The genealogy of self-similar fragmentations with negative index as a continuum random tree

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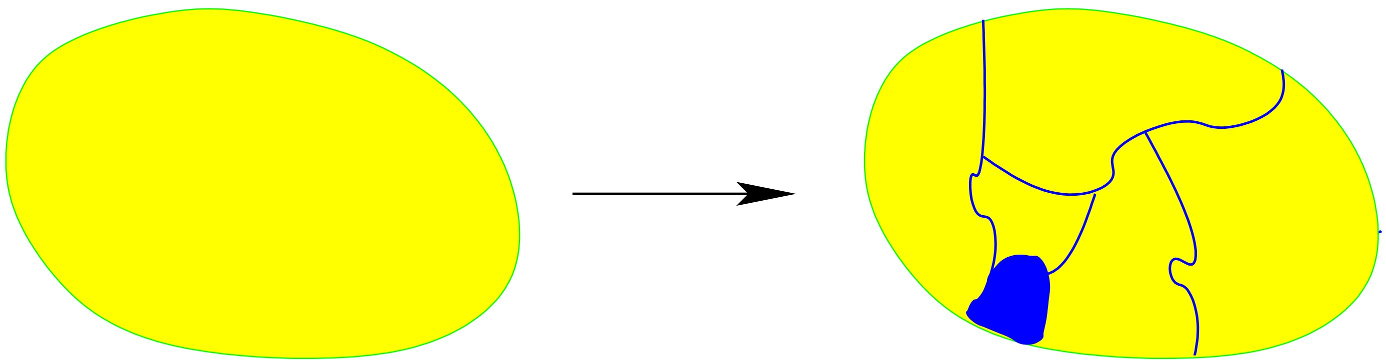

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This allows some loss of mass.

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- the fragments present at time t subsequently evolve independently of the others, in a way similar to that of the original mass 1 fragment, up to a space-time renormalization depending on the mass of each considered fragment.

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- For every $t, t' \geq 0$, given $F(t) = (x_1, x_2, \ldots)$, $F(t + t')$ has the same law as the decreasing rearrangement of the sequences

$$x_1 F^{(1)}({x_1}^{\alpha} t'), x_2 F^{(2)}({x_2}^{\alpha} t'), \ldots$$

where the $F^{(i)}$'s are independent copies of F.

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- Notice $\alpha < 0$, so fragments keep on collapsing faster and faster, implying some loss of mass (even extinction in finite time).

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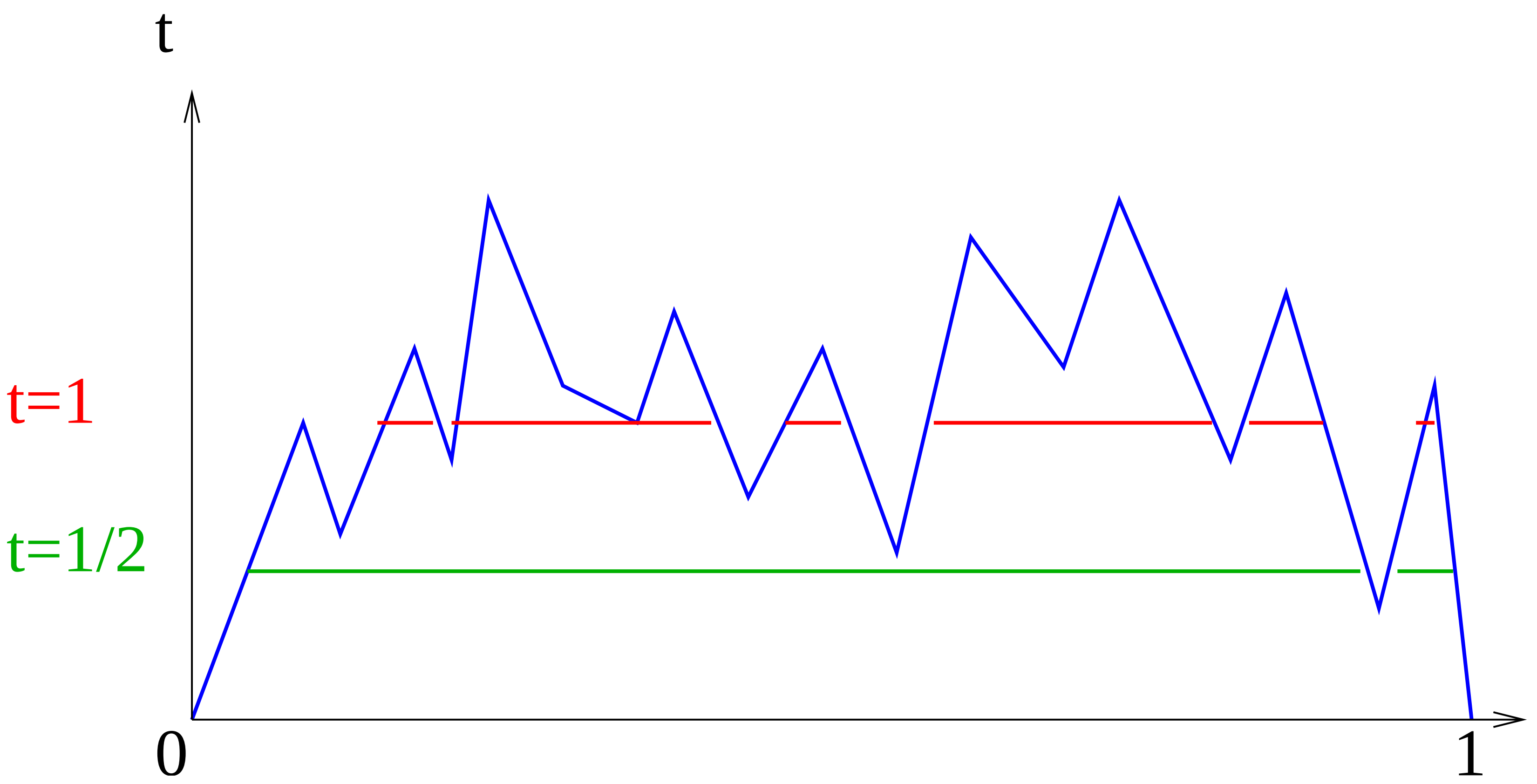

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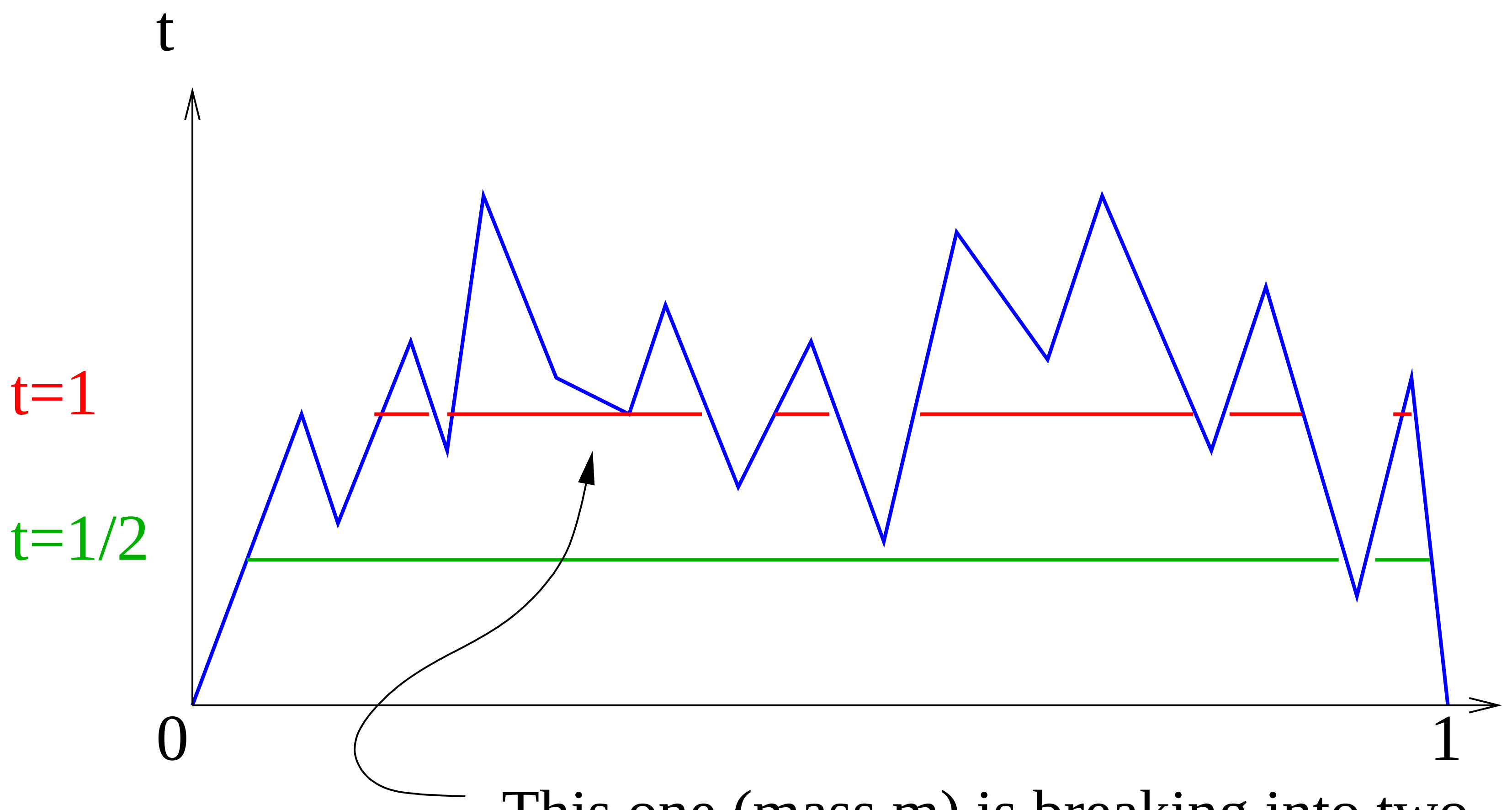

This one (mass m) is breaking into two fragments with sum of masses m.

Continuum Random Trees

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2. Any simple path going from v to w is $[[v, w]]$ (tree property).

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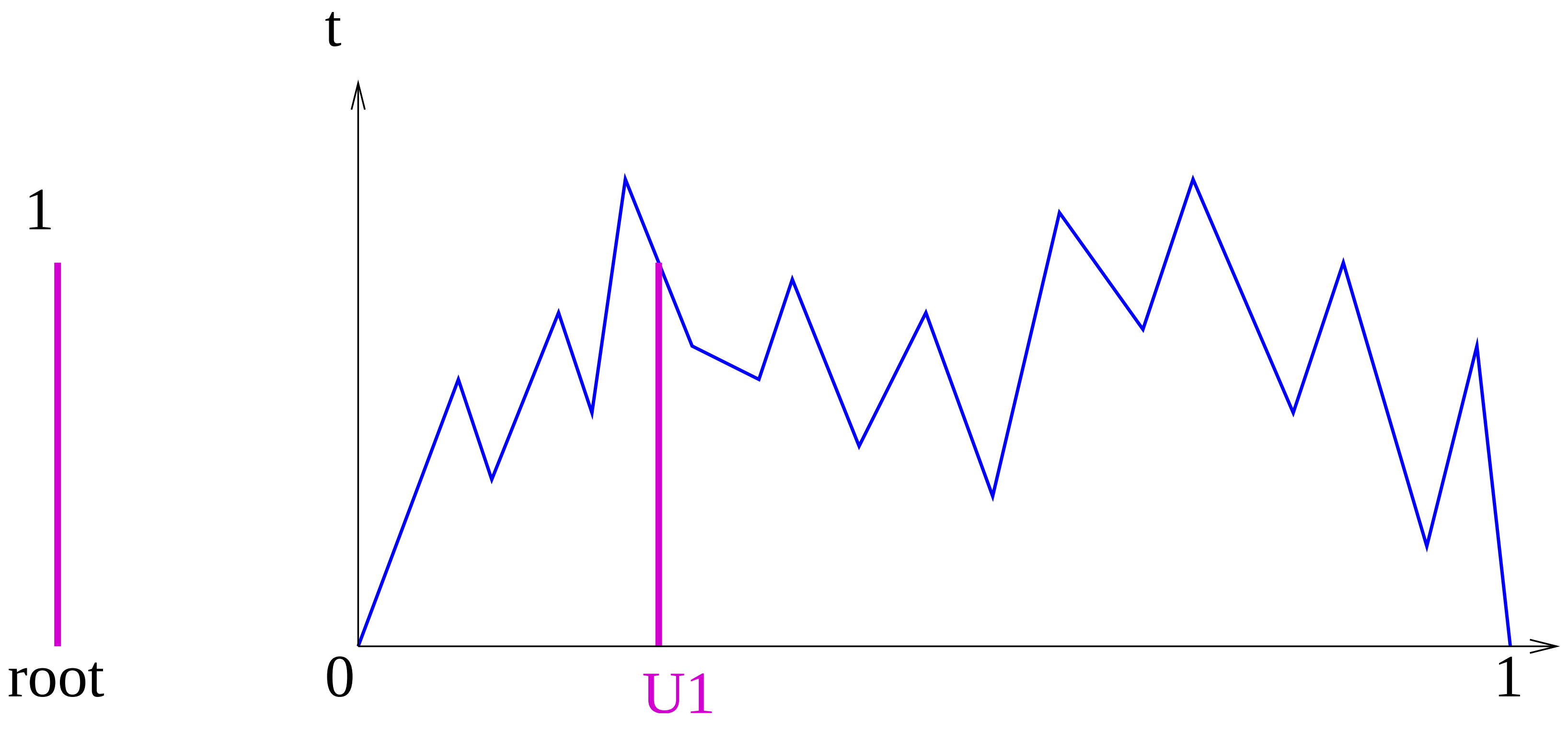

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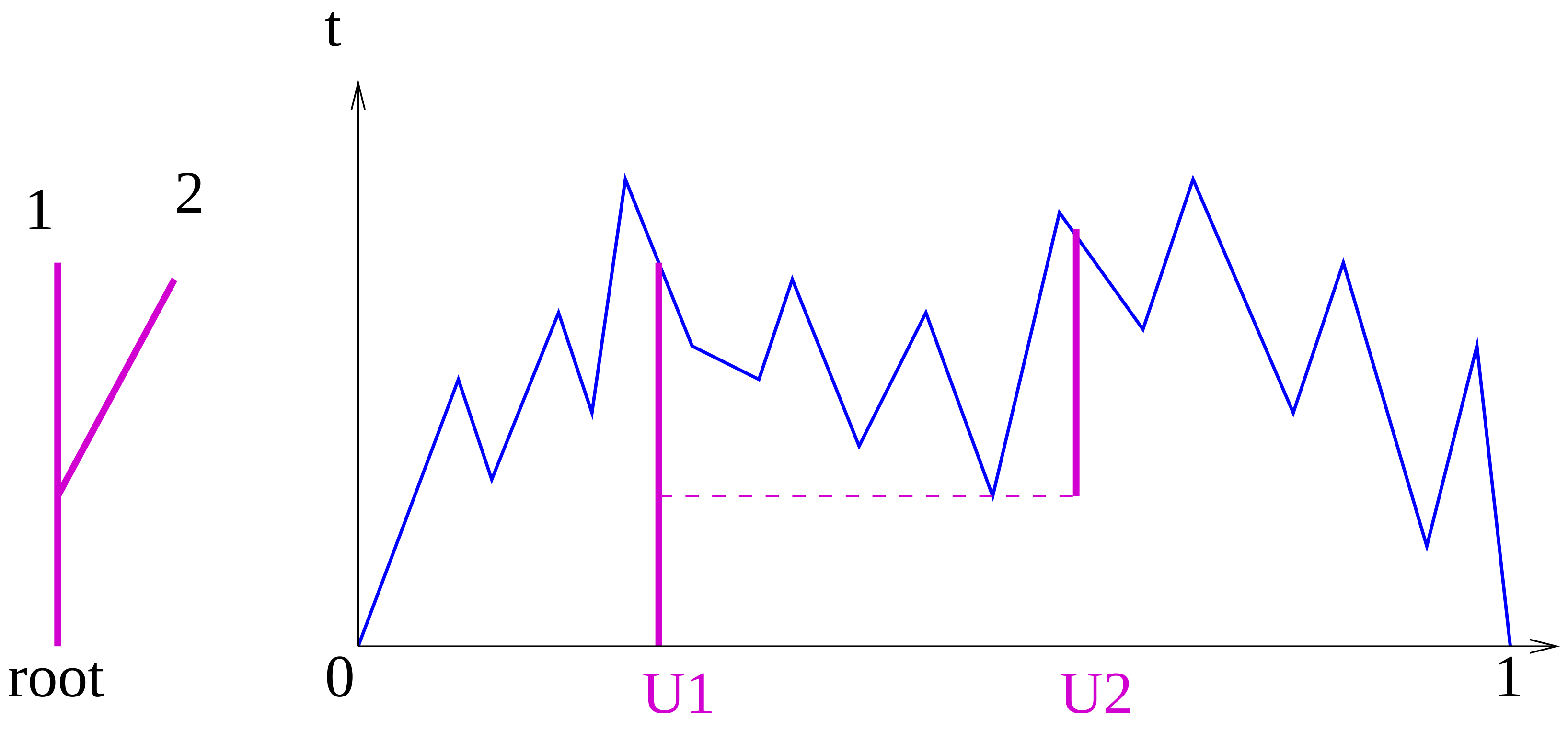

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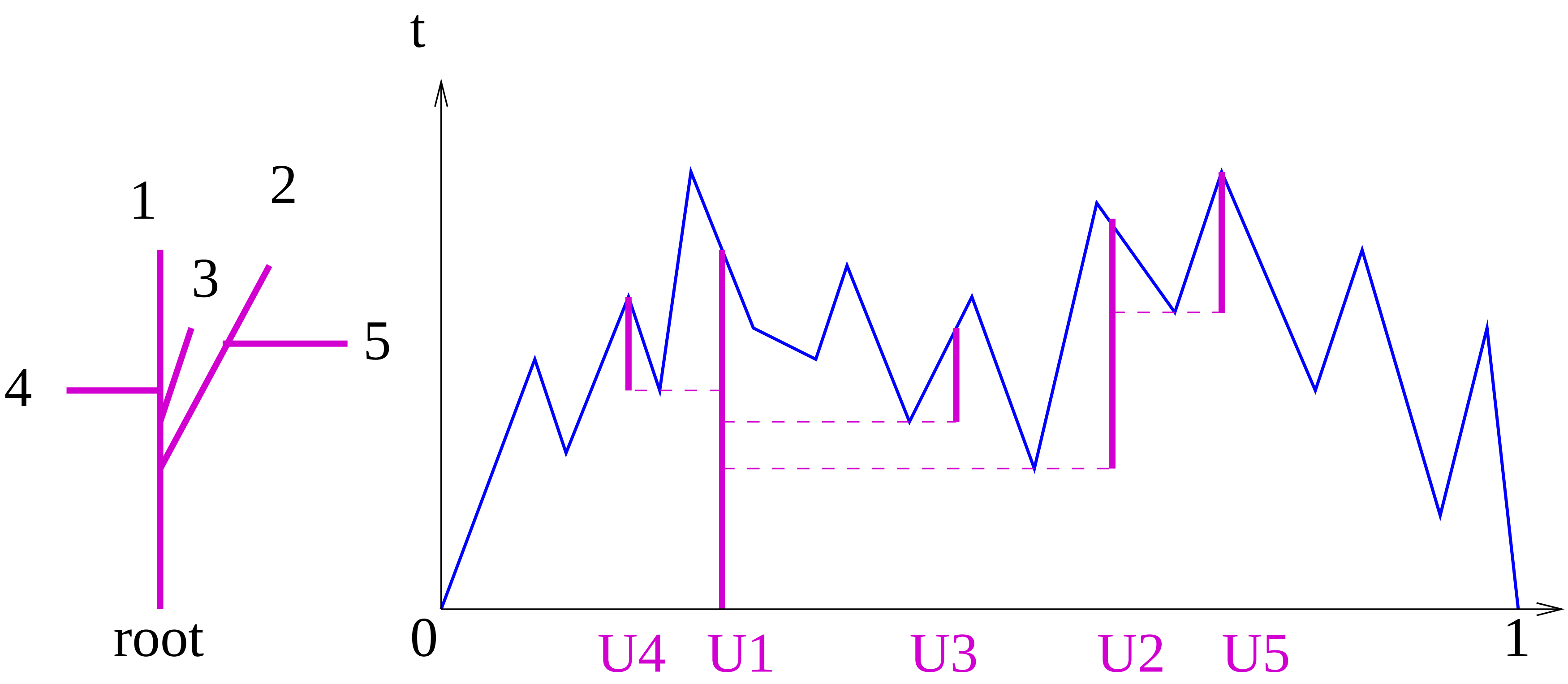

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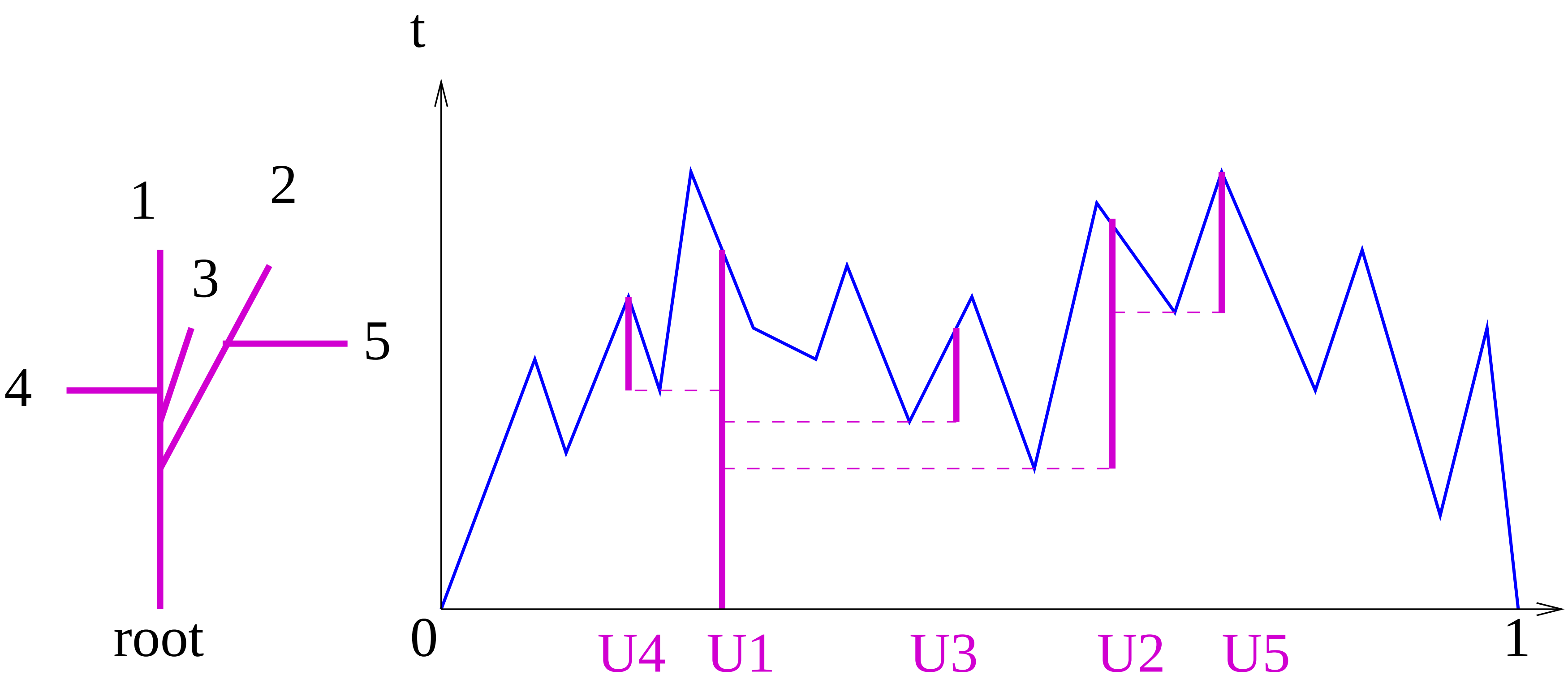

- And so on: the limiting "Brownian CRT" $= [0,1]$, with $D(s,s') = \mathbf{e}(s) + \mathbf{e}(s') - 2\inf_{s\wedge s' \leq u \leq s\vee s'} \mathbf{e}(u)$.

Example: the Brownian tree

- Therefore, the Brownian fragmentation F_{B} can be alternatively obtained from the Brownian CRT $\mathcal{T}_{\mathrm{B}}$, by recording in decreasing order the sizes of tree components of the forest $\{v \in \mathcal{T} : d(\mathrm{root}, v) > t\}$, where the “sizes” are the μ-masses of these components, μ= Lebesgue measure on $[0, 1]$.

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- Is ANY self-similar fragmentation F, which vanishes in finite time (i.e. $\alpha < 0$), of this form, for some kind of measured $\mathbb{R}$-tree $\mathcal{T}_F$?
  Answer: indeed.

Fragmentation trees

Theorem 1 (Haas & M., 2004) Let F be a s.s.f. with no erosion

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This means that every process $F_i(t), t \geq 0$, is pure-jump: the fragments cannot "melt continuously".

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To see this: discretize space.

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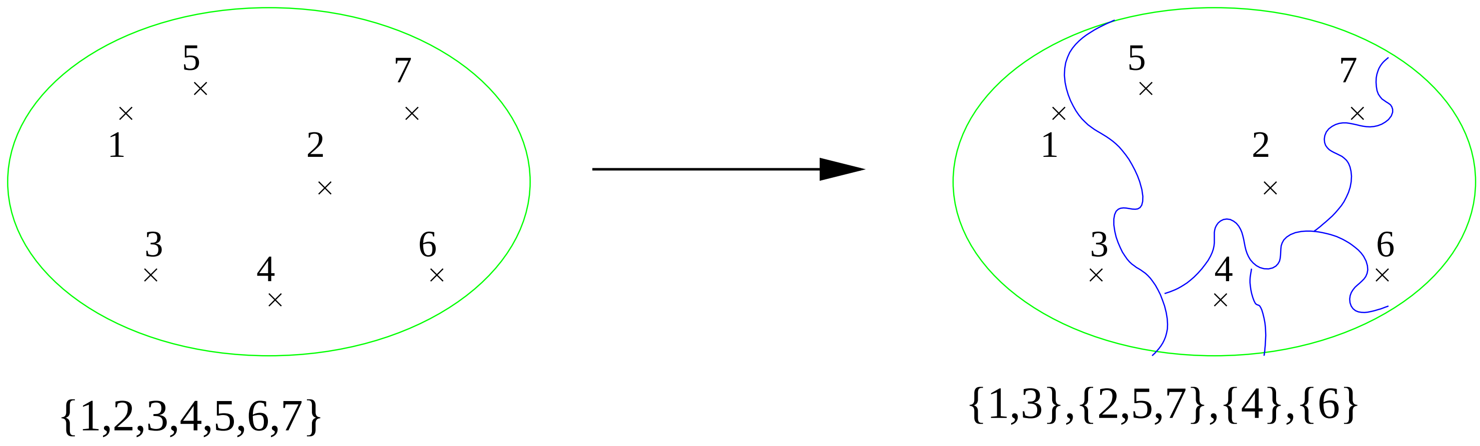

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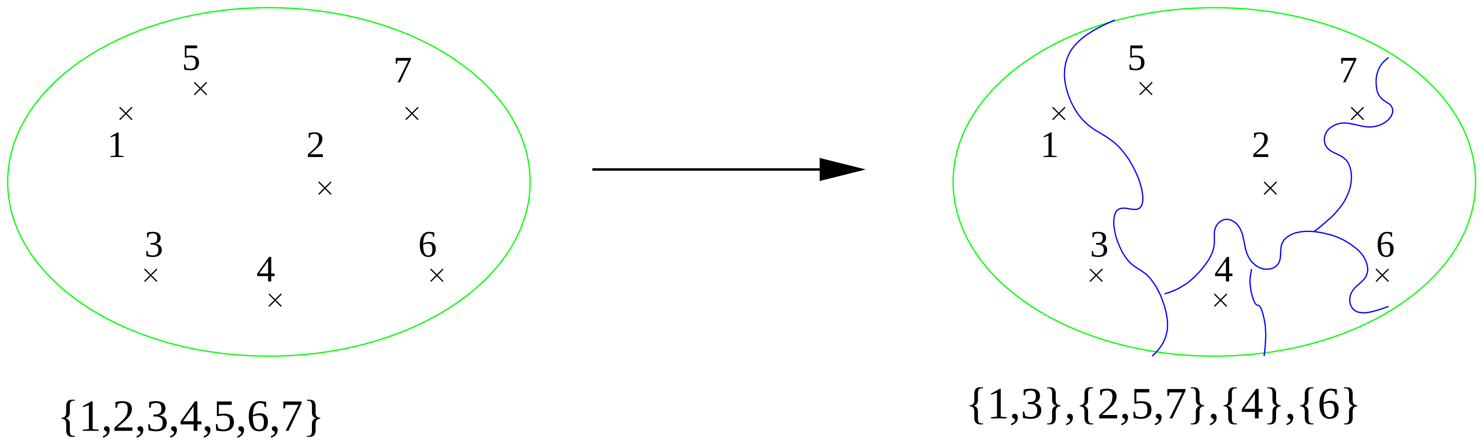

This gives a partition-valued fragmentation $\Pi(t), t \geq 0$.

A tree from a partition

- For $i \in \mathbb{N}$ let $D_i = \inf\{t \geq 0 : \{i\} \in \Pi(t)\}$ (death time, height of leaf i)
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  μ is the weak limit of the empirical measure on the first n leaves.

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- Notice that $\alpha \leq -1$ is thus qualitatively different from $-1 < \alpha < 0$.

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$$E[d(L_1, L_2)^{-\gamma}] \leq 2E[D^{-\gamma}]E[\lambda_1^{\alpha\gamma}; \lambda_1 < \lambda_2].$$

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- Therefore

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Rewrite $(\lambda_1, \lambda_2) = \lambda(D_{\{1,2\}}-)(l_1, l_2)$, where $\lambda(t)$ is the size of the fragment containing L_1, L_2 before separation ($t < D_{\{1,2\}}$).

Idea of proof

- Crucial tool: the dislocation measure of F: a σ-finite measure ν on S s.t. $\int_S (1-s_1)\nu(\mathrm{d}\mathbf{s}) < \infty$. Informally, a fragment with mass x breaks in fragments $x\mathbf{s}$ (with $\mathbf{s} \in S$) at rate $x^\alpha \nu(\mathrm{d}\mathbf{s})$.
- The knowledge of α and ν determines the law of a(n erosionless) F (Bertoin, 2002). It is the jump measure of F. If F has no sudden loss of mass then $\sum_i s_i = 1$, ν-a.e.

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Were it not for the x^α term (i.e. when $\alpha = 0$), the size $F^*(t)$ of the fragment containing L_1 would be $\exp(-\xi(t)), t \geq 0$ where ξ is a subordinator with the "size-biased" Lévy measure

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One gets F^* by a Lamperti time-change (Bertoin, 2002):

$$F^*(t) \stackrel{d}{=} \exp(\xi(\rho(t))) \quad t \geq 0$$

where $\rho(t) = \inf\{s \geq 0 : \int_0^s \mathrm{d}u \exp(-\alpha\xi(u))\}$, notice ρ explodes in finite time ($\alpha < 0$).

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Similarly, $\lambda(t)$ is a time-changed version of an exponential of subordinator with Lévy measure

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We are not done yet!

Indeed, the result is not always $< \infty$, it is in the case ν finite and there exists $N \geq 2$ with $\nu(\sum_{i=1}^{N} s_i < 1) = 0$ (at most N-ary tree).

The idea is then to truncate the tree to make it "look like" a fragmentation tree with the two properties above, and then make the truncation resemble the initial tree more and more.

Possible developments

- Finer analysis of “fragmentation trees”: level sets, local times?
- Case $\alpha \geq 0$ (ultrametrics rather than Brownian CRT’s-looking trees)?
- Extension $x^\alpha \to f(x)$ with regularity assumptions of f near 0 is easy.
- Encoding functions and their Hölder properties (partially done in the paper).

Thank you!