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Resonances and small divisors.

Kolmogorov's heritage in mathematics,
187–213, *Springer, Berlin,* 2007.

This is a very informal and elementary (but enthusiastically written) introduction to the small divisor problems and KAM theory, with an emphasis on celestial mechanics. The author considers the following topics: periodic processes in Nature and approximation of periodic functions by trigonometric polynomials, Kepler's discoveries, almost periodic functions and their approximation by quasi-periodic ones (by the way, the author does not use the term "quasi-periodic function"), perturbations and the averaging principle, the Laplace-Lagrange stability theorem, Poincaré's discovery of chaos in the restricted three-body problem, the mapping

$$(x, y) \mapsto (x + \alpha, y + u(x))$$

of a cylinder as a "toy model" in the perturbation theory (here $x \in \mathbb{R}/\mathbb{Z}$, $y \in \mathbb{R}$, and α is a fixed irrational number), the solvability of the functional equation $u(x) = v(x + \alpha) - v(x)$ as a necessary and sufficient condition for the boundedness of the orbits in this "toy model", Diophantine and Liouville numbers, the formal Fourier series for v and its convergence in the case of a Diophantine α, an abundance of Diophantine numbers in the sense of the Lebesgue measure, the KAM theorem for nearly twist area preserving mappings of an annulus, consequences of KAM theory for the dynamics in the n-body problem with planets of small enough masses, and questions of applicability of KAM theory to our solar system. Many aspects of celestial mechanics and KAM theory are discussed on an intuitive level, whereas in other cases the author provides precise statements of theorems and even rigorous proofs. Several interesting historical and number-theoretical digressions would help the reader to feel the beauty and versatility of the subject.

Unfortunately, the paper is not free from several unpleasant inaccuracies. It starts with the words "During the International Congress of Mathematicians held in Amsterdam in 1954, A. N. Kolmogorov announced an important theorem which was made precise (and proven!) a few years later by V. I. Arnold and J. Moser." Also, it is written on page 209 that "Kolmogorov only gave global indications on the proof and it is Arnold who gave the rigorous proof of this theorem [on invariant curves of area preserving mappings of an annulus]." This is not correct. Kolmogorov's milestone article [Dokl. Akad. Nauk SSSR (N.S.) **98** (1954), 527–530; MR0068687 (16,924c); English translation, in *Stochastic behavior in classical and quantum Hamiltonian systems (Volta Memorial Conf., Como, 1977)*, 51–56, Lecture Notes in Phys., 93, Springer, Berlin, 1979; MR0550888 (81j:34073)] contained a precise formulation of the theorem on invariant tori of analytic nearly integrable Hamiltonian systems, and the proof was sketched with a clear presentation of the key ideas. Of course, the subsequent contribution by Arnold and Moser has been great, but Kolmogorov's theorem was made precise by Kolmogorov himself and proven by Kolmogorov himself. By the way, this opinion is strongly defended by Arnold [in *The Arnoldfest*

(Toronto, ON, 1997), 1–18, Amer. Math. Soc., Providence, RI, 1999; MR1733564 (2001h:01031); Amer. Math. Monthly **111** (2004), no. 7, 608–624; MR2080045; in *Mathematical events of the twentieth century*, 19–47, Springer, Berlin, 2006; MR2182777 (2006j:01011)]. Besides, the author of the paper under review writes on page 209 that "In 1962, Moser succeeded in accomplishing the feat of proving the theorem [on invariant curves of area preserving mappings of an annulus] in the space of infinitely differentiable functions." Here "infinitely" is a misprint and should read "finitely". Finally, the sentence on the same page, "Nowadays, it is known that the theorem is true with 4 derivatives and false with 3," is also in error. As was shown by M. R. Herman, to guarantee the existence of invariant curves in nearly twist area preserving mappings of an annulus, 3 derivatives are enough [Astérisque No. 144 (1986), 248 pp.; MR0874026 (88f:58131)] but 2 are not [*Sur les courbes invariantes par les difféomorphismes de l'anneau. Vol. 1*, Astérisque, 103-104, Soc. Math. France, Paris, 1983; MR0728564 (85m:58062)].

{For the entire collection see MR2376735 (2008g:00024)}

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