A panorama on random maps

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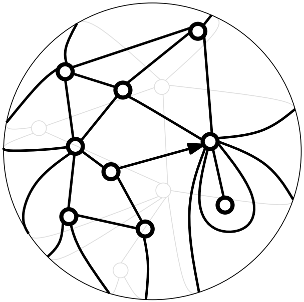
Plane maps

Definition

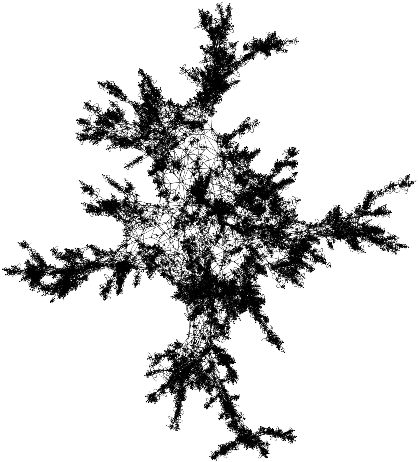
A plane map is an embedding of a connected, finite (multi)graph into the 2-dimensional sphere, considered up to orientation-preserving homeomorphisms of the sphere.



$V(\mathbf{m})$ Vertices
$E(\mathbf{m})$ Edges
$F(\mathbf{m})$ Faces
$d_{\mathbf{m}}(u, v)$ graph distance

A rooted map: distinguish one oriented edge. All maps we consider are rooted.

Simulation of a uniform random plane quadrangulation with 30000 vertices, by J.-F. Marckert



- Q_n uniform random variable in the set $\mathbf{Q}_n$, of rooted plane quadrangulations with n faces (all the faces are quadrangles).
- The set $V(Q_n)$ of its vertices is endowed with the graph distance d_{Q_n}.
- Typically $d_{Q_n}(u, v)$ is of order $n^{1/4}$ (Chassaing-Schaeffer (2004)).

Convergence to the Brownian map

Theorem

There exists a random metric space (S, D), called the Brownian map, such that the following convergence in distribution holds

$$(V(Q_n), n^{-1/4} d_{Q_n}) \xrightarrow[n\to\infty]{(d)} (S, D)$$

as $n \to \infty$, for the Gromov-Hausdorff topology.

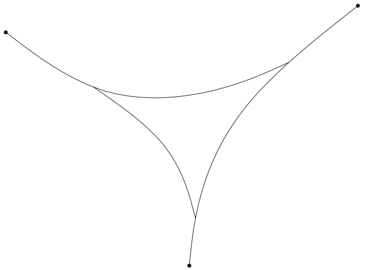
- This means that on some probability space, one can realize Q_n and S as subsets of a common compact metric space, in such a way that their Hausdorff distance tends to 0 a.s. as $n \to \infty$.
- This result has been proved independently by Le Gall (2011) and Miermont (2011), *via* different approaches. Also universality results in Le Gall (2011), discussed later in this talk.

Some of the previous results on scaling limits of random maps

- Chassaing-Schaeffer (2004), based on a bijection between maps and labeled trees (Cori-Vauquelin (1981), Schaeffer),
  - identify $n^{1/4}$ as the proper scaling and
  - compute limiting functionals for random quadrangulations.
- Generalized by Marckert, M., and Weill (2006,2007,2008) to the larger class of Boltzmann random maps.
- Marckert-Mokkadem (2006) introduce the Brownian map.
- Le Gall (2007)
  - Gromov-Hausdorff tightness for rescaled $2p$-angulations
  - the limiting topology is the same as that of the Brownian map.
  - all subsequential limits have Hausdorff dimension 4
- Le Gall-Paulin (2008), and later M. (2008) show that the limiting topology is that of the 2-sphere.
- Bouttier-Guitter (2008) identify the limiting joint law of distances between three uniformly chosen vertices.

Shape of the typical geodesics

- An important ingredient of the proof is to describe the geodesic γ between two "generic" points x_1, x_2: it is a patchwork of small segments of geodesic paths headed toward another generic "root" x_0 (the structure of which was identified by Le Gall 2010).
- So we want to show that B, the set of points x on γ from which we can start a geodesic to x_0 not meeting γ again, is a small set.



Proposition

There exists $\delta \in (0,1)$ such that a.s. for every $\varepsilon > 0$, the set B can be covered with less than $\varepsilon^{-(1-\delta)}$ balls of radius ε. In particular $\dim_{\mathcal{H}}(B) < 1$.

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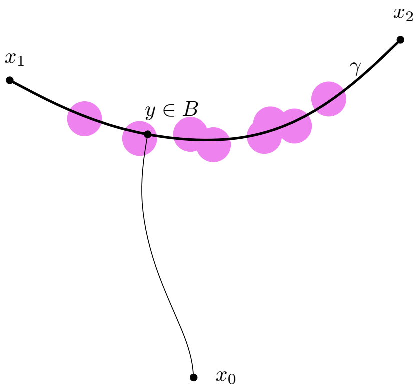

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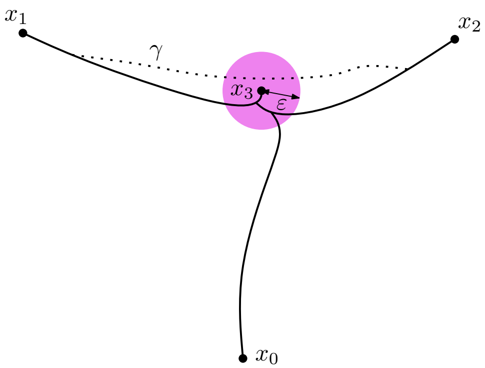
Quickly separating geodesics



- A method to prove the last proposition is to approach points of Γ by points where geodesics perform a quick separation: Evaluate the probability that for 4 randomly chosen points x_0, x_1, x_2, x_3,
  - The three geodesics from x_3 to x_0, x_1, x_2 are disjoint outside of the ball of radius ε around x_3
  - γ passes through the latter ball.

Proposition (codimension estimate)

The probability of the latter event is bounded above by $C\varepsilon^{3+\chi}$ *for some* $\chi > 0$.

Some questions on geodesics

The previous estimate is related to the existence of points x in S from which emanate a *star* with k geodesic arms meeting at x and disjoint elsewhere.

Proposition

Let $x_0, x_1, \dots, x_k$ be uniform randomly distributed points in S. The probability that the geodesics from x_0 to $x_i, 1 \leq i \leq k$ are disjoint outside the ball of center x_0 and radius ε is of order ε^{k-1}.

Question: Is it true that a.s. the set of centers of geodesic stars with k arms

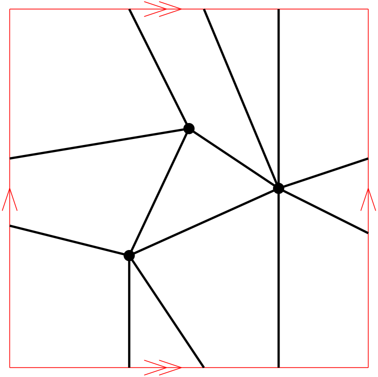
- has Hausdorff dimension $5 - k$ if $k < 5$
- and is empty for $k > 5$?
- What about $k = 5$?

Question: does there exist two geodesics in S that traverse each other at precisely one point?

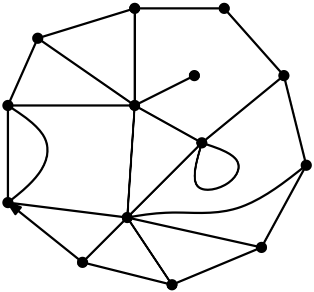
Other topologies

Definition

A map on an orientable surface M is an embedding of a locally finite graph on M, that dissects the latter into topological polygons, and considered up to direct homeomorphisms of M.



On the torus



On a disk

Other topologies: closed surfaces

Partial results for the scaling limit problem have been obtained for bipartite quadrangulations on the closed orientable surface $\mathbb{T}_g$ of genus g. A result by Bettinelli (2011), building on a bijection by Chapuy-Marcus-Schaeffer:

Theorem

Let M_n be a uniform bipartite quadrangulation with n faces of $\mathbb{T}_g$, for some $g > 0$. Then $(V(M_n), n^{-1/4} d_{M_n})$ converges, up to extraction, to a random metric space homeomorphic to $\mathbb{T}_g$.

The uniqueness of the limiting law is not known yet, and is work in progress.

Other topologies: quadrangulations with boundaries

A quadrangulations with a boundary is a rooted map with faces of degree 4, except possibly for the root face which is allowed to have any even degree. A result by Bettinelli (2012), confirming earlier observations by Bouttier-Guitter:

Theorem

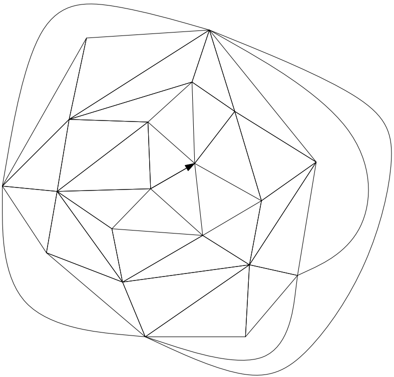
Let $M_{n,k}$ be a uniform quadrangulation with a boundary of length $k = k(n)$, and n internal faces. Then as $n \to \infty$,

- *If $k \ll \sqrt{n}$, the space $(V(M_{n,k}), n^{-1/4} d_{M_{n,k}})$ converges to the Brownian map.*
- *If $\sqrt{n} \ll k$, the space $(V(M_{n,k}), k^{-1/2} d_{M_{n,k}})$ converges to the Brownian continuum random tree*
- *If $k/\sqrt{n} \to \lambda \in (0, \infty)$, then up to extraction the space $(V(M_{n,k}), n^{-1/4} d_{M_{n,k}})$ converges to a limiting metric space with the topology of the disk.*

The uniqueness of the limiting law is work in progress.

Local limits

- In another direction, Angel-Schramm (2002) and Angel (2002) consider local limit results for random triangulations. They construct the so-called uniform infinite planar triangulation (UIPT). See also Krikun (2003,2005), considering quadrangulations.
- Followed by work of Chassaing-Durhuus (2006) Ménard (2008), that generalize the Chassaing-Schaeffer bijective approach in this infinite context.



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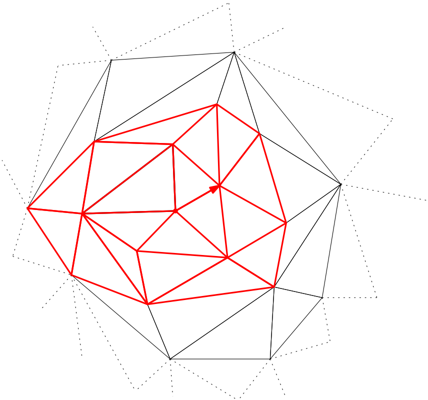

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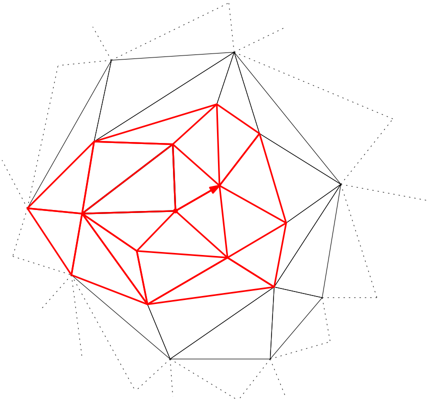


Geometry at infinity of the UIPQ

Theorem (Curien-Ménard-M. (2012))

- *In the UIPQ, there exists a sequence of vertices $p_1, p_2, \dots$ such that any infinite geodesic path goes through every but a finite number of the vertices $p_i, i \geq 1$.*
- *Moreover it holds that for every vertices x, y, $z \mapsto d_{\mathrm{gr}}(x,z) - d_{\mathrm{gr}}(y,z)$ takes the same value for every but a finite number of z's.*

- The Uniform Infinite Planar Quadrangulation has an essentially unique infinite geodesic path, that leads to a single point at infinity.
- Recently Curien-Le Gall (2012) show the convergence of UIPQ to an infinite-volume version of the Brownian map: the Brownian plane.

Boltzmann random maps

- From now on we mostly consider bipartite plane maps (with faces of even degree) for technical simplicity.
- Boltzmann distribution: let $w = (w_k, k \geq 1)$ be a non-negative sequence, $w_1 < 1$ and $w_k > 0$ for some $k \geq 2$. Define a measure by
$$\mathbb{B}_w(\mathbf{m}) = \prod_{f \in F(\mathbf{m})} w_{\deg(f)/2}\,, \qquad \mathbf{m} \text{ rooted, bipartite}\,.$$
- Let
$$\mathbb{B}_w^n(\cdot) = \mathbb{B}_w\Big(\,\cdot \;\Big|\; \{\mathbf{m} \text{ with } n \text{ vertices}\}\Big)\,,$$
defining a probability measure. Uniform on $2p$-angulations with n vertices if $w_k = \delta_{kp}$.

Brownian map universality class

For Boltzmann maps $\mathbb{B}_w^n$ sampled according to "generic" sequences of weights, the Brownian map still prevails in the limit.

Theorem (Le Gall 2011)

If $(w_k, k \geq 1)$ is a weight sequence with finite support, then if M_n has law $\mathbb{B}_w^n$, there exists a constant b_w such that $(V(M_n), b_w n^{-1/4} d_{M_n})$ converges in distribution to the Brownian map.

Questions:

- Let Q_n be a random uniform plane quadrangulation with n faces. Assign each edge length $1+\varepsilon$ or $1-\varepsilon$ with equal probability $1/2$, independently. Let $d_n^{(\varepsilon)}$ be the resulting distance on vertices. Does $(V(Q_n), cn^{-1/4} d_n^{(\varepsilon)})$ converge to the Brownian map (some $c > 0$)?
- Let $Q_n^{(K)}$ be a uniform plane quadrangulation with n faces, in which all degrees are less than some fixed threshold $K > 4$. Does $(V(Q_n^{(K)}), cn^{-1/4} d_{Q_n^{(K)}})$ converge to the Brownian map (some $c > 0$?)

Stable maps

- It is possible to go out of the Brownian map "universality class" for a map with law $\mathbb{B}_w^n$, under certain conditions called non-generic.
- Fix a reference sequence $(w_k^\circ, k \geq 1)$ with

$$w_k^\circ \sim k^{-a}$$

  where $a \in (3/2, 5/2)$ is a parameter, one shows that there exists a unique (c, λ) such that the Boltzmann map with weights $w_k = c\lambda^k w_k^\circ$ is such that
  - the degree of the root face has a heavy tail with exponent $a - 1/2 \in (1, 2)$.
  - the total size of the map also has a heavy tail, with exponent $1/(a - 1/2) \in (1/2, 1)$.

Stable maps

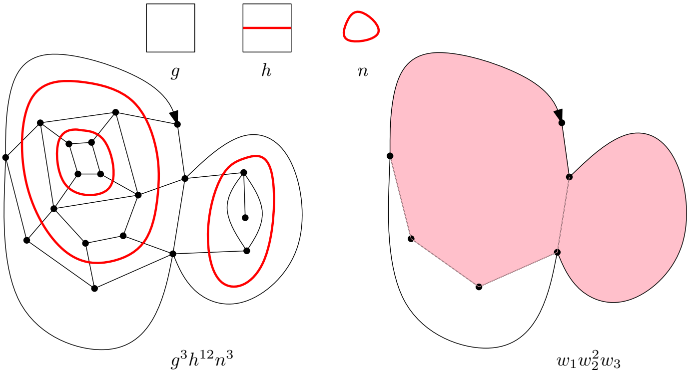
Theorem (Le Gall-Miermont 2009)

For non-generic weights, if M_n has law $\mathbb{B}_w^n$, the sequence $(V(M_n), n^{-1/(2a-1)}d_{M_n})$ converges in distribution, at least along some extraction, to a random metric space (S_a, d_a) with Hausdorff dimension a.s. equal to $2a-1 \in (2,4)$, the stable map with index a.

The topology of stable maps is not known yet. One conjectures that:

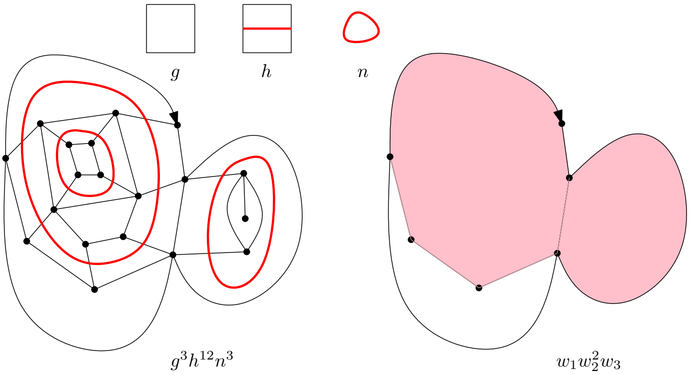
- If $a \in [2, 5/2)$ then (S_a, d_a) is a random Sierpinsky carpet (holes have simple, mutually avoiding boundaries).
- if $a \in [2, 5/2)$ then holes have self and mutual intersections.

The $O(n)$ model on random quadrangulations



Decorate a quadrangulation with simple and mutually avoiding dual loops: pick the configuration $\mathbf{q}$ with probability proprortional to the weight $W^{(n)}_{g,h}(\mathbf{q}) = g^{\#quad}h^{|loops|}n^{\#loops}$.
This is the $O(n)$ model on random quadrangulations.

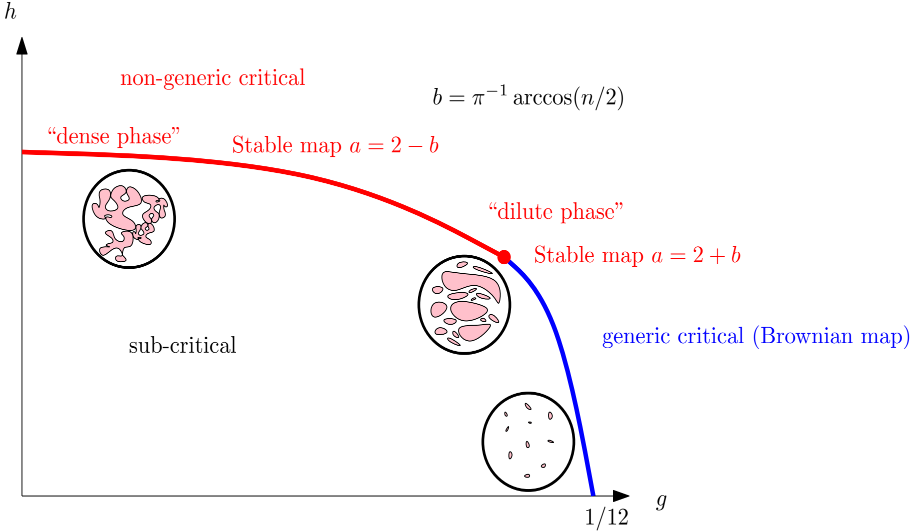
The $O(n)$ model on random quadrangulations



Let $\mathcal{G}(\mathbf{q})$ be the exterior gasket of $\mathbf{q}$, obtained by emptying the interior of the loops. The gasket of the $O(n)$ model is then a Boltzmann random map with parameters

$$w_k = g\delta_{k2} + nh^{2k} \sum_{|\partial\mathbf{q}|=2k} W^{(n)}_{g,h}(\mathbf{q})\,.$$

Phase diagram at fixed $n \in (0, 2]$ [Borot-Bouttier-Guitter 2011]



Motivation

- Maps are seen as discretized 2D Riemannian manifolds.
- This comes from 2D quantum gravity, in which a basic object is the partition function

$$\int_{\mathcal{R}(M)/\mathrm{Diff}^+(M)} [\mathcal{D}g] \exp(-\alpha \, \mathrm{Area}_g(M))$$

  - M is a 2-dimensional orientable manifold,
  - $\mathcal{R}(M)$ is the space of Riemannian metrics on M,
  - $\mathrm{Diff}^+(M)$ the set of orientation-preserving diffeomorphisms,
  - $\mathcal{D}g$ is a “Lebesgue” measure on $\mathcal{R}(M)$. This, and the induced measure $[\mathcal{D}g]$, are the problematic objects.

How to deal with $[\mathcal{D}g]$?

One can replace

$$\int_{\mathcal{R}(M)/\mathrm{Diff}^+(M)} [\mathcal{D}g] \longrightarrow \sum_{T \in \mathbf{Tr}(M)} \delta_T$$

where $\mathbf{Tr}(M)$ is the set of triangulations of M (more popular than quadrangulations in the Physics literature).

- Then one tries to take a scaling limit of the right-hand side, in which triangulations approximate a “smooth”, continuum surface, which in our case is the Brownian map or a related object
- Analog to path integrals, in which random walks can be used to approximate Brownian motion.
- Before scaling limits were considered, the success of this approach came from the rich literature on enumerative theory of maps, after Tutte’s work or the literature on matrix integrals.

The quantum Liouville theory approach

- Another approach: quantum Liouville theory (Polyakov, David, ...)
- Represent $g = e^{2u} h^* g_0(\mu)$ where
  - $u : M \to \mathbb{R}$ (conformal factor)
  - $g_0(\mu)$ is a fixed representative for its complex structure, parameterized by the *moduli* μ
  - $h \in \mathrm{Diff}^+(M)$.
- Computing a formal "Jacobian" for this transformation yields the following partition function for the conformal parameter u:

$$\int \exp(-S_L(u, g_0(\mu)))\mathcal{D}u \,,$$

where $S_L(u, g_0)$ is the Liouville action.

- $S_L(u, g_0)$ can be seen as a Gaussian action (quadratic form), with quadratic term $\int_M \|\nabla u(z)\|^2_{g_0} \mathrm{vol}_{g_0}(\mathrm{d}z)$,
- This indicates that u should be a Gaussian Free Field. This idea has been developed recently by Duplantier-Sheffield.

A diagram on quantum gravity approaches

2D Quantum Gravity

$$\int_{\mathcal{R}(M)/\mathrm{Diff}^+(M)} \exp(-\alpha \mathrm{Vol}_g(M))[\mathcal{D}g]$$

Φ

M

Random maps

$$\int_{\mathcal{R}(M)/\mathrm{Diff}^+(M)} [\mathcal{D}g] \longrightarrow \sum_{\mathbf{m}\in\mathbf{M}} \delta_{\mathbf{m}}$$

Converges to the Browian map

(Random metric spaces)

??

Liouville quantum gravity

"$g \longrightarrow \exp(\gamma h(z))\, g_0(\mu)$"

$$\mathrm{Vol}_g(\mathrm{d}z) \longrightarrow \exp(\gamma h(z))\mathrm{Vol}_{g_0(\mu)}(\mathrm{d}z)$$

h a Gaussian free field

(Random measures on fixed surfaces)

Some questions

- View the random plane quadrangulation Q_n as a Riemannian manifold, by declaring its faces to be copies of the unit Euclidean square. Viewing Q_n as a Riemann surface, there exist a conformal map

$$\phi_n : Q_n \to \mathbb{S}^2$$

unique up to Möbius transformations. Let μ_n be the image under ϕ_n of the uniform measure on Q_n.
    - show that μ_n converges to a limit, which is related to the exponential of a Gaussian free field on $\mathbb{S}^2$ (Gaussian multiplicative chaos).
    - show that the function $d_n(u, v) = d_{M_n}(\phi_n^{-1}(u), \phi_n^{-1}(v)), u, v \in \mathbb{S}^2$, converges to a limiting random metric on $\mathbb{S}^2$.
- Do the same with a (non-generic) critical $O(n)$ model on a quadrangulation: this gives a conformal map $\phi_n : Q_n \to \mathbb{S}^2$. It is expected that $\phi_n(\mathcal{G}(Q_n))$, where $\mathcal{G}(Q_n)$ is the gasket of Q_n, converges in distribution to a Conformal Loop Ensemble (Sheffield, Werner)