Workshop on Complexity — FoCM
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Exact computation of the determinant and of the inverse of a matrix

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Parts of this work have been done with Claude-Pierre Jeannerod (INRIA, LIP ENS Lyon) and with Erich Kaltofen (North Carolina State University).

Problems.

A, $n \times n$ matrix.
Determinant and inverse computation.

.. Algebraic complexity over $\mathsf{K}[x]$, $\mathsf{K}(x)$
.. Bit complexity over $\mathbb{Z}, \mathbb{Q}$
.. Algebraic complexity without divisions over R

▷ Deterministic or
Monte Carlo or Las Vegas randomized algorithms.
▷ Time complexity (& practical algorithms).
▷ (Space complexity).

$\longrightarrow$ A new algorithm and a survey of recent results.

The talk

1. Classical facts in linear algebra

2. Matrix inversion over K(x)

3. Determinant without divisions over R

4. Bit complexity

Remaining problems

1. Classical facts in linear algebra

Classical facts I/ Algebraic complexity

- ▷ Mm : matrix multiplication (or n^ω) .
- ▷ Det : determinant.
- ▷ Inv : inversion.
- ▷ SysLin : linear system solution.

- [Strassen 69, Bunch and Hopcroft 74] :

$$\mathsf{Det} \preceq \mathsf{Mm}, \quad \mathsf{SysLin} \preceq \mathsf{Mm}$$

(Recursive factorization).

$$\mathsf{Mm} \preceq \mathsf{Inv}$$

- [Strassen 73, Baur & Strassen 83] :

$$\mathsf{Inv} \preceq \mathsf{Det}$$

(Automatic differentiation).

$$\longrightarrow \quad \mathsf{SysLin} \preceq \mathsf{Mm} \approx \mathsf{Det} \approx \mathsf{Inv}$$

$A \in \mathsf{K}[x]^{n \times n}$ of degree d,

one must take into account the degrees of the polynomials involved in the computations.

The determinant has degree nd at worst.

- [Mc Clellan 73] :

The determinant can be computed in $\tilde{O}(n^\omega \times nd)$ arithmetic operations.

Proof : evaluation / interpolation scheme.

Linear system solution

$Ay = b$

• [Moenck & carter 79, Dixon 82, Storjohann 2002] :

If the polynomial matrix A is invertible then $Ay = b$ can be solved in $\tilde{O}(n^\omega d)$ arithmetic operations.

"x-adic lifting"

For i from 1 to $2nd$

Computation of $\bar{y} = \bar{y}_0 + \bar{y}_1 x + \bar{y}_2 x^2 + \ldots + \bar{y}_i x^i$:

$\bar{y}_i = A^{-1} b_i \mod x$ // *Correction*

$b_{i+1} = (b_i - A\bar{y}_i)/x$ // *New residue*

Reconstruction (e.g. Padé approximation) of the rational solution y.

• For generic matrices [Pan 87] :

When the Smith normal form is trivial.

The polynomial matrix determinant can be computed in $\tilde{O}(n^\omega d)$ arithmetic operations.

$A \in \mathbb{Z}^{n \times n}$, with entries of bit-lengths bounded by $O(\log \|A\|)$

one must take into account the lengths of the integers involved in the computations.

The determinant has length $\tilde{O}(n \times \log \|A\|)$ at worst.

- [Mc Clellan 73] :

The determinant can be computed in $\tilde{O}(n^\omega \times n \log \|A\|)$ bit operations.

Proof : Chinese remaindering.

Classical facts IV/ Algebraic complexity without divisions

- [Strassen 73] :

Removing divisions from programs that compute polynomials.

The determinant can be computed in $\tilde{O}(n^\omega \times n)$ ring operations.

Proof.

- Gaussian elimination without division for the identity matrix.
- Homotopy :

$$B(z) = I + z(A - I),$$

$$\text{avec : } B_{|z=0} = I, \quad B_{|z=1} = A$$

- Multiplications instead divisions :

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \ldots = (1+z)(1+z^2)(1+z^4)\ldots$$

- Computations on truncated power series.

Worst case classical complexities

(Constant input size d or $\log \|A\|$), up to $\log$ factors)

	K	Ring	K$[x]$	$\mathbb{Z}$
$Ax = b$	n^ω	—	n^ω	n^ω
Det	n^ω	$n^{\omega+1}$	$n^{\omega+1}$	$n^{\omega+1}$
Inv (Adj)	n^ω	$n^{\omega+1}$	$n^{\omega+1}$	$n^{\omega+1}$

Avoid the product algebraic complexity $\times$ size ?

2. Matrix inversion over $\mathsf{K}(x)$

$A \in \mathsf{K}[x]^{n \times n}$ of degree d,

A^{-1} has size $n^3 d$

Theorem. The inverse of a generic matrix polynomial can be computed in $\tilde{O}(n^3 d)$ arithmetic operations.

Recursive block algorithm ?

$$\left[\begin{array}{cc} & \\ \hline \times & \times \end{array}\right] \cdot \left[\begin{array}{c|c} \times & \times \\ \times & \times \end{array}\right] = \left[\begin{array}{c|c} \times & \times \\ \hline 0 & \times \end{array}\right] \quad ?$$

The usual approach based on Schur complement leads to high degrees,

$$\left[\begin{array}{cc} I & 0 \\ \hline -CA^{-1} & I \end{array}\right] \cdot \left[\begin{array}{c|c} A & B \\ C & D \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline 0 & D - CA^{-1}B \end{array}\right]$$

$$\left[\begin{array}{cc} & \\ \hline \times & \times \end{array}\right] \cdot \left[\begin{array}{c|c} \times & \times \\ \times & \times \end{array}\right] = \left[\begin{array}{c|c} \times & \times \\ \hline 0 & \times \end{array}\right] \quad ?$$

Sub-problem : $B \in \mathsf{K}[x]^{n \times (n/2)}$
Compute a basis of the kernel of B as a $\mathsf{K}[x]$-module.

$$\left[\begin{array}{cc} \times & \times \end{array}\right] \cdot \left[\begin{array}{c} \times \\ \times \end{array}\right] = \left[\begin{array}{c} 0 \end{array}\right] \quad ?$$

Definition. A basis is minimal with row degrees d_i if any other basis has degrees $d'_i \geq d_i$.

Sub-problem : $B \in \mathsf{K}[x]^{n \times (n/2)}$, compute a minimal basis of the kernel of B as a $\mathsf{K}[x]$-module.

Example,

6×3 matrix polynomial of degree 2 over GF(7)[x]

Schur complement $\longrightarrow$ degrees 6.

Minimal polynomial basis of degree 2 :

$$B = \begin{bmatrix} 4x^2+5x+6 & 4x^2+3x & 4x^2+x+6 \\ 5x^2+5x & 3x^2+2x+1 & 4x \\ 5x^2+6 & 5x^2+5x+3 & x^2+2x+6 \\ 6x+3 & 4x^2+3x+2 & 2x^2+5x \\ 5x^2+3x+1 & 6x^2+4x & 4x^2+2x+1 \\ 3x^2+4 & 2x^2+6 & 5x^2+1 \end{bmatrix}$$

$$\begin{bmatrix} 4x & x+6+5x^2 & 4x+2x^2 & 6x^2+5 & 5+x & 2+4x \\ 2x+4+x^2 & 6x+3+3x^2 & 4x+1 & 5+x & 6x^2+6x+3 & 2 \\ 1 & 4x^2+5x+6 & 6x+3+5x^2 & 6x+2 & 5x+6 & 6x^2+4x+5 \end{bmatrix} \cdot B = 0.$$

Lemma. A minimal polynomial basis for the kernel of a generic matrix $B \in \mathsf{K}[x]^{n\times(n/2)}$ of degree d has all its row degrees equal to d and can be computed in $\tilde{O}(n^\omega d)$.

Proof.

1. Existence.

By linearization (matrix resultants), the coefficients of such a minimal basis may be computed from linear systems involving the block Toeplitz matrix

$$T = \begin{bmatrix} B_d & B_{d-1} & \dots & B_0 & 0 & \dots & 0 \\ 0 & B_d & B_{d-1} & \dots & B_0 & \dots & 0 \\ & & \ddots & \ddots & \ddots & \ddots & \\ 0 & \dots & 0 & B_d & B_{d-1} & \dots & B_0 \end{bmatrix} \in \mathsf{K}^{(nd)\times(nd)}.$$

T is generically invertible.

2. Computation. Block Toeplitz system solution or matrix Padé approximation (minimal basis computation) or Knuth-Schönage's half-gcd (matrix generalization) ...

$$\left[\begin{array}{c}\text{Min. basis } 2\\ \hline \text{Min. basis } 1\end{array}\right] \cdot \left[\begin{array}{c|c} A_{11} & A_{12} \\ A_{21} & A_{22}\end{array}\right] = \left[\begin{array}{c|c} \times & 0 \\ \hline 0 & \times\end{array}\right] \quad ?$$

Degrees : d d 2d

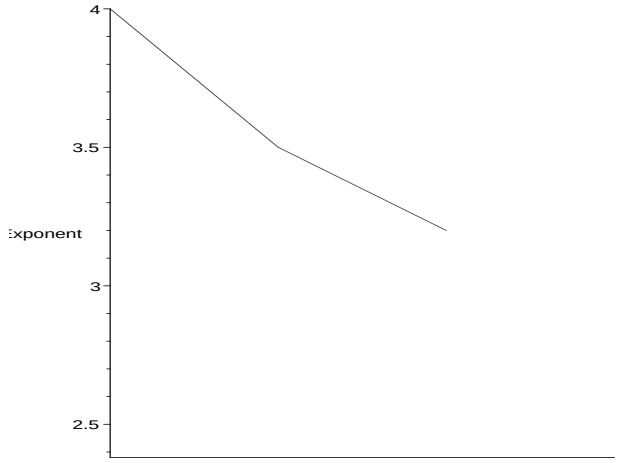
- One step :

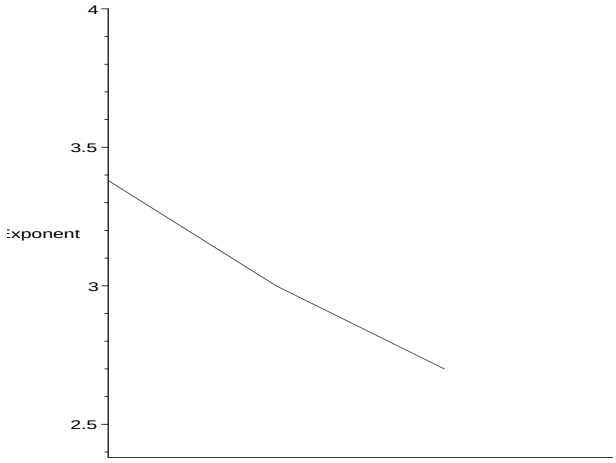
Compute 2^{k+1} minimal bases, each in $\tilde{O}((\frac{n}{2^k})^\omega 2^k d)$.

Update of the transformation, $(\frac{n}{2^k} \times \frac{n}{2^k})$-block diagonal matrix times a matrix of degree $2^k d$ in $\tilde{O}((\frac{n}{2^k})^\omega 2^{3k} d)$.

- $O(\log n)$ steps, $\tilde{O}(n^3 d + (n/2)^3(2d) + \ldots + nd) = \tilde{O}(n^3 d)$.

3. Determinant without divisions





Matrix multiplication in n^3, in n^ω

$A \in \mathsf{K}^{n \times n}$.

[Strassen 73], $\tilde{O}(n^4)$, $\tilde{O}(n^{\omega+1})$

Removal of the divisions.

[Kaltofen 92], $\tilde{O}(n^{3+1/2})$, $O(n^{3.03})$

- Lanczos-Wiedemann approach, - removal of the divisions,
- baby step / giant step strategy.

[Kaltofen and Villard 2001-2002], $\tilde{O}(n^{3+1/5})$, $O(n^{2.7})$

- block Wiedemann strategy in above approach,
- Knuth-Schönage's half gcd computation.

Theorem. The determinant, the characteristic polynomial and the adjoint of an $n \times n$ matrix over R can be computed in $O(n^{2.698})$ (or $\tilde{O}(n^{3+1/5})$ or $\tilde{O}(n^{3+1/3})$) ring operations. The algorithm is deterministic.

Corollary. The determinant, the characteristic polynomial and the Smith normal form of an $n \times n$ matrix over $\mathbb{Z}$ can be computed in $O(n^{2.698})$ bit operations. The algorithm is Las Vegas randomized.

Lanczos-Wiedemann approach

$$u^t v, u^t A v, u^t A^2 v, \ldots, u^t A^{2n} v \quad \text{Berlekamp-Massey}$$

Minimal polynomial of A,
$$\pi(x) = x^d + p_{d-1}x^{d-1} + \ldots + p_1 x + p_0$$

$$\downarrow$$

$B = DA$ has distinct eigenvalues, then,

Characteristic polynomial of B = Minimal polynomial of B

$$\chi(x) = x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0$$

$$\det B = c_0 \quad \rightarrow \quad \det A$$

Block generalization.
$U, V \in \mathsf{K}^{n \times m}, \quad U^t V, \; U^t A V, \; U^t A^2 V, \ldots, \; U^t A^{2n/m} V$

Matrix minimal polynomial :
$x^d I + x^{d-1} C_{d-1} + \ldots + C_0 \in \mathsf{K}[x]^{m \times m}, \quad \det A = \det C_0.$

$\rightarrow$ Less iterates thus smaller order truncated power series in Strassen's removing of divisions + block normal point.

Step 1. /* Pre-conditioning */

The minimal polynomial is different from the characteristic polynomial, its Frobenius form is :

$$F = \begin{bmatrix} 0 & 1 & & & & \\ 1 & 2 & & & & \\ & & 0 & 1 & & \\ & & 1 & 2 & & \\ & & & & 0 & 1 \\ & & & & 1 & 2 \end{bmatrix}$$

$$\downarrow$$

Random diagonal matrix
$\mathrm{diag}(1, 2, 3, 4, 5, 6)$

$$\downarrow$$

The new matrix B and its Frobenius form :

$$B = \begin{bmatrix} 2 & -1 & -5 & -3 & -4 & 2 \\ -20 & -14 & 14 & 14 & 4 & -18 \\ -12 & -9 & 33 & 21 & 15 & -15 \\ -12 & -16 & -24 & -8 & -20 & -8 \\ 60 & 55 & -55 & -50 & -10 & 60 \\ 30 & 30 & 0 & -12 & 12 & 24 \end{bmatrix}, \quad F_B = \begin{bmatrix} & & & & & 720 \\ 1 & & & & & 2136 \\ & 1 & & & & -3480 \\ & & 1 & & & -1588 \\ & & & 1 & & 43 \\ & & & & 1 & 27 \end{bmatrix}.$$

Step 2.

/* Computation of the minimal matrix polynomial */

/* Computation of the matrix sequence */

$$C(x) = \begin{bmatrix} 13 & 8 \\ 8 & 14 \end{bmatrix}, \begin{bmatrix} 217 & 185 \\ 107 & 97 \end{bmatrix}, \begin{bmatrix} 11280 & 9937 \\ -109 & -137 \end{bmatrix}, \begin{bmatrix} 376483 & 329832 \\ 44410 & 39843 \end{bmatrix}$$

/* Matrix half-gcd */

$$\begin{bmatrix} -\frac{174528786}{3457457} - \frac{8507719079}{17287285}\,x - \frac{3160837177}{17287285}\,x^2 + x^3 & -\frac{175402998}{3457457} - \frac{7511942372}{17287285}\,x - \frac{2766671371}{17287285}\,x^2 \\ -\frac{209510760}{3457457} + \frac{2483844884}{17287285}\,x + \frac{3074596254}{17287285}\,x^2 & -\frac{161245200}{3457457} + \frac{2070367712}{17287285}\,x + \frac{2694080482}{17287285}\,x^2 + x^3 \end{bmatrix}$$

Étape 3.

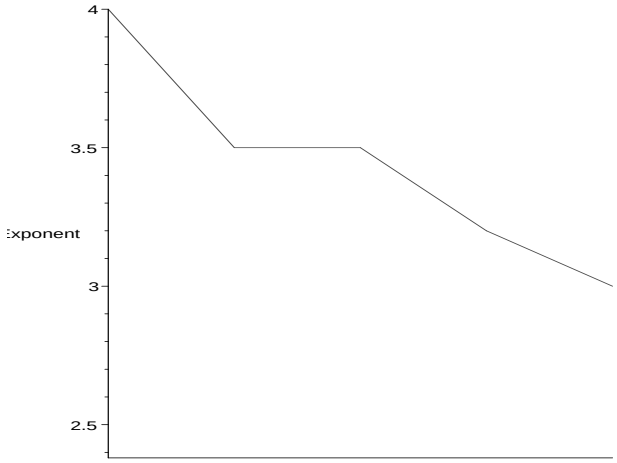
/* Recursive call, determinant of smaller dimension */

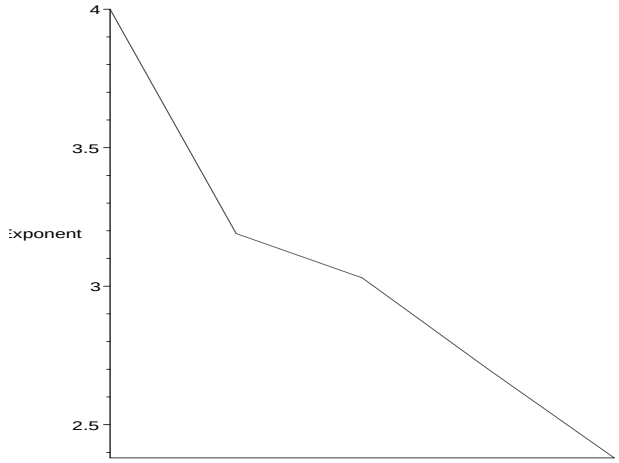
$$\det C(0) = -720$$

Back to A *i.e.* "undo" the pre-conditioning :

$$\rightarrow \qquad \det A = -720/720 = -1.$$

4. Bit complexity





Matrix multiplication in n^3, in n^ω

$A \in \mathbb{Z}^{n \times n}$, up to $\tilde{O}(\log \|A\|)$

[Eberly, Giesbrecht and Villard, 2000], $\tilde{O}(n^{3.5})$, $\tilde{O}(n^{3.19})$.
Additive (rank k) matrix perturbations.

[Kaltofen 92-2000], $\tilde{O}(n^{3.5})$, $O(n^{3.03})$.
Baby step / giant step strategy.

[Kaltofen and Villard 2001-2002], $\tilde{O}(n^{3+1/5})$, $O(n^{2.7})$
Block strategy in above approach.

[Kaltofen 2002, Emiris and Pan 2002].
Output sensitive approach.

[Storjohann 2002], $\tilde{O}(n^3)$, $O(n^\omega)$
High order lifting.

Extensions and remaining problems

Extensions

- Black box (structured, sparse . . .) matrix characteristic polynomial.
- Smith normal form computation.
- Parallel algorithms.

Remaining problems

	K	Ring	K$[x]$	$\mathbb{Z}$
$Ax = b$	n^ω	—	n^ω	n^ω
Det	n^ω	$\underline{n^{2.7}}$	n^ω	n^ω
Inv (Adj)	n^ω	$\underline{n^{2.7}}$	$\underline{\underline{n^{\omega+1}}}$	$\underline{\underline{n^{\omega+1}}}$