# Geometric aspects of scaling limits of random planar maps

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# Planar maps

## Definition

A planar map is a proper embedding of a connected graph into the two-dimensional sphere, considered up to orientation-preserving homeomorphisms of the sphere.

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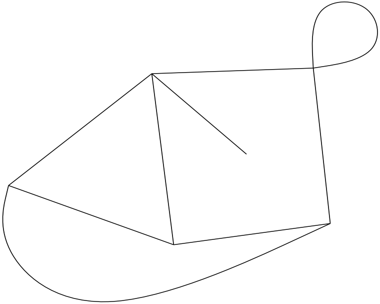

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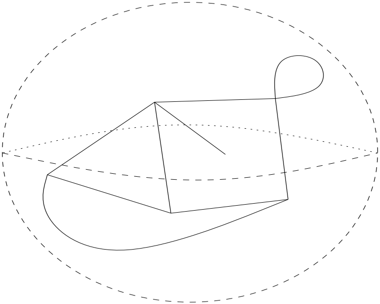

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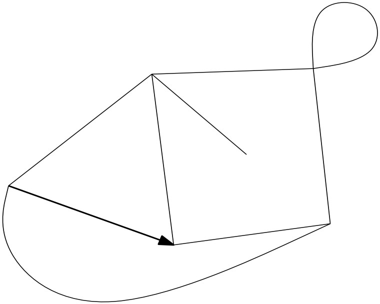

A rooted map

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# Enumeration results

A basic enumeration result is [Tutte 1963]

**Theorem**

$$\#\mathbf{M}_n = \frac{2}{n+2} 3^n \,\mathrm{Cat}_n \sim \frac{2}{\sqrt{\pi}} 12^n n^{-5/2},$$

*$\mathbf{M}_n$ the set of rooted planar planar maps with n edges.*

Counting maps:

- Bijective recursive equations for generating functions, leads to Tutte's quadratic method
- The limiting free energy of matrix models are generating functions for planar maps [t'Hooft 1974, Brézin, Itzykson, Parisi & Zuber 1978]
- Enumerating factorizations of permutations and branching covers, representation theory [Goulden-Jackson 2008]
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# Enumeration by bijective methods

These are, at present, the most suited to understand why random maps should "approximate a random surface" (discretization for 2DQG, [Ambjørn *et al.* 90's]).

- Bijective methods allow to encode maps within tree structures with various decorations: buds, distinguished types of vertices, labels, and so on.
- Advantage: can keep track of geodesic distances to a base vertex
- Using this, [Chassaing & Schaeffer, 2004] show that the typical distance between two vertices of a $n$-faces quadrangulation is of order $n^{1/4}$, and exhibit limit distributions involving the ISE (Integrated Super-Brownian Excursion).

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# A guess

Let $\mathbf{Q}_n$ be the set of rooted planar quadrangulations with $n$ faces, and $\mathbf{q}_n$ be a uniformly distributed element in $\mathbf{Q}_n$. Endow the set $V_n$ of its vertices with the graph distance $d_{\mathrm{gr}}$.

One expects

$$(V_n, n^{-1/4} d_{\mathrm{gr}}) \underset{n\to\infty}{\longrightarrow} (S, d),$$

where $(S, d)$ is a random metric space.

A conjectured limit space arises in [Marckert & Mokkadem 2006], [Le Gall 2007]

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# Topologies on metric spaces

A natural framework for random metric spaces is to compare them using the Gromov-Hausdorff distance. If $(X, d), (X', d')$ are compact metric, let

$$\mathrm{d_{GH}}(X, X') = \inf_{\phi,\phi'} \delta_H(\phi(X), \phi'(X')),$$

the infimum being taken over isometric embeddings of $X, X'$ into a common metric space $(Z, \delta)$ and $\delta_H$ is the usual Hausdorff distance between compact subsets of $Z$.

**Proposition**

*This endows the space $\mathbb{M}$ of isometry classes of compact spaces with a complete, separable distance.*

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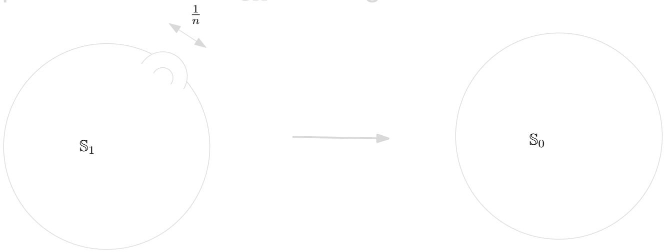
# Gromov-Hausdorff convergence

- This topology yields powerful approximation methods, a bit like weak convergence for measures. For instance, every space is well-approximated by its $\varepsilon$-nets. $\mathrm{d}_{\mathrm{GH}}$-convergence also preserves closed metric conditions, like being an $\mathbb{R}$-tree, or being a path metric space.
- However, a *caveat* is that topological/dimension properties are not preserved under $\mathrm{d}_{\mathrm{GH}}$-convergence.



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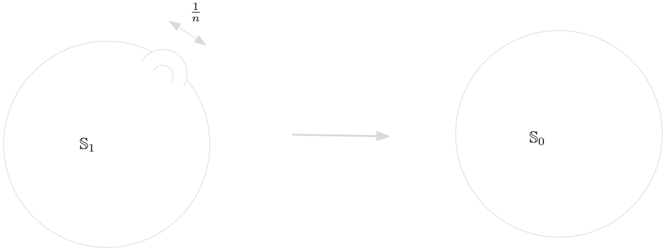

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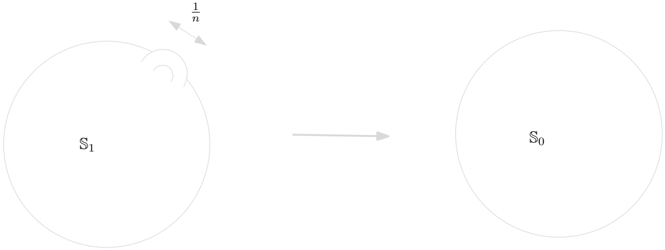

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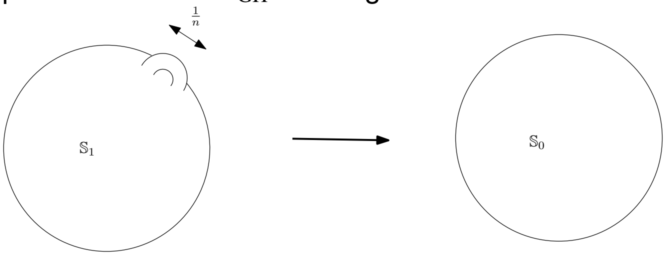
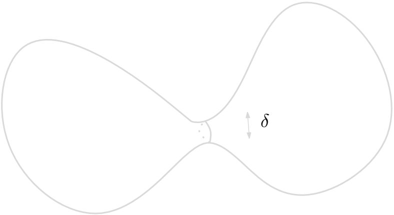


# Regular convergence

Whyburn (1935) gave the following criterion for 2-sphere topology preservation under Hausdorff convergence:




**Proposition**

*Let $\mathcal{X}_n \to \mathcal{X}$ in $(\mathbb{M}, \mathrm{d}_{\mathrm{GH}})$, where $\mathcal{X}_n, n \geq 1$ is a path metric space homeomorphic to the 2-sphere. Assume that for every $\varepsilon > 0$, there exists $\delta, N > 0$ such that for $n \geq N$, every loop $\gamma$ on $\mathcal{X}_n$ with diameter $\leq \delta$ is contractible in its $\varepsilon$-neighborhood. Then either $\mathcal{X}$ is homeomorphic to the 2-sphere, or is a singleton.*

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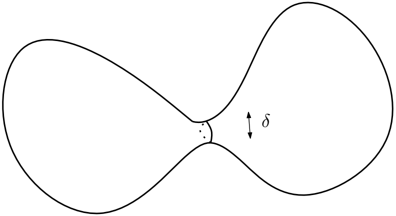


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# A sphericity theorem

Partial answers to the approximation of random surfaces by random maps have been brought recently by [Le Gall 2007], [Le Gall & Paulin, 2008]. We are going to focus on the following result.

**Theorem**

*The family of laws of (isometry classes of) $(V_n, n^{-1/4} d_{\mathrm{gr}})$ is relatively compact in the set of probability measures on $(\mathbb{M}, \mathrm{d}_{\mathrm{GH}})$, endowed with the weak topology.*
*Moreover, any limiting point for this family is supported by spaces that are homeomorphic to the 2-sphere.*

Le Gall & Paulin obtain this result by reasoning in a "continuous" framework. We discuss an alternative, "discrete-world" based approach.

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# Schaeffer's bijection

- Let $\mathbf{T}_n$ be the set of rooted plane trees with $n$ edges,
- $\mathbb{T}_n$ be the set of labelled trees $(\mathbf{t}, \mathbf{l})$ where $\mathbf{l} : V(\mathbf{t}) \to \mathbb{Z}$ satisfies $\mathbf{l}(\mathrm{root}) = 1$ and

$$|\mathbf{l}(u) - \mathbf{l}(v)| \leq 1 , \qquad u, v \text{ neighbors} .$$

- Last, let $\overline{\mathbb{T}}_n$ be those $(\mathbf{t}, \mathbf{l}) \in \mathbb{T}_n$ for which $\mathbf{l} \geq 1$ (well-labeled trees).

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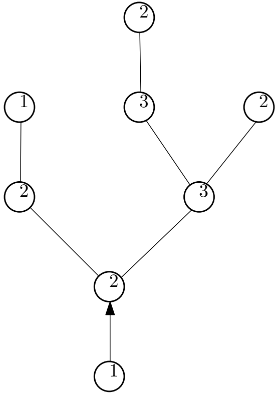

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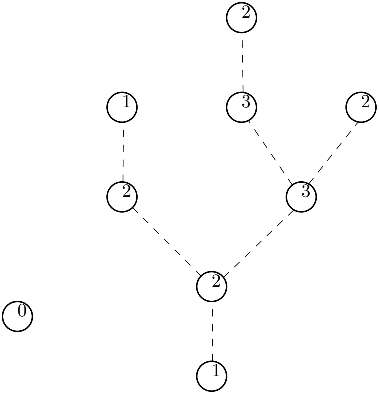

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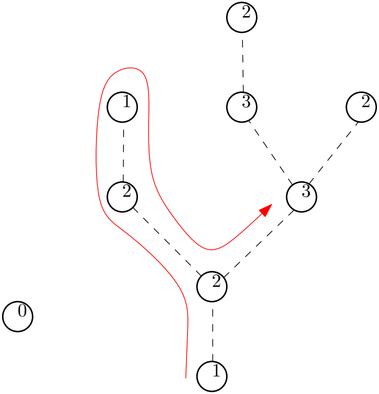

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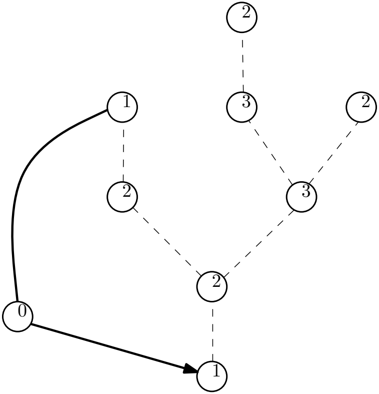

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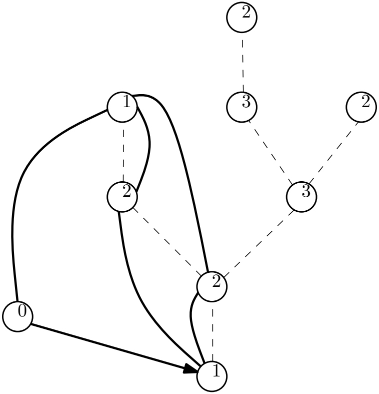

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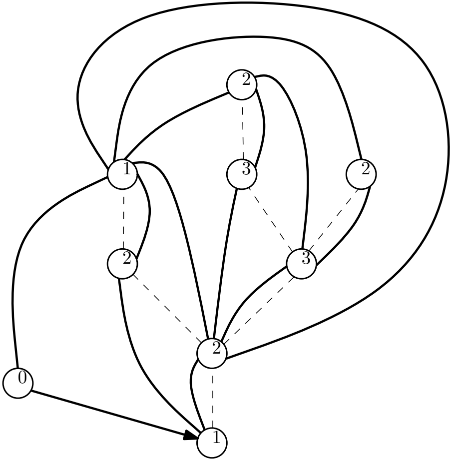


Note that the labels are geodesic distances in the resulting map

# The Brownian tree

The Brownian tree arises as the scaling limit of many discrete random tree models, e.g. uniform random element $T_n$ of $\mathbf{T}_n$:

$$(V(T_n), (2n)^{-1/2} d_{\mathrm{gr}}) \to \mathcal{T} ,$$

for the Gromov-Hausdorff distance.

- Build $\mathcal{T}_\circ$ as an $\mathbb{R}$-tree, by grafting segments drawn from a Poisson measure on $\mathbb{R}_+$ with intensity $t\mathrm{d}t$ recursively at a uniform location in the tree constructed at each stage.
- Then let $\mathcal{T}$ be (the isometry class of) the metric completion of $\mathcal{T}_\circ$.
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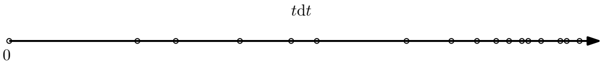
- Build $\mathcal{T}_\circ$ as an $\mathbb{R}$-tree, by grafting segments drawn from a Poisson measure on $\mathbb{R}_+$ with intensity $t\mathrm{d}t$ recursively at a uniform location in the tree constructed at each stage.
- Then let $\mathcal{T}$ be (the isometry class of) the metric completion of $\mathcal{T}_\circ$.
- $\mathcal{T}$ has a well-know construction from the Brownian excursion [Aldous 1993], [Le Gall 1993], hence its name.

# Pictures



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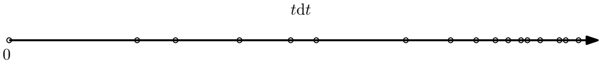


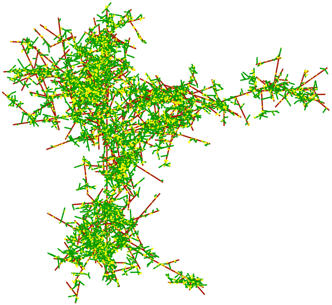

# Brownian labels on the Brownian tree

- Once the tree is build, one can consider a white noise supported by the tree, or, equivalently, branching Brownian paths.
- Informally, we let $Z$ be a centered Gaussian process run on $\mathcal{T}$, with covariance function

$$\mathrm{Cov}\,(Z_a, Z_b) = d_{\mathcal{T}}(\mathrm{root}, a \wedge b)\,,$$

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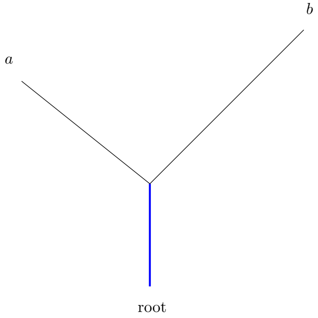

# Convergence of labelled trees

- Let $(T_n, L_n)$ be uniform in $\mathbb{T}_n$, then

$$\left(\frac{1}{(2n)^{1/2}} T_n, \left(\frac{9}{8n}\right)^{1/4} L_n\right) \longrightarrow (\mathcal{T}, Z)$$

- Let $(\overline{T}_n, \overline{L}_n)$ be uniform in $\overline{\mathbb{T}}_n$, then

$$\left(\frac{1}{(2n)^{1/2}} \overline{T}_n, \left(\frac{9}{8n}\right)^{1/4} \overline{L}_n\right) \longrightarrow (\overline{\mathcal{T}}, \overline{Z}),$$

  the Brownian tree with Brownian labels, conditioned on the labels being non-negative.

- The latter is the same tree with labels $(\mathcal{T}, Z)$, but re-rooted at the point $a_*$ where $Z$ attains its infimum [Le Gall & Weill, 2005].

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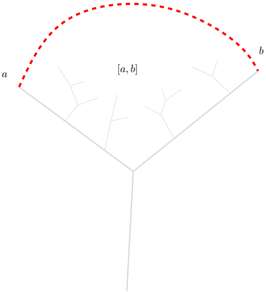
# Identification of points in the limit

Start from the convergence of rescaled $(\overline{T}_n, \overline{L}_n)$ to $(\overline{\mathcal{T}}, \overline{Z})$. Let $Q_n \in \mathbf{Q}_n$ be encoded by $(\overline{T}_n, \overline{L}_n)$, assume $(V(Q_n), n^{-1/4} d_{\mathrm{gr}}) \to (S, d)$.

Take points $a_n, b_n \in \overline{T}_n$ "converging" to $a, b \in \overline{\mathcal{T}}$, identify $a_n, b_n$ with vertices of $Q_n$. If

$$\overline{Z}_a = \overline{Z}_b = \inf_{[a,b]} \overline{Z},$$

then $a_n, b_n$ become identified in the limit ($d_{\mathrm{gr}}(a_n, b_n) = o(n^{1/4})$).




A theorem by Le Gall says that these are the only identifications to be made: points $a_n, b_n$ in $\overline{T}_n$ such that $\overline{Z}_a + \overline{Z}_b - 2\min_{[a,b]} \overline{Z} > 0$ will be far away in the $n^{1/4}$ scale.

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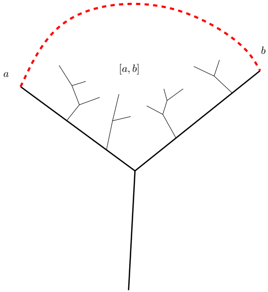


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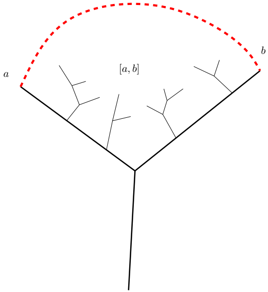


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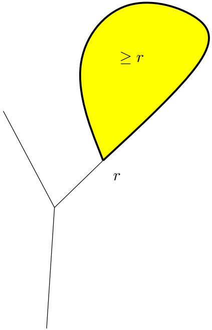
# A.s. forbidden configurations in the tree with labels

## Proposition

*A.s. there does not exist a point a in $\overline{\mathcal{T}}$ (besides the root) such that $\overline{Z}_b \geq \overline{Z}_a$ for all the descendents of a.*




As a consequence of this and the previous discussion, two points $a_n, b_n \in \overline{T}_n$ converging to $a, b \in \overline{\mathcal{T}}$ with $a$ an ancestor of $b$ are not identified in the limit, i.e. are far away in scale $n^{1/4}$.

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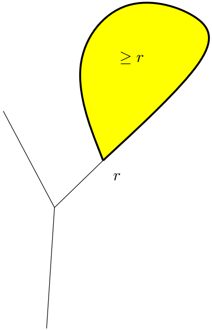


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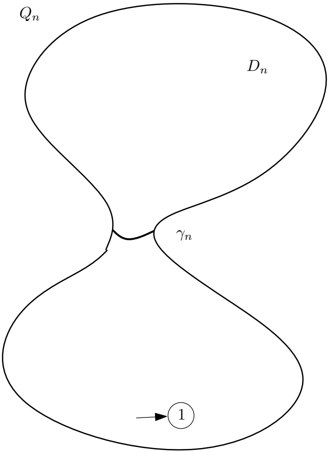
# Proof of the main theorem

Assume the existence of $o(n^{1/4})$-length loops $\gamma_n$ in $Q_n$.




1. Let $D_n$ be the component separated from the root by $\gamma_n$
2. The tree $\overline{T}_n$ must enter $D_n$, in the limit all labels inside $D_n$ are $\geq \ell$
3. To avoid forbidden configurations, the tree must have a branch leaving $D_n$ after entering
4. This allows to find ancestors $a_n, b_n$ far away in $\overline{T}_n$ but at $o(n^{1/4})$-distance in $Q_n$, which is excluded.

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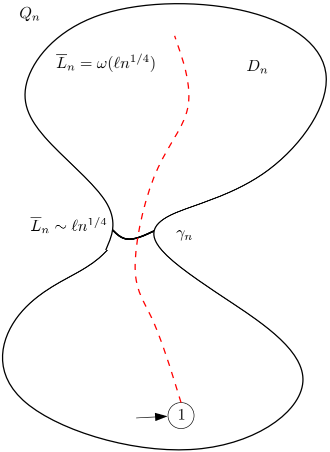


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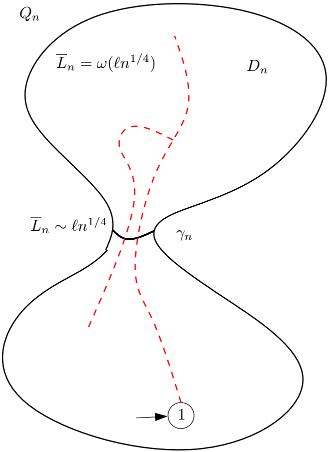


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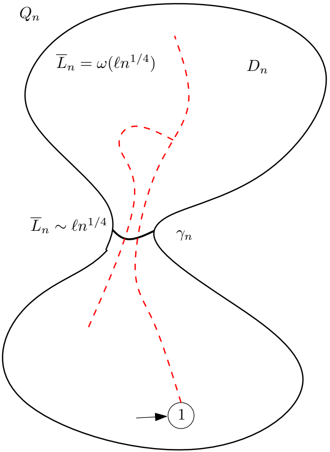


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