La structure conforme des cartes planaires

Nicolas Curien

CNRS

Université Paris 6

9 ème Journée Cartes (Lyon). 14/03/2014.

Un petit bout de la structure conforme de certaines cartes planaires sous une hypothèse technique

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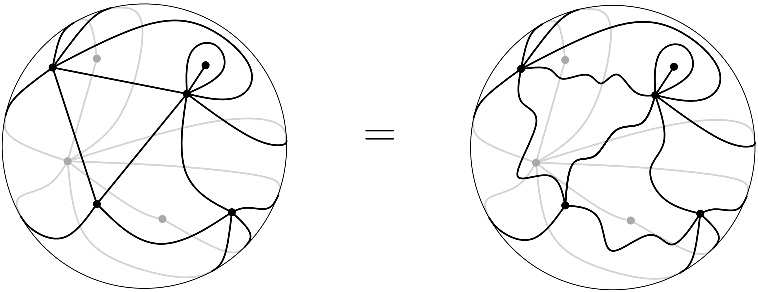
One question

What does a random metric on the 2d-sphere/plane look like ?
(and how to define it ?)

Triangulations

Definition

Triangulation := finite connected graph properly drawn on the sphere (seen up to continuous deformations) such that the faces are all triangles.



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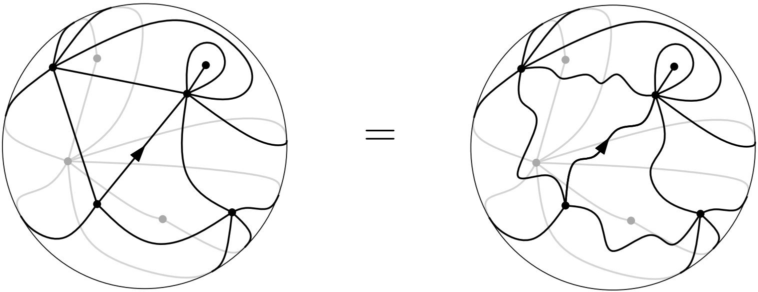
Triangulation := finite connected graph properly drawn on the sphere (seen up to continuous deformations) such that the faces are all triangles + one distinguished oriented edge.



Enumeration

William Thomas Tutte (1917 – 2002)

$$\#\ \text{Triangulations with } n \text{ faces} = \frac{2 \cdot 4^{n-1}(3n)!!}{(n+1)!(n+2)!!}.$$

Other ways to count maps $\rightarrow$ Matrix integrals, bijective methods (Cori-Vauquelin-Schaeffer type bijections).

Large scale structure

Let T_n be a random uniform triangulation with n faces.

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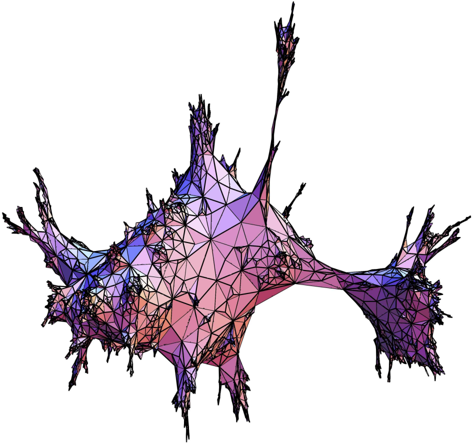

Figure : A (non isometric) embedding of T_{21237}.

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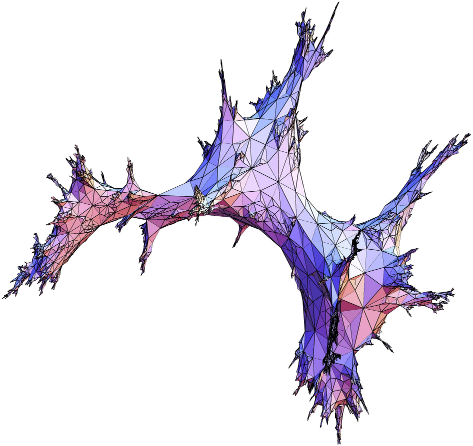

Figure : A (non isometric) embedding of T_{17429}.

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Theorem (Le Gall (11), see also Miermont)

We have the following convergence in distribution for the Gromov-Hausdorff topology

$$\underbrace{\left(T_n, n^{-1/4}\mathrm{d_{gr}}\right)}_{\textit{random metric space}} \xrightarrow[n\to\infty]{(d)} \underbrace{(\mathrm{m}_\infty, D)}_{\substack{\textit{random metric space} \\ \textit{the Brownian map}}}.$$

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$\to$ a random fractal metric space with the topology of the sphere

End of the story?

Although homeomorphic to $\mathbb{S}_2$, the Brownian map does not canonically embed onto the sphere (not yet).
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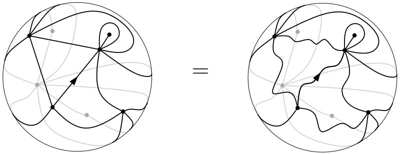
By definition a triangulation has no canonical embedding :



Really ?

A triangulation as a Riemann surface

A triangulation can be seen as a topological surface but also as a compact (simply connected) Riemann surface. See Gill & Rohde.

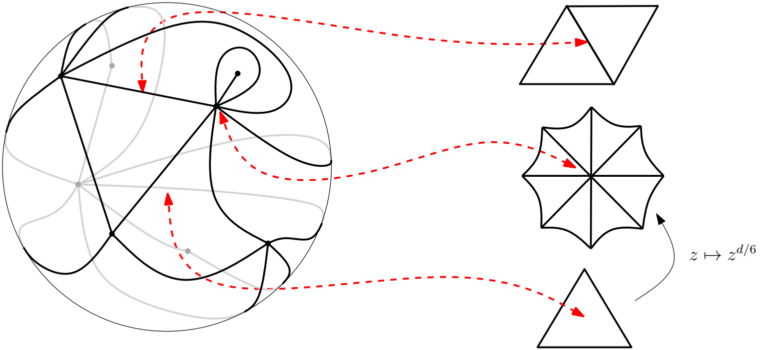
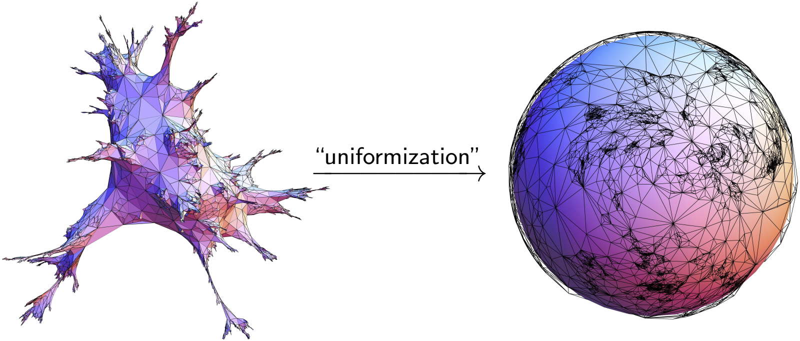


Figure : The three different types of charts

Rieman uniformization theorem

→ ∃! a conformal map from the triangulation onto $\mathbb{S}_2$ (up to Möbius transformations) : canonical drawing.



Riemann uniformization theorem

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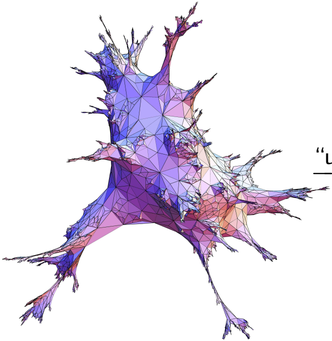
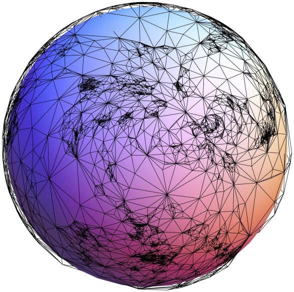
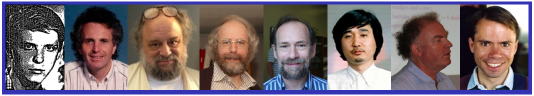


$\xrightarrow{\text{“uniformization”}}$

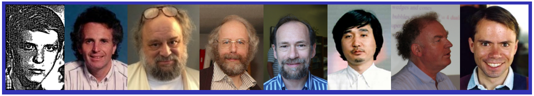


Define $$\mu_n = \frac{1}{\#\text{Vertices}} \sum_{v \in \text{Vertices}} \delta_v .$$



Well-known open conjectures. This form is due to [Duplantier-Sheffield 2008] :

- Prove that μ_n converges as $n \to \infty$ towards a random probability measure μ_∞.



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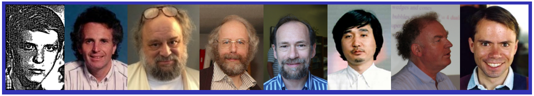

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- Prove that μ_∞ "$=$" $\exp(\sqrt{\frac{8}{3}}\mathrm{GFF})dxdy$ where GFF is the Gaussian Free Field on the 2-sphere.
- This would imply the very famous KPZ relations between critical exponents of statistical mechanics models on deterministic and random lattices (at least for percolation, SAW, SRW).

Setting of today

Defining infinite planar maps

We will define infinite planar triangulations as limit of finite triangulations. If t_1 and t_2 are two triangulations define their local distance as

$$\mathrm{d}_{\mathrm{loc}}(t_1, t_2) = \left(1 + \sup\{r \geqslant 0 : B_r(t_1) = B_r(t_2)\}\right)^{-1},$$

where $B_r(t)$ is the graph made of all the vertices and edges of t which are within distance r from the root edge.

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Distance OK. $\{$Finite triangulations$\}$ is not a closed set
$\rightarrow$ add infinite triangulations.

Prototype : UIPT

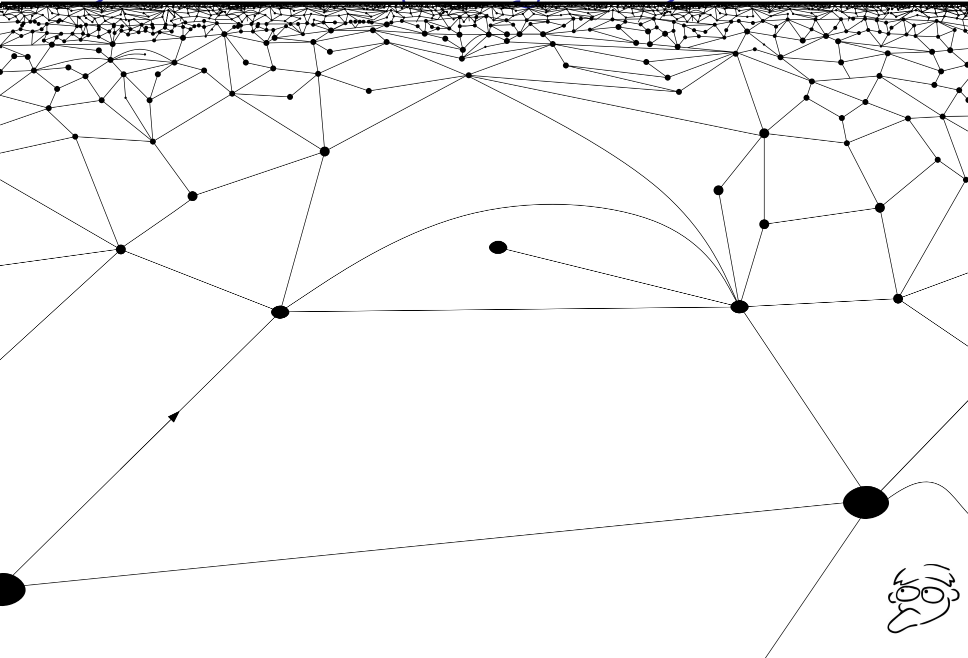
Theorem (Angel & Schramm 03)

If T_n is a uniform (rooted) triangulation with n faces then we have the following convergence in distribution for $\mathrm{d}_{\mathrm{loc}}$

$$T_n \xrightarrow[n\to\infty]{(d)} T_\infty,$$

where T_∞ is an infinite triangulation of the plane called the Uniform Infinite Planar Triangulation.

Illustration (in the quadrangular case)

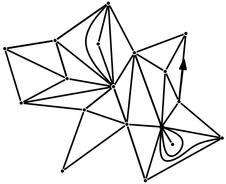


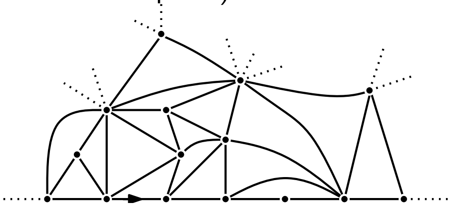
UIHP

If now $T_{n,p}$ is a uniform triangulation of the p-gon with n faces then we have [Angel]

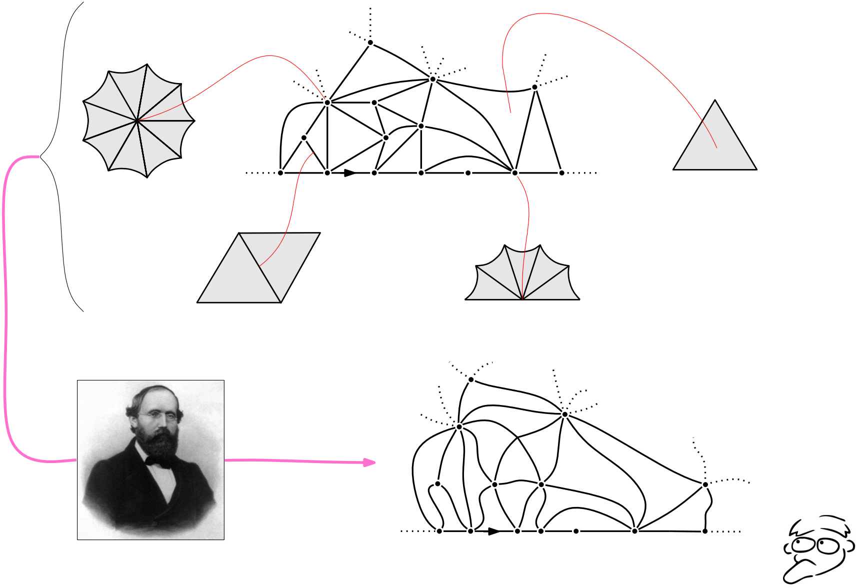
$$T_{n,p} \xrightarrow[n\to\infty]{(d)} T_{\infty,p} \xrightarrow[p\to\infty]{(d)} T_{\infty,\infty},$$

where the latter is an infinite triangulation with an infinite simple boundary (or triangulation of the half-plane).





Uniformization



Uniformization

See $T_{\infty,\infty}$ as a Riemann surface and use RUT by sending the root edge to $[-1/2; 1/2]$ and $\infty \to \infty$ to get a canonical drawing :

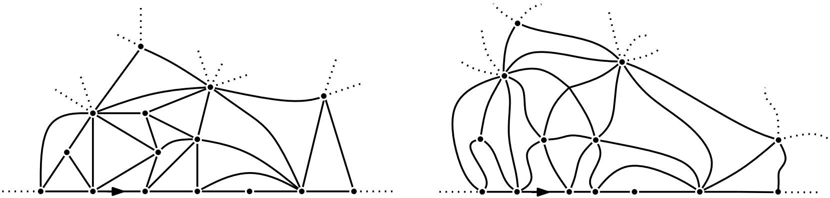


Figure : Uniformization of the UIHPT (artistic drawing)

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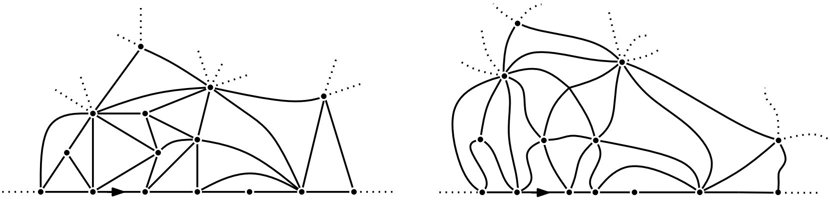

Figure : Uniformization of the UIHPT (artistic drawing)

In the uniformization, denote by $\mathcal{X}_n$ the location of the nth vertex on the right of the root edge and consider the random probability measure on $[0, 1]$:

$$\mu_n = \frac{1}{n} \sum_{k=1}^{n} \delta_{\mathcal{X}_k / \mathcal{X}_n}.$$

The result

Theorem (Under a reasonable technical assumption $\star$)

The sequence of random probability measures μ_n is tight and any subsequential limit μ_∞ satisfies a.s.

- μ_∞ *is not atomic,*
- μ_∞ *has topological support* $= [0, 1]$,
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Conjecture [Duplantier-Sheffield 08] :

$$\mu_\infty(du) = \underbrace{\exp(\sqrt{\frac{8}{3}}X_u)du\mathbf{1}_{[0,1]}(u)}_{\text{normalized}},$$

where $(X_u)_{u\in[0,1]}$ is a centered Gaussian process with covariance $E[X(x)X(y)] = \frac{1}{2}\log^{-1}|x - y|$.

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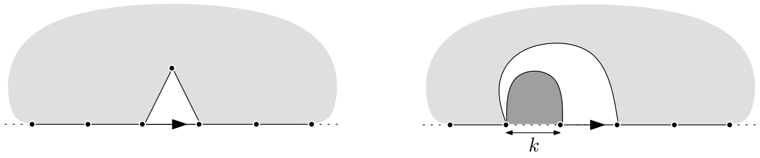
Elements of proof

First ingredient : Spatial Markov property

Assume you reveal the root triangle in the UIHPT (one step peeling) :



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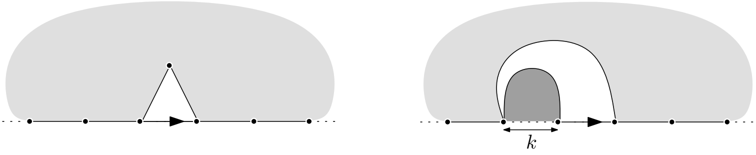

The probability of the first event is $\frac{2}{3}$ and the second is

$$q_{-k} = \frac{(2k-2)!}{4^k (k-1)!(k+1)!} \sim C.k^{-5/2},$$

and furthermore the unexplored region in light gray is independent of the explored region (with holes filled-in) and is distributed as an UIHPT.

Exploration process

An exploration process is Markovian if for every $i \geqslant 0$ the edge to peel at time i is chosen using a (possibly random) algorithm that can use the knowledge of the discovered part but does not depend on the "unknown" part.

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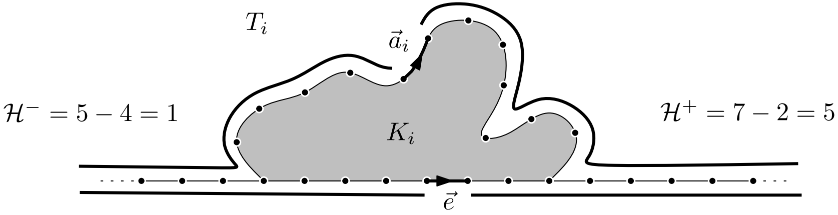
In this case the peeling steps are i.i.d.

Distances along the boundary

We keep track of the position of the peeling position with respect to $-\infty$ and $+\infty$ by using the distances $\mathcal{H}_-$ and $\mathcal{H}_+$ from $\pm\infty$.



Second ingredient : SLE_6 process

We *define* the SLE_6 process on the UIHPT using its conformal representation in $\mathbb{H}$. This induces an exploration of the UIHPT.

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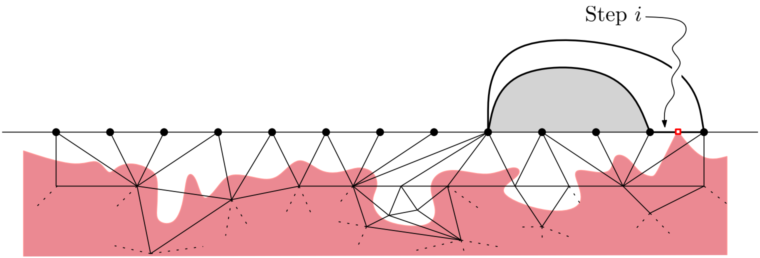
From now on, we focus on the SLE_6 exploration of the UIHPT and by the above remark the peeling steps are i.i.d.

We decompose the variation of the distances from $\pm\infty$ as

$$\Delta\mathcal{H}_i^{\pm} = \underbrace{P_i^{\pm}}_{\textit{peeling}} + \underbrace{\eta_i^{+}}_{\textit{SLE}_6\ \text{noise}} .$$



Proposition

We have

- *The $P_i^{\pm}$ are i.i.d, bounded by 1, have zero expectation and $\mathbb{P}(P_i^{\pm} = -k) \sim C \cdot k^{-5/2}$.*
- *The $\eta_i^{\pm}$ are centered and have (almost) exponential tails.*

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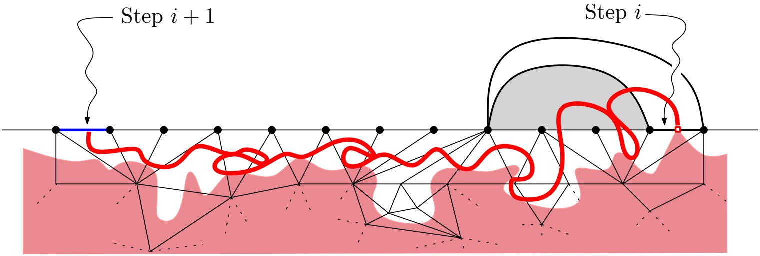

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Starring $(\star)$

Unfortunately the $\eta_i^{\pm}$ are not independent (though they decorrelate but we have no explicit rate). We introduce our working assumption

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Theorem ($\star$)

We have the following convergence in distribution

$$\left(\frac{\mathcal{H}^+(nt)}{n^{2/3}}, \frac{\mathcal{H}^-(nt)}{n^{2/3}} \right) \xrightarrow[n\to\infty]{(d)} 3^{-2/3} \cdot (S_t^+, S_t^-)_{t \geqslant 0}$$

in the Skorokhod sense where (S^+, S^-) is a pair of independent standard $\frac{3}{2}$-stable processes with no positive jumps.

3/2-stable processes with no positive jumps

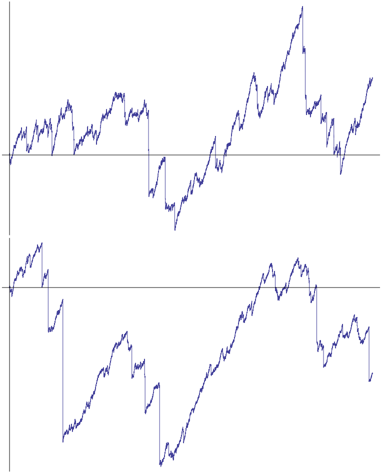
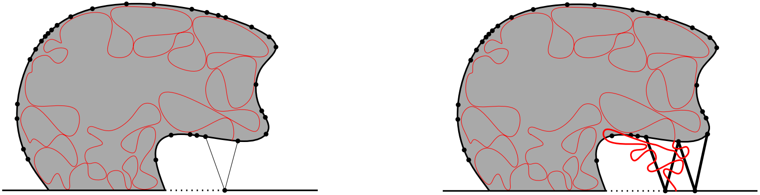


Figure : Two (approximated) samples of the process S.

Last ingredient : bouncing off the walls

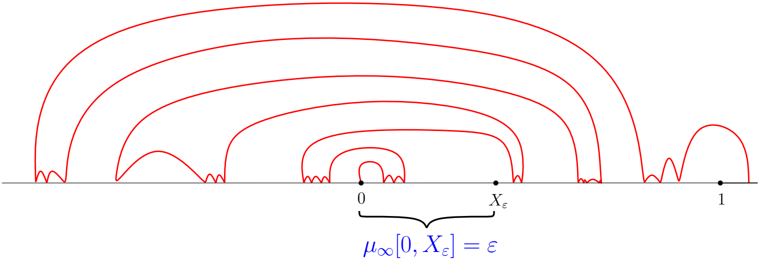
A time when the SLE_6 hits $\mathbb{R}_+$...



... approximately corresponds to a new minimum of the horodistance $\mathcal{H}^+$ and vice-versa.

Proof

Compute in two ways the number N_ε of commutings done by the SLE_6 between $\mathbb{R}_+$ and $\mathbb{R}_-$ after having swallowed mass ε :



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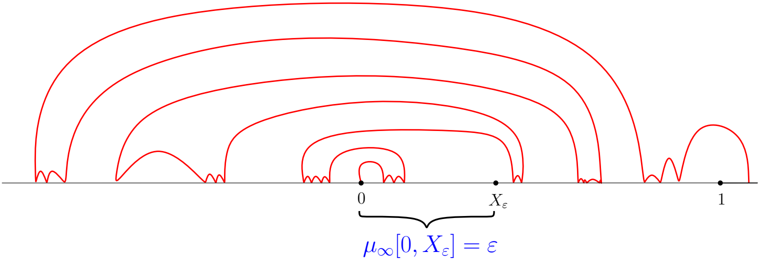

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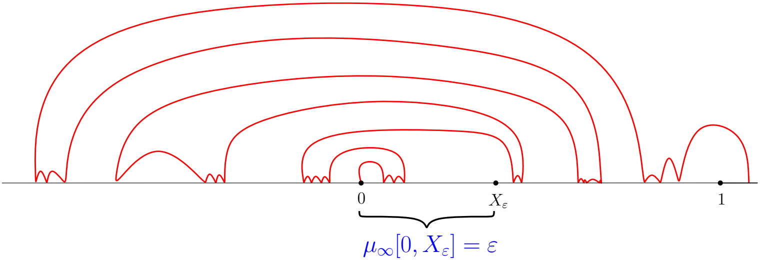

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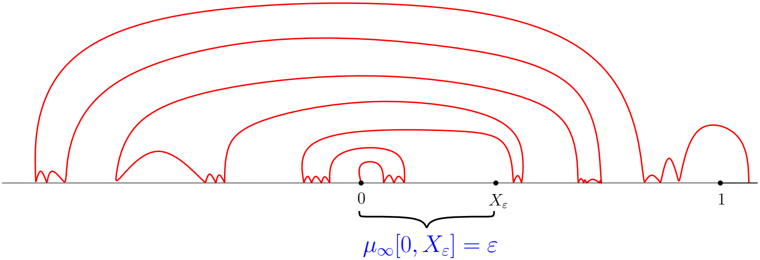

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$$X_\varepsilon \approx \varepsilon^{3+o(1)}.$$

Future works, directions

1. Remove *
2. Prove that the SLE_6 is the scaling limit of percolation interfaces
3. Go the the full-plane (using winding number instead of bouncing on the walls)
4. Does the law of the bouncings characterize the measure ? If yes, then KPZ and the convergence $\mu_n \to \mu_{GFF}$ would follow (since the $\frac{3}{2}$-stable process appears in the SLE_6 exploration of a GFF).
5. Otherwise, branching structure of SLE_6 exploration ?
6. What about planar maps decorated with “matter”?

...


Merci