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Sur l'uniformisation des laminations paraboliques. [On the uniformization of parabolic laminations]

Integrable systems and foliations/Feuilletages et systèmes intégrables (*Montpellier*, 1995), 73–91, *Progr. Math.*, 145, *Birkhäuser Boston, Boston, MA*, 1997.

An n-dimensional lamination Λ is a topological space decomposed into n-manifolds which fit together in local charts of the form $U \times T$, where U is an open n-ball. The topological space T, the local transversal of the chart, may vary from chart to chart. By imposing extra structure on leaves, and requiring the structure to vary continuously in the transverse direction, one can define smooth structures, conformal structures, Riemannian metrics, etc. on a lamination. In foliation theory, the ambient space Λ is a manifold and the transversal T is an open ball of complementary dimension to U, but laminations have become interesting in their own right because of many examples where Λ is not a manifold.

Many tools of topology, analysis, and dynamics have been generalized to laminations. A. Connes proved an index theorem which, among other things, relates curvature to the Euler characteristic, in analogy to the Gauss-Bonnet theorem [in *Operator algebras and applications. I* (Kingston, ON, 1980), 521–628, Proc. Sympos. Pure Math., 38, Amer. Math. Soc., Providence, R.I., 1982; MR0679730 (84m:58140)]. L. Garnett applied ergodic theory to laminations [J. Funct. Anal. **51** (1983), no. 3, 285–311; MR0703080 (84j:58099)]. A. Candel studied uniformization of conformal structures [Ann. Sci. École Norm. Sup. (4) **26** (1993), no. 4, 489–516; MR1235439 (94f:57025)].

The question studied by Candel is the following: Given a conformal structure on a compact, 2-dimensional lamination Λ, is there a Riemannian metric on Λ, in the given conformal class, whose Gaussian curvature is constant? The problem breaks naturally into cases, depending on the classification of leaves as elliptic, parabolic, or hyperbolic Riemann surfaces, depending on whether the universal cover is the 2-sphere, the complex plane, or the unit disc. If elliptic leaves exist, or if all leaves are hyperbolic, Candel's paper completely settled the problem. Candel also gave a necessary and sufficient condition in the case where parabolic leaves exist.

The current paper focuses on the case where every leaf of Λ is parabolic (the author says that he does not know if there is an example with mixed parabolic and hyperbolic leaves such that the parabolic leaves do not form a closed subset; such an example was given by the reviewer and U. Oertel in ["Spaces which are not negatively curved", Comm. Anal. Geom., to appear]). The Reeb foliation of the 3-sphere is a parabolic lamination with no flat metric, and so the paper focuses mainly on the case where each leaf of Λ is dense. The first theorem says that if g is a Riemannian metric on Λ, then there is a sequence of metrics conformally equivalent to g whose curvature forms tend uniformly to zero. The proof is a beautiful combination of Garnett's harmonic measures, Connes' index theory, and the Hahn-Banach theorem. The first theorem is not definitive, because one does not know if the sequence of metrics converges. Indeed, the second theorem gives a conformal lamination Λ all of whose leaves are dense and parabolic, but for which there is no flat Riemannian metric in the given conformal class; in fact, one cannot find such a metric even if it is allowed to vary measurably in the transverse direction. The third theorem says that the natural affine structure on each leaf of a parabolic lamination varies continuously in the transverse direction. The fourth theorem studies linear foliations of the 3-torus: if the slope of such a foliation satisfies a certain Diophantine condition, then each smooth metric has a flat metric in the same conformal class.

The paper does not say whether the counterexample in the second theorem has a flat Riemannian metric in the given smooth class, and so the following question is left open: Given a conformal lamination all of whose leaves are dense and parabolic, is there a flat Riemannian metric in the given smooth class?

Reviewed by *Lee Mosher*