Geometric aspects of scaling limits of random maps

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Maps

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A map is a proper embedding of a graph into a two-dimensional (compact, oriented, connected) surface, cutting the latter into topological disks, and considered up to orientation-preserving homeomorphisms of the surface.

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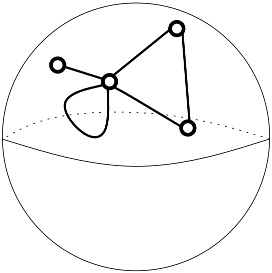

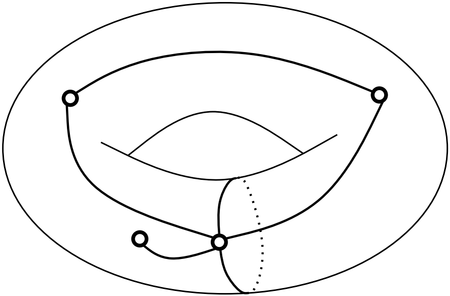

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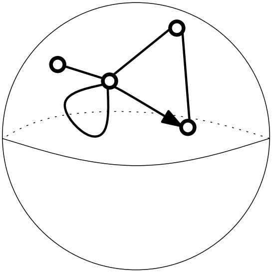

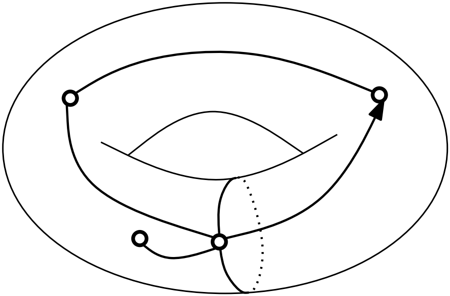

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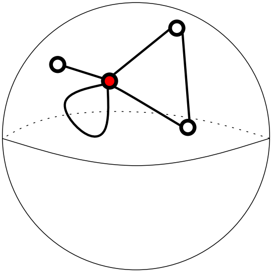

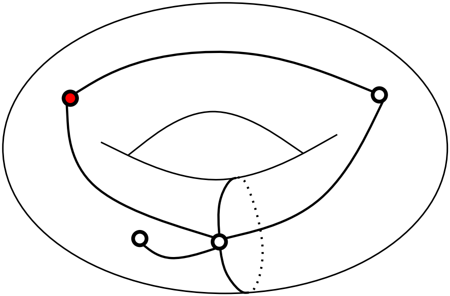

Pointed maps

Enumeration results

Theorem

Let $\mathbf{Q}_n^g$ be the set of genus-g rooted bipartite quadrangulations with n faces ($\sim$ genus-g rooted maps with n edges), then

$$\#\mathbf{Q}_n^0 = \frac{2}{n+2} 3^n \operatorname{Cat}_n \sim \frac{2}{\sqrt{\pi}} 12^n n^{-5/2}$$ *[Tutte 1963]*

$$\#\mathbf{Q}_n^g \sim t_g \, 12^n \, n^{-5\chi(g)/4}$$ *[Bender and Canfield 1986]*

Counting maps:

- Bijective recursive equations for generating functions, leads to Tutte's quadratic method
- Gaussian matrix integrals are generating functions for maps [t'Hooft 1974, Brézin, Itzykson, Parisi & Zuber 1978]
- Enumerating branching covers [Goulden-Jackson 2008]
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Enumeration by bijective methods

These are, at present, the most suited to understand why random maps should "approximate a random surface" (discretization for 2DQG, [Ambjørn *et al.* 90's]).

- Bijective methods allow to encode maps within tree structures with various decorations: buds, distinguished types of vertices, labels, and so on.
- Advantage: can keep track of geodesic distances to a base vertex
- Using this, [Chassaing & Schaeffer, 2004] show that the typical distance between two vertices of a n-faces planar quadrangulation is of order $n^{1/4}$, and exhibit limit distributions involving the ISE (Integrated Super-Brownian Excursion).

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A guess

Let $\mathbf{Q}_n^g$ be the set of genus-g rooted bipartite quadrangulations with n faces, and $\mathbf{q}_n$ be a uniformly distributed element in $\mathbf{Q}_n^g$. Endow the set $V(\mathbf{q}_n)$ of its vertices with the graph distance d_{gr}.

One expects

$$(V(\mathbf{q}_n), n^{-1/4} d_{\mathrm{gr}}) \underset{n\to\infty}{\longrightarrow} (S, d)\,,$$

where (S, d) is a random metric space.

A conjectured limit space arises in [Marckert & Mokkadem 2006], [Le Gall 2007] in the case $g = 0$.

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Topologies on metric spaces

A natural framework for random metric spaces is to compare them using the Gromov-Hausdorff distance. If $(X, d), (X', d')$ are compact metric, let

$$\mathrm{d_{GH}}(X, X') = \inf_{\phi,\phi'} \delta_H(\phi(X), \phi'(X')),$$

the infimum being taken over isometric embeddings of X, X' into a common metric space (Z, δ) and δ_H is the usual Hausdorff distance between compact subsets of Z.

Proposition

This endows the space $\mathbb{M}$ of isometry classes of compact spaces with a complete, separable distance.

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Gromov-Hausdorff-Prokhorov convergence

In order to handle measured metric spaces, we extend the definition to isometry classes of (X, d, μ) where μ is a Borel probability measure on (X, d). The Gromov-Hausdorff-Prokhorov distance is defined by

$$\mathrm{d}_{\mathrm{GHP}}(X, X') = \inf_{\phi,\phi'} \delta_H(\phi(X), \phi'(X')) \vee \delta_P(\phi_*\mu, \phi'_*\mu'),$$

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$$\delta_P(\nu, \nu') = \inf\{\varepsilon : \nu(C) \leq \nu'(C^\varepsilon) + \varepsilon \text{ for all closed } C\},$$

where $C^\varepsilon = \{x \in Z : \delta(x, C) < \varepsilon\}$.

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For convenience, we consider quadrangulations with random sizes, distributed according to Boltzmann distributions. Set $V_{\mathbf{q}} = \#V(\mathbf{q})$ (volume). Define a σ-finite measure

$$\mathcal{Q}^g(\{\mathbf{q}\}) = 12^{-\#F(\mathbf{q})}, \qquad \mathbf{q} \in \mathbf{Q}^g := \bigcup_{n \geq 0} \mathbf{Q}^g_n,$$

and for $\lambda > 0$ let

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Main result

Endow a quadrangulation **q** with the weight

$$\mu_{\mathbf{q}} := \frac{1}{V_{\mathbf{q}}} \sum_{v \in V(\mathbf{q})} \delta_v \,.$$

Theorem

1. *Fix $\lambda > 0$. The laws on $\mathbb{M}^{\mathrm{wt}}$ of the spaces $(V(\mathbf{q}), a^{-1/4} d_{\mathrm{gr}}, \mu_{\mathbf{q}})$ under $\mathcal{P}^{g}_{\lambda/a}$ with $a > 1$, form a tight family of probability measures.*
2. *Any limiting law $\mathcal{S}_\lambda$ is supported by spaces (X, d, μ) such that*
   - *(X, d) is a path metric space*
   - *μ is diffuse with $\mathrm{supp}\,(\mu) = X$*
   - *For $\mu \otimes \mu$-a.e. (x, y), there exists a unique geodesic path between x and y.*

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Discussion

1. One of the main questions is to have uniqueness of the limiting law. A natural conjectured Brownian map has been defined by [Marckert & Mokkadem 06, Le Gall 07] ($g = 0$).
2. At present, only the laws of the mutual distances between 2 [Chassaing-Schaeffer 04] or 3 [Bouttier-Guitter 08] randomly chosen points have been obtained ($g = 0$).
3. A fine study of geodesics to a fixed points, implying the above essential uniqueness result, has been recently done in the $g = 0$ case [Le Gall 08], by different means.
4. When $g = 0$ the limiting space is a.s. homeomorphic to the sphere [Le Gall & Paulin, M.]. A similar 'non-degeneracy' property is expected to hold in higher genera, and is work in progress.
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2. At present, only the laws of the mutual distances between 2 [Chassaing-Schaeffer 04] or 3 [Bouttier-Guitter 08] randomly chosen points have been obtained ($g = 0$).
3. A fine study of geodesics to a fixed points, implying the above essential uniqueness result, has been recently done in the $g = 0$ case [Le Gall 08], by different means.
4. When $g = 0$ the limiting space is a.s. homeomorphic to the sphere [Le Gall & Paulin, M.]. A similar 'non-degeneracy' property is expected to hold in higher genera, and is work in progress.
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The Marcus-Schaeffer bijection

- Let $\mathbf{T}_n^g$ be the set of *g*-trees, i.e. maps of genus *g* with one face, and with *n* edges,
- $\mathbb{T}_n^g$ be the set of labelled *g*-trees $(\mathbf{t}, \mathbf{l})$ where $\mathbf{l} : V(\mathbf{t}) \to \mathbb{Z}$ is defined up to a (global) additive constant and

$$|\mathbf{l}(u) - \mathbf{l}(v)| \leq 1 , \qquad u, v \text{ neighbors} .$$

Theorem

The construction to follow yields a bijection between $\mathbb{T}_n$ and pointed bipartite quadrangulations of genus g with n faces.
(Induces an two-to-one mapping between rooted, pointed and rooted labelled g-trees)

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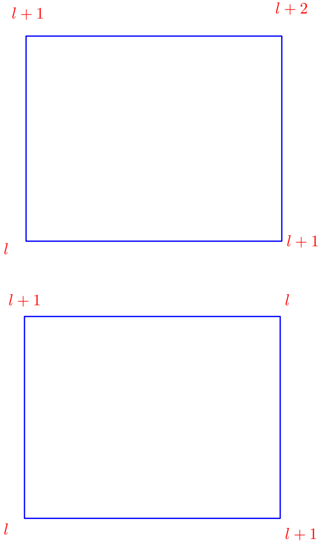

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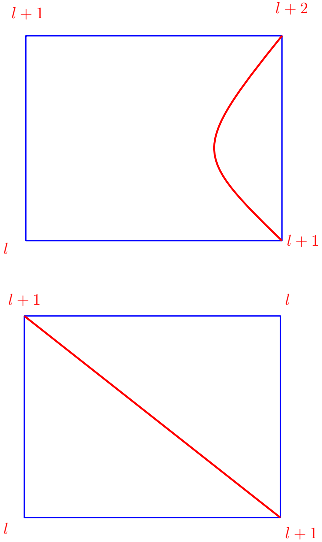

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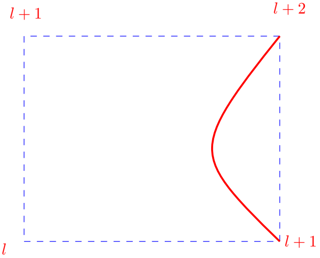

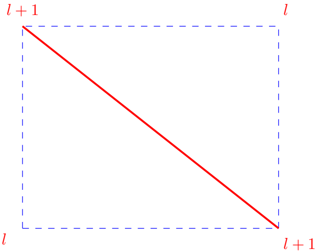

A multi-pointed analogue

Let $\mathbf{Q}^{g,k}$ be the set of triples $(\mathbf{q}, \mathbf{x}, D)$ such that:

1. $\mathbf{q}$ a bipartite genus-g quadrangulation
2. $\mathbf{x} = (x_1, \dots, x_k) \in V(\mathbf{q})^k$, a sequence of k *sources*
3. $D = (d_1, \dots, d_k) \in \mathbb{Z}^k$ a sequence of *delays*, defined up to an additive constant and satisfying
   - $|d_i - d_j| < d_{\mathrm{gr}}(x_i, x_j)$ for $1 \le i \ne j \le k$
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The following construction yields a bijection between $\mathbf{Q}^{g,k}$ and the set of pairs $(\mathbf{m}, \mathbf{l})$ with $\mathbf{m}$ a genus-g map with k faces $f_1, \dots, f_k$ and $\mathbf{l} : V(\mathbf{m}) \to \mathbb{Z}$, defined up to an additive constant, and such that $|\mathbf{l}(u) - \mathbf{l}(v)| \le 1$ for u, v neighbors.

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1. Label the vertices of the quadrangulation **q** by $\mathbf{l}(v) = \min_{1 \leq i \leq k}(d_{\mathrm{gr}}(v, x_i) + d_i)$,
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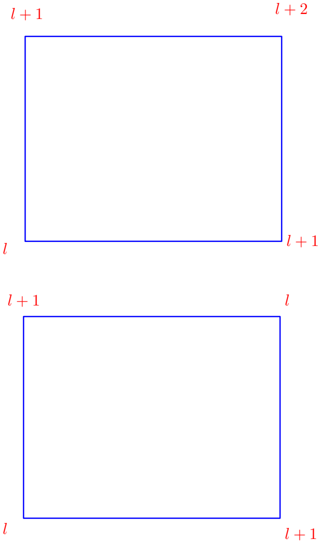

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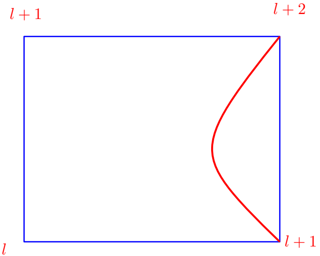

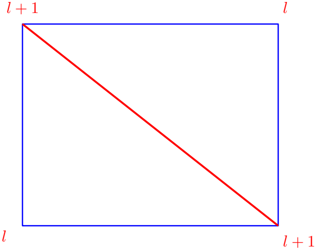

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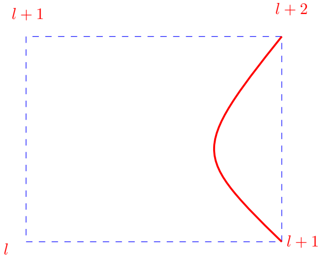
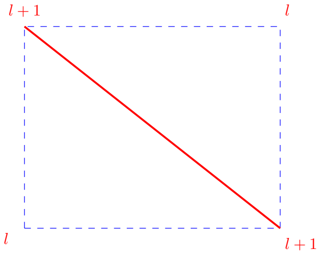
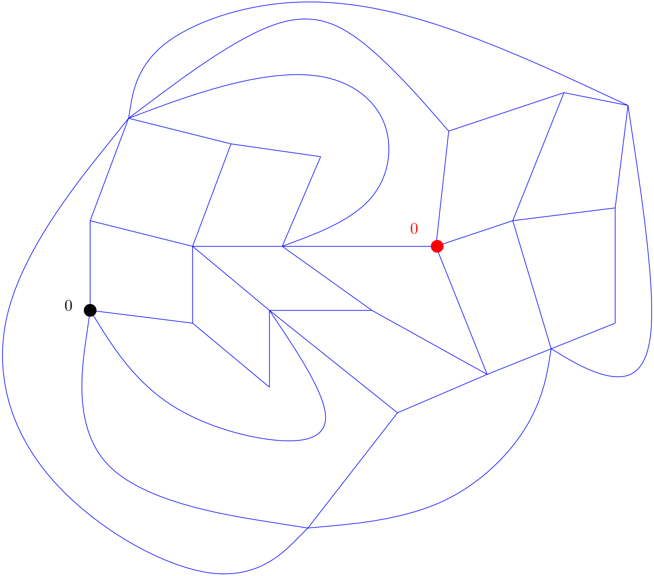




Example



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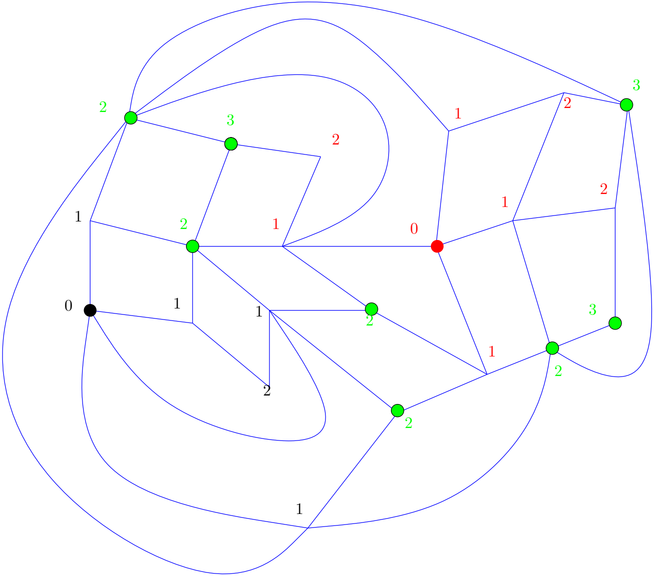

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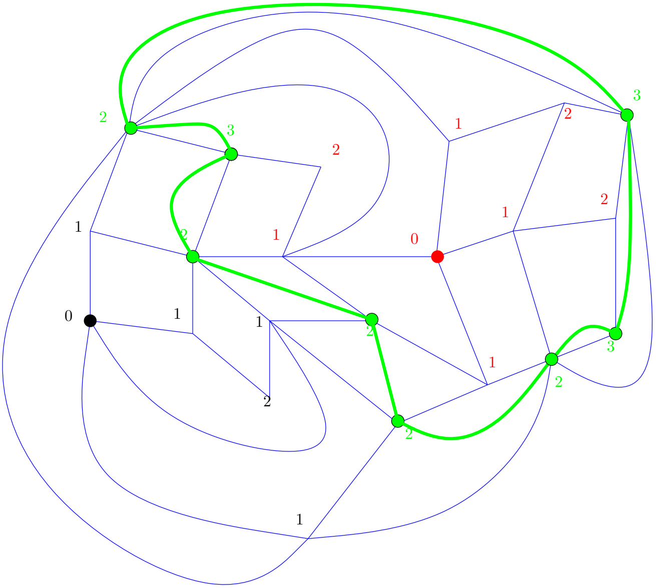

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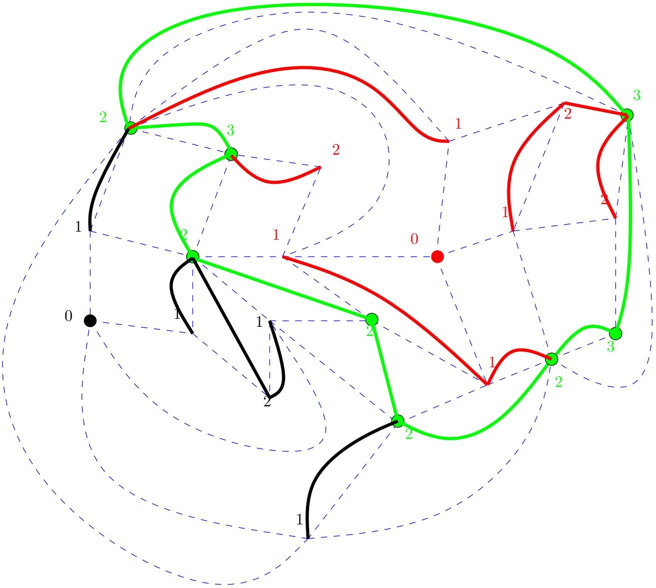
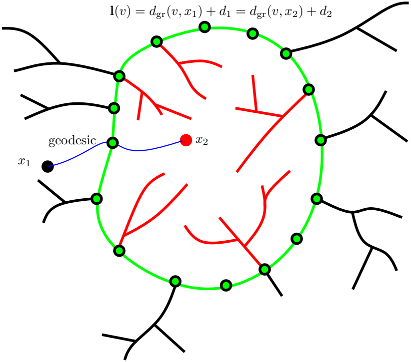


The general aspect of a $g = 0$, two-source situation



- A green, labelled cycle, on which labelled trees are grafted.
- Take a geodesic path in the quadrangulation between the sources x_1 and x_2.
- This geodesic crosses the green cycle at some x.

Note

$$d_{gr}(x, x_1) + d_{gr}(x, x_2) = d_{gr}(x_1, x_2)$$
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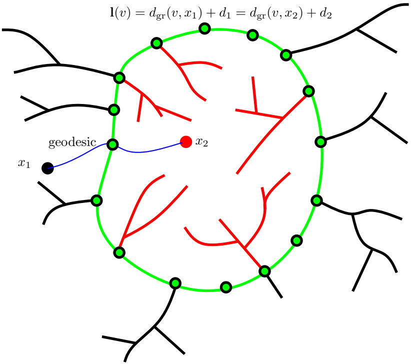

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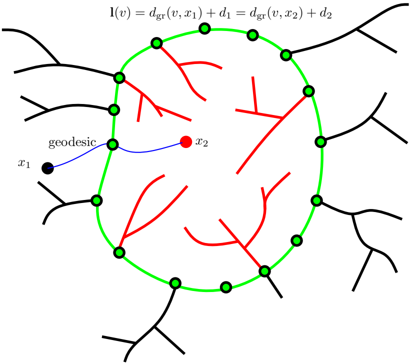

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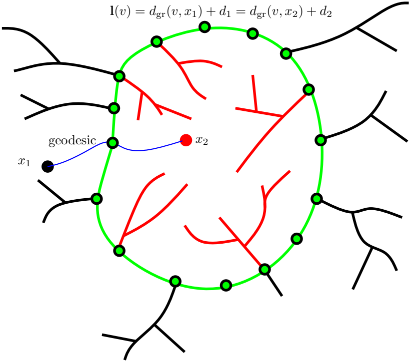

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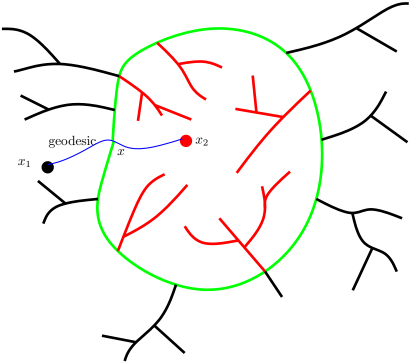
The continuum limit



- In the continuum limit, under $\mathcal{P}^0_{\lambda/a}$ with $a \to \infty$ the length of the green loop scales as $a^{1/2}$.
- The labels on the loop are a discrete random bridge, which scales like $a^{1/4}$ and has a Brownian bridge as a limit.
- The latter attains its infimum at a unique location.

This entails that in a scaling limit (X, d, μ) of $\mathbf{q}$, and for μ-a.e. x_1, x_2 and a.e. D unique x such that

$$d(x_1, x) = (d(x_1, x_2) + D)/2$$
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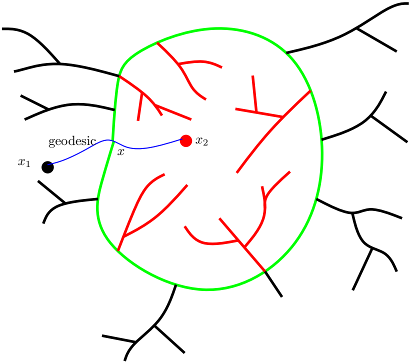

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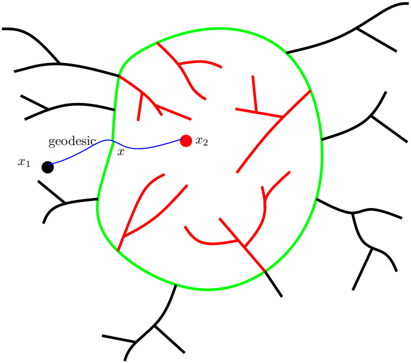

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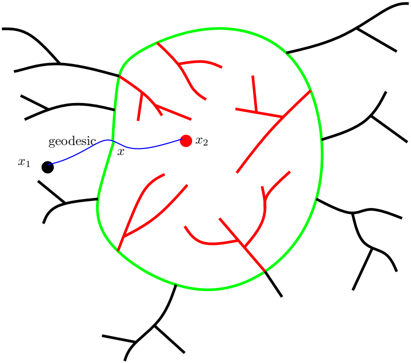

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