienne** (F-ENSLY-PM)
The local linearization problem for smooth $\mathrm{SL}(n)$**-actions. (English, French summaries)**
Enseign. Math. (2) **43** (1997), *no. 1-2,* 133–171.

Let (M, x_0) be a pointed C^k-manifold and let G be a group with a C^k-action $\Phi\colon G \times (M, x_0) \to (M, x_0)$. For every element $g \in G$, let $Dg = g_{*x_0}$ be the linear tangent map at x_0. Suppose that there exists a local C^k-diffeomorphism $\varphi\colon (M, x_0) \to (T_{x_0}M, 0)$, viz $\varphi(x_0) = 0$, such that $D_g\varphi = \varphi g$ for every $g \in G$; then Φ is linearizable at x_0. The existence problem for φ is called the C^k-linearization problem for Φ at x_0. The main concern of the work under review is the linearization problem for actions of special linear groups $\mathrm{SL}(n, \mathbf{R})$ acting on $(\mathbf{R}^m, 0)$.

To begin with, here are some exciting results (in the work) that provide matter to stimulate the interest in reading the whole paper: (1) For all $n > 1$ and all $k = 1, \cdots, \infty$, every C^k-action of $\mathrm{SL}(n, \mathbf{R})$ on $(\mathbf{R}^n, 0)$ is C^k-linearizable (Theorem 1.1). (2) There is a C^ω-action of $\mathrm{SL}(2, \mathbf{Z})$ on $(\mathbf{R}^2, 0)$ which is not linearizable. For $n > 2$ and $m > 2$ every C^ω-action of $\mathrm{SL}(n, \mathbf{Z})$ on $(\mathbf{R}^m, 0)$ is C^ω-linearizable (Theorem 1.2, (c),(d)). (3) Every C^ω-action of $\mathrm{SL}(n, \mathbf{R})$ on $(\mathbf{R}^m, 0)$ is C^ω-linearizable (Theorem 2.6). (4) Let n and m be such that $n > m$; then every C^1-action of $\mathrm{SL}(n, \mathbf{Z})$ on $(\mathbf{R}^m, 0)$ is nonfaithful.

Below is an overview of the whole paper, which contains 10 sections. In §1 and §2 the authors introduce their main concern. They overview the main known results that are closely related to their concern, such as the Bochner-Cartan theorem, the Sternberg local linearization theorem at a resonanceless hyperbolic fixed point for C^∞-maps, the Thurston stability theorem for nontrivial C^1-actions, and so on. In many instances the proofs of the theorems are given. In §3 the authors deal with actions of $\mathrm{SL}(n, \mathbf{R})$. They prove many results that are useful for the understanding of the linearization problem for $\mathrm{SL}(n, \mathbf{R})$-actions. Theorem 3.5 is one of the highlighted results in the paper. Roughly speaking, let M be an m-dimensional connected topological manifold; Theorem 3.5 tells us that for $n > m + 1$ the only C^0-action of $\mathrm{SL}(n, \mathbf{R})$ on M is the trivial one, and every nontrivial C^0-action of $\mathrm{SL}(m + 1, \mathbf{R})$ on M is transitive. Moreover §3 contains a complete classification of C^0-manifolds that are homogeneous under C^0-actions of $\mathrm{SL}(n, \mathbf{R})$. Indeed, let M be a compact C^0-manifold with a transitive C^0-action of $\mathrm{SL}(m + 1, \mathbf{R})$, where $m = \dim M$; then, up to conjugation, M is either S^m or $\mathbf{R}\mathrm{P}^m$ with the canonical projective action of $\mathrm{SL}(m + 1, \mathbf{R})$. Assume that M is noncompact and $m > 2$; then every transitive C^0-action of $\mathrm{SL}(m, \mathbf{R})$ on M is equivalent to the canonical action on $\mathbf{R}^m - \{0\}$ or $\mathbf{R}\mathrm{P}^{m-1} \times \mathbf{R}$.

Note that every action $\mathrm{SO}_n \times F \to F$ can be extended to an action of $\mathrm{SL}(n, \mathbf{R})$; it suffices to set

$$E = \mathrm{SL}(n, \mathbf{R}) \times_{\mathrm{SO}_n} F$$

(the total space of the associated bundle). Clearly one obtains an action of $\mathrm{SL}(n, \mathbf{R})$ on E and the $\mathrm{SL}(n, \mathbf{R})$-equivariant fiber bundle $E \to \mathrm{SL}(n, \mathbf{R})/\mathrm{SO}_n$ as well. This construction is the so-called "suspension".

In §4 the authors mainly deal with actions of $\mathrm{SL}(n,\mathbf{R})$ on $(\mathbf{R}^n,0)$. Results in §2 and in §3 help to bring under control the linearization problem for C^k-actions of $\mathrm{SL}(n,\mathbf{R})$ on $(\mathbf{R}^n,0)$. For instance, the authors parametrize the set of C^0-actions of $\mathrm{SL}(n,\mathbf{R})$ on $(\mathbf{R}^n,0)$ (Theorem 4.1). They prove that for $n\ge 3$ and $k=1,\cdots,\infty$, every C^k-action of $\mathrm{SL}(n,\mathbf{R})$ on $(\mathbf{R}^n,0)$ is C^k-linearizable (Theorem 4.2).

In §5 and §6 the authors deal with actions of $\mathrm{SL}(2,\mathbf{R})$. To begin with, they recall the orbit structure of the adjoint representation of $\mathrm{SL}(2,\mathbf{R})$. Each orbit is either a single point or a surface. Because the orbit structure is well understood, the authors use it to obtain the classification (up to conjugation with automorphisms of $\mathrm{SL}(2,\mathbf{R})$) of faithful actions of $\mathrm{SL}(2,\mathbf{R})$ on $(\mathbf{R}^2,0)$. In §7, §8 and §9 the C^0-actions of $\mathrm{SL}(2,\mathbf{R})$ on $(\mathbf{R}^m,0)$, $m\ne 2$, are studied. There exist C^∞-actions of $\mathrm{SL}(2,\mathbf{R})$ on $\mathbf{R}^3$ which are not linearizable; the authors give an example obtained by deforming the adjoint representation. In §10 the authors deal with actions of the discrete groups $\mathrm{SL}(n,\mathbf{Z})$. For m and n with $1\le m\le n$, $\mathrm{SL}(n,\mathbf{Z})$ has no faithful C^1-action on $(\mathbf{R}^m,0)$. In contrast to the local linearization theorem for C^k-actions of $\mathrm{SL}(2,\mathbf{Z})$ on $(\mathbf{R}^2,0)$ (Theorem 6.3), the authors prove that there exist nonlinearizable analytic actions of $\mathrm{SL}(2,\mathbf{Z})$ on $\mathbf{R}^2$; one example is given. So lattices in semi-simple Lie groups may admit nonlinearizable C^ω-actions, unless some restrictive conditions are satisfied. For instance, the authors prove the following (Theorem 10.4): Let Γ be an irreducible lattice in a semi-simple Lie group G which is connected with finite center; suppose that G has no nontrivial compact factor group and $\mathrm{rank}(G)>1$; then every analytic action of Γ on $(\mathbf{R}^n,0)$ is linearizable.

The last section contains many relevant remarks.

Reviewed by *Michel Nguiffo B. Boyom*