Random maps and continuum random 2-dimensional geometries

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Abstract. In the recent years, much progress has been made in the mathematical understanding of the scaling limit of random maps, making precise the sense in which random embedded graphs approach a model of continuum surface. In particular, it is now known that many natural models of random plane maps, for which the faces degrees remain small, admit a universal scaling limit, the *Brownian map*. Other models, favoring large faces, also admit a one-parameter family of scaling limits, called *stable maps*. The latter are believed to describe the asymptotic geometry of random maps carrying statistical physics models, as has now been established in some important cases (including the so-called rigid $O(n)$ model on quadrangulations).

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1. Introduction

A map is a finite graph that is properly embedded into a 2-dimensional oriented topological surface, and that dissects the latter into a collection of topological polygons, called the *faces* of the map. Two maps are equivalent (and henceforth are identified) if they can be put in correspondence via a direct homeomorphism between the underlying surfaces. When the underlying surface is the 2-dimensional sphere $\mathbb{S}^2$, we say that the map is *plane*.

There are many other equivalent definitions for maps, reflecting the central role they play in many different branches of mathematics, including graph theory (e.g. the 4-color theorem), combinatorics, representation theory, algebraic geometry, mathematical physics. The book [33] gives a very accessible introduction to a variety of topics featuring maps as a key concept.

Starting in the years 1980, it was recognized by theoretical physicists that maps could provide a useful tool in the theory of 2-dimensional quantum gravity. See for instance the survey [24] or the book [3]. A basic object in this theory is a partition function defined as the integral of a certain action functional over the space of all Riemannian metrics on a 2-dimensional surface, considered up to diffeomorphisms. This integral, which can be seen as a 2-dimensional analog of path integrals, is a problematic and ill-defined object on a mathematical point of view. It was therefore suggested that the integral could be approximated by a (finite) sum over maps with a fixed, but large size. Here the family of maps over which one takes a sum (and the notion of size), is not specified: It is expected that any "reasonable" choice for such a family will provide an acceptable approximation of the same "universal" object. Keeping in mind the fact that path integrals can be formulated in terms of Brownian motion, and that the latter is (by Donsker's theorem) the scaling limit of any random walk with centered, independent increments having a finite variance, this last assertion does not seem unreasonable.

From a probabilistic point of view, the above questions can be formulated as follows: Taking a map at random in a certain collection, and letting its size go to infinity, can one approximate a continuum random 2-dimensional geometry?

Let us be more specific. We say that a map is *rooted* if one of the edges is distinguished and given an orientation. We are only going to consider rooted maps in the sequel, which is only a matter of mathematical convenience: All results discussed below are believed to hold also for non-rooted maps. Let $\mathbf{Q}_n$ be the set of rooted plane *quadrangulations* with n faces, meaning that all faces are bounded by 4 edges — here, an edge that lies entirely in one face should be counted twice. Since equivalent maps are identified, the set $\mathbf{Q}_n$ is a finite set, and in fact [55]

$$\mathrm{Card}(\mathbf{Q}_n) = \frac{2}{n+2}\frac{3^n}{n+1}\binom{2n}{n}. \tag{1}$$

Let Q_n be a random variable that is uniformly distributed in $\mathbf{Q}_n$. We can view Q_n as a metric space by naturally endowing the set $V(Q_n)$ of its vertices with the graph distance d_{Q_n}: For $u, v \in V(Q_n)$, $d_{Q_n}(u, v)$ is the minimal number of edges needed to go from u to v in Q_n. We want to understand the geometry of the space $(V(Q_n), d_{Q_n})$ as $n \to \infty$.

At this point, we choose to consider properly renormalized versions of these spaces as $n \to \infty$. Without a renormalization, these spaces become unbounded, and their discrete limits, called *local limits*, describe so to speak a random infinite lattice, the Uniform Infinite Planar Quadrangulation, which was introduced (for the slightly different case of triangulations) by Angel and Schramm, see [6, 5, 32, 20, 44]. The theory of local limits is very rich, but to obtain such limits, it is not necessary to have a very detailed understanding of the distance d_{Q_n}. Rather, we want to consider *scaling limits* and obtain bounded, continuum limiting objects.

In order to address such problems, a natural approach is to try to count quadrangulations with vertices at certain prescribed distances. The enumerative theory of maps started with the "census" works of Tutte [55], who established (1) alongside many other similar results for other families of maps. A striking connection between map enumeration and matrix integrals was established in the years 1970 [54, 17] and spawned a huge literature, see [24] for a survey. However, these approaches do not seem to allow to keep track of the extra information of graph distances. Despite a spectacular semi-rigorous computation by Ambjørn and Watabiki [4, 3] for the two-point function of triangulations, there was no clear way to to attack the problem from a mathematical angle.

Yet another, more direct approach to the enumeration of maps had been noted in 1981 by Cori and Vauquelin [22]. Motivated by the simple form of formulas such as (1), they were able to provide a bijection between rooted maps and a family of trees, called *well-labeled.* Despite some notable exceptions such as [7], these techniques were mostly put to sleep until they reached their full potential starting with the PhD thesis of Schaeffer [51], who developed a more systematic study of bijective enumeration techniques for maps. A key feature of the Cori-Vauquelin-Schaeffer bijections is that the labels of a well-labeled tree allow to keep track of certain graph distances in the associated map. Based on this observation and on the fact that random labeled trees are a relatively common object in the probability literature, Chassaing and Schaeffer [21] were able to derive rigorously the limit distribution of the *radius* of Q_n, as well as other interesting functionals. Specifically, they proved in that if u_* is the origin of the root edge of Q_n, then

$$\left(\frac{9}{8n}\right)^{1/4} \max_{u\in V(Q_n)} d_{Q_n}(u,u_*) \underset{n\to\infty}{\longrightarrow} \Delta \tag{2}$$

in distribution, where Δ is a random variable that can be defined in terms of Le Gall's Brownian snake [34] (or Aldous' Integrated SuperBrownian Excursion [1]) — here, the normalization $(8/9)^{1/4}$ is a matter of convention. Using generalizations of the Cori-Vauquelin-Schaeffer bijections by Bouttier, Di Francesco and Guitter [14], it was shown in [42, 56, 45, 49] that (2) holds for models of plane maps that are far more general than uniform quadrangulations, with the same random variable Δ arising in the limit and the same normalization exponent $n^{1/4}$, but with possibly different scaling constants. These models include uniform p-angulations for any $p \geq 3$, i.e. uniform maps with n faces all of degree p.

The work by Chassaing and Schaeffer suggested that the whole renormalized metric space $(V(Q_n),(9/8n)^{1/4}d_{Q_n})$ should approach a limiting metric space as $n\to\infty$ in some sense. Such a result was obtained by Marckert and Mokkadem [43], but for an *ad hoc* topology that does not take fully into account the metric structure. However, the limit object that they introduced is a well-defined random metric space, that they called the *Brownian map*. A series of papers by Le Gall [35, 36] and by Le Gall and Paulin [41] set important milestones in the theory. In particular, [35] shows that the laws of the random metric spaces $((V(Q_n),(9/8n)^{1/4}d_{Q_n}),n\geq 1)$ form a relatively compact family of probability laws on metric spaces, endowed with the Gromov-Hausdorff topology (see below). This shows that these spaces converge, at least along proper extractions, to a limiting space (S,D). Further, Le Gall showed that the topology of *any such* limit is a.s. the same as Marckert-Mokkadem's Brownian map, and [41] showed that the latter is none other than that of the 2-dimensional sphere (see also [46] for an alternative approach). Results on uniqueness of typical geodesics in the subsequential limits of random quadrangulations were obtained in [36, 47]. We also mention that Bouttier and Guitter [16] derived the 3-point function of quadrangulations, i.e. the limit law of $n^{-1/4}(d_{Q_n}(u_1,u_2),d_{Q_n}(u_1,u_3),d_{Q_n}(u_2,u_3))$, where u_1,u_2,u_3 are three random points chosen uniformly and independently in $V(Q_n)$. Other references and surveys on the topic include [8, 10, 9, 15, 19, 23, 40].

These works left open the question of the uniqueness of the limiting distribution of $(V(Q_n),(9/8n)^{1/4}d_{Q_n})$, and this problem was only solved recently in two independent works by Le Gall and the author [37, 48], where it is proved that the limit is indeed the random metric space introduced by Marckert and Mokkadem.

We postpone an exact statement to Section 2.3. We also mention that Le Gall obtains universality results in the same context as in [42], as well as for uniform random triangulations. The "uniqueness of the Brownian map" somehow justifies the initial statement of physicists that random maps approximate a continuum random surface.

The story is far from ending here, though. It is noted in [39] that other natural models of random maps admit scaling limits that are different from the Brownian map, when one allows models where the degrees of faces are large. These limits, the "stable maps", which form a one-parameter family of mutually singular random spaces, are not topological spheres anymore, but should rather be thought of as random fractal carpets. One initial motivation for this work was that maps with large faces were believed to describe the interfaces of statistical physics models on random maps. This has been partially established in two recent works by Borot, Bouttier and Guitter [12, 11], for the so-called $O(n)$ model on quadrangulations and certain variants, giving a renewed viewpoint on these well-studied models [29, 30, 13].

Let us conclude this introduction by stressing that there is another, purely continuum approach to quantum gravity, the so-called Liouville quantum gravity. The mathematical grounds for this theory are starting to emerge after the work by Duplantier and Sheffield [26]. This theory crucially involves conformal invariance, as opposed to the random maps approach, yet it is believed that the two approaches have very deep connections.

The rest of the paper is organized as follows. In Section 2.1 we introduce the Cori-Vauquelin-Schaeffer bijection, and use it to motivate the definition of the Brownian map. We introduce the latter in section 2.3, as well as the main convergence result. Then in Section 3 we briefly introduce the model of stable maps, and motivate its connection to the $O(n)$ model on quadrangulations. Section 4 gives some final remarks.

2. Convergence to the Brownian map

In this section, we explain how to construct plane quadrangulations from labeled trees. This will motivate the definition of the Brownian map in the next section.

2.1. The Cori-Vauquelin-Schaeffer bijection. A *plane tree* is a plane map with a single face, or equivalently, a finite tree that is embedded in the sphere. If the tree, considered as a map, is rooted, then the root vertex is by definition the origin of the root-edge.

A *well-labeled tree* is a pair $(\mathbf{t}, \mathbf{l})$, where $\mathbf{t}$ is a rooted plane tree, and $\mathbf{l} : V(\mathbf{t}) \to \mathbb{Z}$ is a labeling function on the set of vertices of $\mathbf{t}$, that satisfies the following properties:

- $\mathbf{l}(\text{root}) = 0$
- $|\mathbf{l}(u) - \mathbf{l}(v)| \leq 1$ if u, v are adjacent vertices in $\mathbf{t}$.

It is well-known that there are $\frac{1}{n+1}\binom{2n}{n}$ rooted plane trees with n edges, and it follows that there are $\frac{3^n}{n+1}\binom{2n}{n}$ well-labeled trees with n edges. We let $\mathbb{T}_n$ be the set of such well-labeled trees.

Let $(\mathbf{t}, \mathbf{l})$ be a fixed well-labeled tree with n edges. As we go around the tree in cyclic order, one encounters exactly once each of the $2n$ corners incident to the vertices of the tree. For every such corner c, we draw an arc from c to its *successor*, which is the first corner $s(c)$ coming after c in cyclic order around $\mathbf{t}$, whose label is strictly smaller than the label of the vertex incident to c. If there is no such corner, then we draw an arc from c to an extra vertex named v_*. The vertex v_* should not belong to the support of the embedding of $\mathbf{t}$, and the arcs can be drawn in such a way that they do not cross, and do not intersect the support of $\mathbf{t}$ (except of course at its vertices). Finally, consider the embedded graph $\mathbf{q}$ whose edge-set is the set of arcs, and whose vertex-set is $V(\mathbf{q}) = V(\mathbf{t}) \cup \{v_*\}$. It is easy to see that $\mathbf{q}$ is a map. An edge of this map is naturally distinguished: It is the arc going from the corner incident to the root-edge of $\mathbf{t}$ to its successor. This edge can be given two orientations, yielding two different rooted maps. The choice of the orientation can be specified using a parameter $\epsilon \in \{-1, 1\}$. The map $\mathbf{q}$ is also naturally *pointed*, in the sense that it has a distinguished vertex v_*. We let $\mathbf{Q}_n^\bullet$ be the set of pointed, rooted plane quadrangulations with n faces.

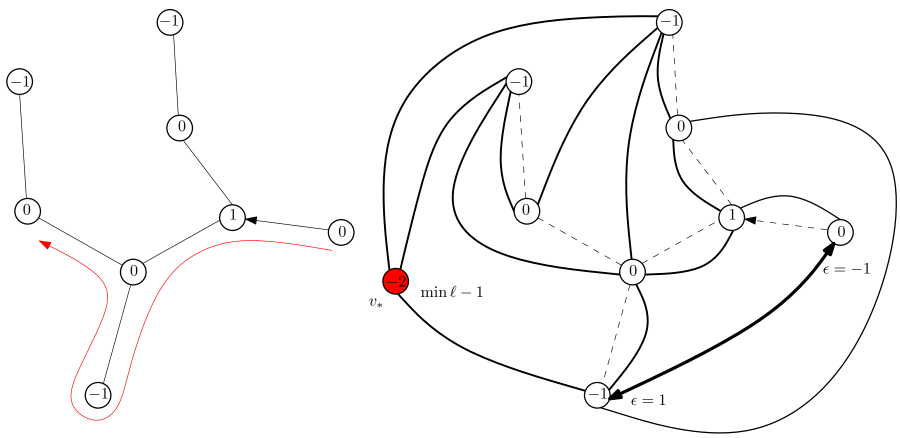

Figure 1. Illustration of the Cori-Vauquelin-Schaeffer bijection

Theorem 2.1 ([22, 21]). *The map $\mathbf{q}$ is a plane quadrangulation with n faces, and the mapping $((\mathbf{t}, \mathbf{l}), \epsilon) \mapsto \mathbf{q}$ defined above is a bijection between $\mathbb{T}_n \times \{-1, 1\}$ and $\mathbf{Q}_n^\bullet$. Moreover, if $v \in V(\mathbf{t}) = V(\mathbf{q}) \setminus \{v_*\}$, then the graph distance in $\mathbf{q}$ from v to the distinguished vertex v_* is given by the formula*

$$d_{\mathbf{q}}(v, v_*) = \mathbf{l}(v) - \min_{u \in V(\mathbf{t})} \mathbf{l}(u) + 1\,. \tag{3}$$

Let us draw some immediate consequences of this theorem. For every $\mathbf{q} \in \mathbf{Q}_n$, it holds that $\mathrm{Card}(V(\mathbf{q})) = n + 2$, which is a simple consequence of the Euler formula. Each choice of a vertex in $\mathbf{q}$ yields a distinct element of $\mathbf{Q}_n^\bullet$, so that $\mathrm{Card}(\mathbf{Q}_n^\bullet) = (n+2)\mathrm{Card}(\mathbf{Q}_n)$. Together with the previous theorem and the formula we derived earlier for $\mathrm{Card}(\mathbb{T}_n)$, this entails the counting formula (1).

A second consequence is that if (T_n, L_n) is a uniform random element in $\mathbb{T}_n$, then the pointed rooted quadrangulation (Q_n, v_*) associated with it (choosing the orientation of the root uniformly at random) is such that Q_n is uniform over $\mathbf{Q}_n$, and v_* is uniform over $V(Q_n)$ conditionally given Q_n. Hence, we see that (3) implies roughly that through the Cori-Vauquelin-Schaeffer bijection, the label function $(L_n(v), v \in V(T_n))$ describe graph distances to a uniformly chosen vertex, in a uniformly chosen random quadrangulation with n edges.

2.2. Scaling limits of well-labeled trees. In order to derive the large n behavior of Q_n, we are thus led to understand that of the random labeled tree (T_n, L_n). A convenient way to do this is by describing the latter using the so-called contour and label processes. Let $u_0, u_1, \dots, u_{2n-1}, u_{2n} = u_0$ be the sequence of vertices of T_n visited in contour order, starting from the root corner (some vertices are visited more than once), so in particular, u_0 is the root vertex. Let $C_n(i)$ be the graph distance in T_n from u_0 to u_i, and let $L_n(i) = L_n(u_i)$, with a slight abuse of notation. Both C_n, L_n are extended to continuous processes from $[0, 2n]$ to $\mathbb{R}$ by linear interpolation between integer times.

Recall that the *normalized Brownian excursion* $\mathbf{e} = (\mathbf{e}_t, 0 \le t \le 1)$ is a random continuous process, which is so to speak an excursion of Brownian motion away from 0, conditioned to have total duration 1. For $s, t \in [0,1]$, let

$$d_{\mathbf{e}}(s,t) = \mathbf{e}_s + \mathbf{e}_t - 2 \inf_{s\wedge t \le u \le s \vee t} \mathbf{e}_u \,, \qquad 0 \le s, t \le 1 \,.$$

This quantity, denoted by $d_{\mathbf{e}}(s,t)$, defines a pseudo-distance on $[0,1]$, and the quotient metric space $\mathcal{T}_{\mathbf{e}} = [0,1]/\{d_{\mathbf{e}} = 0\}$ is an important probabilistic object, the *Continuum Random Tree* [2]. It is a random $\mathbb{R}$-tree, see [28].

Conditionally given $\mathbf{e}$, we define a *label process* $Z = (Z_t, 0 \le t \le 1)$ as a continuous centered Gaussian process satisfying $Z_0 = 0$ and

$$E[|Z_s - Z_t|^2 \,|\, \mathbf{e}] = d_{\mathbf{e}}(s,t) \,, \qquad 0 \le s, t \le 1 \,.$$

It is easy to see that Z is a class function for the relation $\{d_{\mathbf{e}} = 0\}$, and hence that it induces a random function on $\mathcal{T}_{\mathbf{e}}$, which we still denote by Z. Informally, Z should be understood as a Brownian motion indexed by the tree $\mathcal{T}_{\mathbf{e}}$.

The pair $(\mathcal{T}_{\mathbf{e}}, Z)$ is the continuum counterpart of the tree (T_n, L_n), which can be formalized in the following statement.

Theorem 2.2. *As $n \to \infty$, the following convergence in distribution holds in the space $\mathcal{C}([0,1], \mathbb{R})^2$ (endowed with the uniform norm):*

$$\left(\left(\frac{C_n(2nt)}{\sqrt{2n}} \right)_{0 \le t \le 1}, \left(\left(\frac{9}{8n} \right)^{1/4} L_n(2nt) \right)_{0 \le t \le 1} \right) \longrightarrow (\mathbf{e}, Z) \,.$$

The renormalization that appear in this statement is relatively transparent: The diffusive rescaling $\sqrt{n}$ of the first component comes from the fact that C_n is very similar to a simple random walk (it is in fact conditioned to be positive and to be back to the origin at time $2n$), while the $n^{1/4}$ rescaling of the second component comes from the fact that L_n describes a family of centered random walks indexed by the branches of the tree, which have lengths of order $\sqrt{n}$. The exact scaling constants come from applications of the central limit theorem.

2.3. The Brownian map. Starting from the continuum labeled tree $(\mathcal{T}_{\mathbf{e}}, Z)$, one can try to define a continuous analog of the Cori-Vauquelin-Schaeffer bijection. Due to the fact that we want to rescale distances in Q_n, the arcs involved in the construction of Q_n from (T_n, L_n) should become smaller and smaller, and in the limit they correspond to certain identifications of points in $\mathcal{T}_{\mathbf{e}}$.

Similarly to $d_{\mathbf{e}}$, one can define a pseudo-distance d_Z on $[0,1]$ by the formula

$$d_Z(s,t) = Z_s + Z_t - 2\max\Big(\inf_{u\in I(s,t)} Z_u, \inf_{u\in I(t,s)} Z_u\Big),$$

where $I(s,t) = [s,t]$ if $s \le t$, and $I(s,t) = [s,1]\cup[0,t]$ if $t < s$: This is the circular arc from s to t if $[0,1]$ is seen as a circle by identifying 0 with 1. In turn the quotient $([0,1]/\{d_Z = 0\}, d_Z)$ is a random real tree.

Following [43, 35], the Brownian map is the metric space obtained by quotienting the pseudo-metric space $([0,1], d_Z)$ with respect to the two equivalence relations $\{d_{\mathbf{e}} = 0\}$ and $\{d_Z = 0\}$. Formally, define a pseudo-metric on $[0,1]$ by letting

$$D^*(s,t) = \inf\Big\{\sum_{i=1}^k d_Z(s_i,t_i) : k \ge 1, s = s_1, t = t_k, d_{\mathbf{e}}(t_i, s_{i+1}) = 0, 1 \le i \le k-1\Big\},$$

and let $S = [0,1]/\{D^* = 0\}$.

Definition 2.3. The Brownian map is the random metric space (S, D^*).

We can now state the main convergence result of [37, 48]. Recall that the Gromov-Hausdorff distance between two compact metric spaces is the infimum Hausdorff distance between isometric embeddings of these two spaces in a common metric space [18]. In order to give a more complete picture, we also include results from [35, 41, 46] on the Hausdorff dimension and topology of the limiting metric space at the end of the following statement.

Theorem 2.4. *We have the following convergence in distribution in the Gromov-Hausdorff topology:*

$$\Big(V(Q_n), \Big(\frac{9}{8n}\Big)^{1/4} d_{Q_n}\Big) \underset{n\to\infty}{\longrightarrow} (S, D^*)\,.$$

The Brownian map is a random geodesic metric space which is almost surely homeomorphic to the 2-dimensional sphere, and has Hausdorff dimension 4.

Let us give some intuition on this result and elements of the proof. The existence of limits of $(V(Q_n), (9/8n)^{1/4} d_{Q_n})$ along subsequences can be obtained as a consequence of Theorem 2.2 and Gromov's compactness theorem [18]. Then, from the description of Q_n in terms of (T_n, L_n), it is not too difficult to see that any subsequential limit should be described as a pseudo metric D on $[0,1]$, satisfying the key properties, for $s, t \in [0,1]$:

$$D(s, s_*) = Z_s - \inf Z\,, \qquad D(s,t) \leq d_Z(s,t)\,, \qquad D(s,t) = 0 \text{ if } d_{\mathbf{e}}(s,t) = 0\,,$$

where s_* is the point of $[0,1]$ at which the process Z attains its overall minimum. The first formula is a continuum analog of (3), and the second can be easily obtained from the discrete picture by building explicitly a path with length $d_Z(s,t)$ from s to t in $([0,1], D)$, by gluing two pieces of geodesic paths from s, t towards s_*. Such geodesics are obtained as continuum analogs of the chain from a given corner to its consecutive successors until reaching v_*. In the continuum, they correspond to negative records of the process Z from s to s_*. Finally, the last constraint is obvious from the discrete picture: It just says that two corners incident to the same vertex of T_n also correspond to a single vertex in Q_n.

It can then be checked that D^* is the maximal pseudo-distance on $[0,1]$ that satisfies the three constraints above. In particular, it always holds that $D \leq D^*$, and the uniqueness property of the Brownian map boils down to showing that $D^* \leq D$. To this end, one must show that in the metric space $([0,1], D)$, any given points $s, t \in [0,1]$ can be joined by a path made of pieces of geodesic paths pointing towards s_*, whose total length can be made arbitrarily close to $D(s,t)$. This property can look surprising (it is certainly wrong in Euclidean geometry), but it turns out to be true in our situation. More precisely, if γ is a geodesic path from s to t in $([0,1], D)$, it so happens that for almost every point u on γ, a geodesic from u to s_* intersects γ along a non-trivial segment, so that geodesics in $([0,1], D)$ tend to stick together (a property related to the coalescence of geodesics studied in [36]). Most of the work in [48] is to show that the "bad" set Γ of exceptional points u on γ from which a geodesic to s_* does not re-intersect γ is small, in the sense that its box-counting dimension is strictly bounded by 1. See Figure 2. This is proved by essentially counting arguments based on a bijection developed in [47], which is a generalization of the Cori-Vauquelin Schaeffer bijection taking into account several distinguished vertices rather than one. This bijection allows to estimate the probability of certain star-shaped configurations of geodesics, which can be related to the event that a uniformly random point being close to Γ.

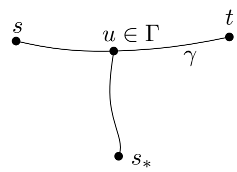

Figure 2. Illustration of a bad point $u \in \Gamma$: The geodesic from u to s_* branches away from the geodesic from s to t immediately.

3. Boltzmann maps and $O(n)$ models

A natural model of random maps, generalizing the model of uniform quadrangulations considered so far, consists in fixing a family of non-negative local weights and choosing a map with probability proportional to the product of these local weights indexed by the faces of the map. Here we are going to focus only on bipartite maps, where all the faces have even degree, as it is a technically simpler situation.

3.1. Boltzmann maps. We fix a family $w = (w_1, w_2, \ldots)$ of non-negative real numbers, and assume that $w_i > 0$ for some $i > 1$. By convention, fix $w_0 = 1$. Let $\mathbf{M}$ be the set of rooted bipartite plane maps, and $\mathbf{M}_n$ the subset of such maps with n vertices. It is assumed that $\mathbf{M}_1$ contains a single element, the vertex map with one vertex, one face and no edge.

For every $\mathbf{m} \in \mathbf{M}$, let $F(\mathbf{m})$ be the set of faces of $\mathbf{m}$ and $\deg(f)$ be the degree of an element $f \in F(\mathbf{m})$. Then, let

$$W_w(\mathbf{m}) = \prod_{f \in F(\mathbf{m})} w_{\deg(f)/2} \,.$$

This defines a σ-finite, non-negative measure on $\mathbf{M}$, with total mass $Z_w = W_w(\mathbf{M}) \in (0, \infty]$. If it is finite, then we can define a *Boltzmann probability distribution* by letting $P_w = W_w/Z_w$. For technical reasons, it is useful to require slightly more than the finiteness of Z_w, so we say that w is *admissible* if

$$Z_w^\bullet = W_w(\mathrm{Card}(V(\cdot))) = \sum_{\mathbf{m} \in \mathbf{M}} \mathrm{Card}(V(\mathbf{m})) W_w(\mathbf{m}) < \infty \,.$$

It turns out that some of the key features of Boltzmann measures can be obtained in terms of the function

$$f_w(x) = \sum_{k \geq 0} \binom{2k+1}{k} w_{k+1} x^k \,, \qquad x \geq 0 \,. \tag{4}$$

For instance, it is shown in [42] that w is admissible if and only if the equation

$$f_w(x) = 1 - \frac{1}{x}$$

admits a solution in $(1, \infty)$. In this case, the smallest such solution z_w^+ is equal to $(Z_w^\bullet + 1)/2$.

The interesting situation occurs when the two graphs of the functions f_w and $x \mapsto 1 - x^{-1}$ are tangent at z_w^+, which is then necessarily the unique solution. One says that w is *critical.*

3.2. Regular critical maps. Moreover, w is said to be *regular critical* if it is critical, and if the radius of convergence R_w of f_w satisfies $R_w > z_w^+$.

Regular critical random maps behave like uniform quadrangulations in the scaling limit, as the following result by Le Gall shows, generalizing Theorem 2.4.

Theorem 3.1 ([37]). *Let w be a regular critical sequence, and let M_n be a random element of $\mathbf{M}_n$ with distribution $P_w(\cdot \mid \mathbf{M}_n)$. Then there is a constant $C_w \in (0,\infty)$ such that $(V(M_n), C_w n^{-1/4} d_{M_n})$ converges as $n \to \infty$ to the Brownian map (S, D^*), in distribution for the Gromov-Hausdorff topology.*

Partial results toward this theorem had been obtained in [42], which extends the Chassaing-Schaeffer results of [21], such as the convergence (2). See also [45, 56, 49].

3.3. Maps with large faces. However, an interesting phenomenon occurs in certain situations where w is a critical, non-regular weight. More precisely, we want to look at situations where the second derivative of f_w explodes at z_w^+. This happens when, so to speak, the distributions of face degrees under P_w is heavy-tailed. To look for such situations we follow [39] and introduce a base weight sequence $w^\circ = (w_k^\circ, k \geq 1)$ satisfying

$$\lim_{k\to\infty} k^a w_k^\circ = 1\,, \tag{5}$$

for some positive parameter $a > 3/2$. Set $f_\circ = f_{w^\circ}$, so by (4,5), the radius of convergence of $f_\circ$ is $1/4$. The fact that $a > 3/2$ guarantees that $f_\circ(1/4)$ and $f_\circ'(1/4)$ are finite. Set

$$c = \frac{4}{4f_\circ(1/4) + f_\circ'(1/4)}\,, \qquad \beta = \frac{f_\circ'(1/4)}{4f_\circ(1/4) + f_\circ'(1/4)}\,,$$

and consider the weight sequence $w = (w_k, k \geq 1)$ defined by

$$w_k = c(\beta/4)^{k-1} w_k^\circ\,.$$

Then [39, Proposition 2] shows that w is admissible, critical and that $z_w^+ = R_w = \beta^{-1}$. Moreover, these choices for c, β are the only ones for which these properties hold.

We finally assume that a is less than $5/2$. Under these hypotheses, we consider a random map M_n with distribution $P_w(\cdot \mid \mathbf{M}_n)$. Then it holds that the largest degree of a face of M_n is of order $n^{1/\alpha}$, and the typical graph distances in M_n are of order $n^{1/2\alpha}$, where $\alpha = a - 1/2 \in (1,2)$. We obtain in [39] the following partial scaling limit result.

Theorem 3.2. *From every subsequence, we can extract a further subsequence along which the following convergence in distribution holds in the Gromov-Hausdorff topology:*

$$(V(M_n), n^{-1/2\alpha} d_{M_n}) \underset{n\to\infty}{\longrightarrow} (S_\alpha, D_\alpha)\,.$$

The limit (S_α, D_α) is a random metric space called stable map *of exponent α. Its Hausdorff dimension equals 2α almost-surely.*

Note that the convergence in this statement holds only along appropriate subsequences, it is still an open question to show that the distribution of (S_α, D_α) is uniquely defined. We see that the laws of these spaces are mutually singular when

α varies, because they have different dimensions, and also mutually singular with respect to the law of the Brownian map, which has Hausdorff dimension 4.

The stable maps are described as random quotients, similarly to the Brownian map, but the processes that encode these objects are more elaborate than the Brownian snake $(\mathbf{e}, Z)$. Using a bijection by Bouttier, Di Francesco and Guitter [14], one can relate P_w-distributed maps with certain models of Galton-Watson trees with two types, and with labeled vertices. Under the hypotheses of Theorem 3.1, these trees still admit the Brownian snake as a scaling limit. But under the hypotheses of Theorem 3.2, the trees converge to the so-called *stable trees* of Duquesne, Le Gall and Le Jan [38, 27, 28], which are models of random $\mathbb{R}$-trees with branchpoints of infinite degrees. The stable maps are random quotients of these trees.

Many questions remain on the topological nature of the spaces (S_α, D_α). It is expected that these are random fractal carpets, i.e. spheres minus a countable collection of mutually disjoint open subsets. Depending on the value of α, it is believed that these "holes" have simple and mutually non-intersecting boundaries a.s., or have self and mutual intersections a.s., the critical value for α being $3/2$. These conjectures come from analogies with the so-called *conformal loop ensembles* CLE from [52, 53], which are believed to describe the interfaces of conformally invariant statistical physics models on regular lattices. In the next section, we explain how stable maps play a similar role in the situation where the lattices are random rather than regular.

The reason why one believes that topological aspects of CLEs and stable maps should be similar comes from physical motivation, namely, the so-called Knizhnik-Polyakov-Zamolodchikov correspondence [31, 25], which relates conformally invariant models to models in random metrics. Despite spectacular recent progress [26] towards its mathematical understanding, this correspondence is still quite mysterious, and far from being well understood on the side of random maps.

3.4. The $O(n)$ model on quadrangulations. As an *a posteriori* justification for the model of maps introduced around (5), let us discuss the *rigid $O(n)$ model* introduced in [12]. Here, we consider maps made of two building blocks: Plain quadrangles and quadrangles traversed by a piece of arc from a side to the opposite side. The rigid $O(n)$ configurations are maps made of these two building blocks with obvious compatibility conditions, namely, that pieces of arcs should connect to form a collection of closed loops. See Figure 3. For given positive parameters g, h, n, we assign weight g, h to the two building blocks respectively, and weight n to every loop of the configuration. The total weight of a configuration c is then the product $W^{(n)}_{g,h}(c)$ of weights of its blocks and loops. If the sum of these total weights over all configurations is finite, we can consider a probability measure $P^{(n)}_{g,h}$ by renormalizing $W^{(n)}_{g,h}$. A partial account on a $P^{(n)}_{g,h}$-distributed random map can be given by the *exterior gasket*, which is obtained by removing the interior of the loops (i.e. the part that does not contain the root-edge) as well as the faces traversed by the loops. It is then an easy exercise to check that this exterior gasket has a Boltzmann law P_w, with $w_k = nh^{2k}F_k + g\mathbf{1}_{\{k=2\}}$, where F_k is the sum of total weights of $O(n)$ configurations with a boundary of size $2k$. Such maps are made of the usual building blocks, but the face incident to the root is a polygon of degree $2k$, which is not traversed by a loop.

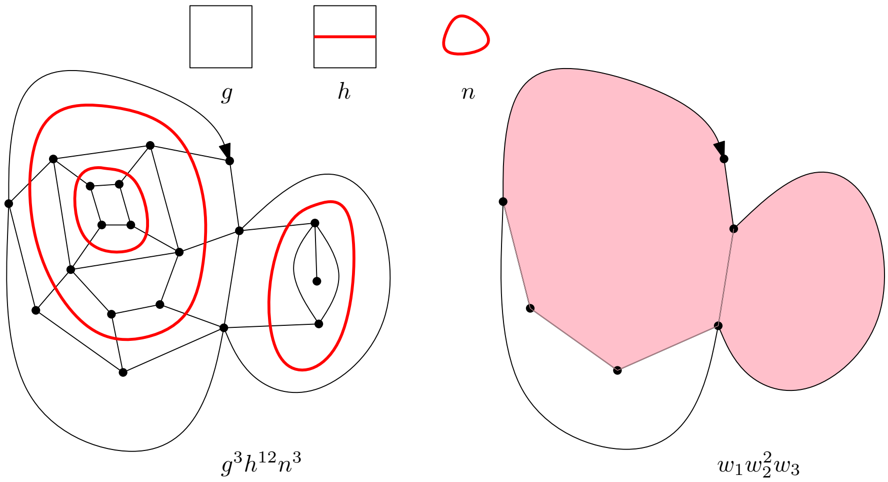

Figure 3. A example of rigid $O(n)$ configuration, and its exterior gasket

It is shown in [12] that for any fixed $n \in (0,2]$, the weights w_k are exactly of the form discussed after (5) if and only if the parameters (g,h) belong to a concave *critical line*, which we assume from now on. If h is smaller than a value $h_c = h_c(n) > 0$, then the hypotheses of Theorem 3.1 hold, and the scaling limit of the exterior gasket is the Brownian map. If h is larger than or equal to h_c then the hypotheses of Theorem 3.2 are in force, with $a = 3/2 + \pi^{-1} \arcsin(n/2) \in (3/2, 2]$ if $h > h_c$, and $a = 5/2 - \pi^{-1} \arcsin(n/2) \in [2, 5/2)$ if $h = h_c$, these situations being called *dense and dilute* phases of the $O(n)$ model in physics. Note that for $n = 2$, the dense and dilute phase coincide in a single phase corresponding to $a = 2$.

4. Conclusion

There are many interesting aspects of the geometry of maps that we have not covered in this short review. One of them is the analogous problem of scaling limit of maps on other surfaces than the sphere. In the case of the g-torus or of the disk, Chapuy [19] and Bettinelli [8, 10, 9] have set the first milestones in this problem (see also [47]), but the analog of Theorem 2.4 in this context is still open so far.

Arguably, the most crucial question in the theory of random maps and their scaling limits is to relate these to other approaches of 2-dimensional quantum gravity, and in particular, to discover connections with the approaches based on conformal geometry or the moduli space of curves [50]. The genuine combinatorial nature of the bijections underlying the study of random maps make these potential links quite mysterious, but this participates to the intrinsic beauty of the topic.

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