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Multipliers and approximation properties of groups

Multiplicateurs et propriétés d'approximation de groupes

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Multipliers and approximation properties of groups

This thesis focusses on some approximation properties which generalise amenability for locally compact groups. These properties are defined by means of multipliers of certain algebras associated to the groups. The first part is devoted to the study of the p-AP, which is an extension of the AP of Haagerup and Kraus to the context of operators on L_p spaces. The main result asserts that simple Lie groups of higher rank and finite centre do not satisfy p-AP for any p between 1 and infinity. The second part concentrates on radial Schur multipliers on graphs. The study of these objects is motivated by some connections with actions of discrete groups and weak amenability. The three main results give necessary and sufficient conditions for a function of the natural numbers to define a radial multiplier on different classes of graphs generalising trees. More precisely, the classes of graphs considered here are products of trees, products hyperbolic graphs and finite dimensional CAT(0) cube complexes.

Multiplicateurs et propriétés d'approximation de groupes

Cette thèse porte sur des propriétés d'approximation généralisant la moyennabilité pour les groupes localement compacts. Ces propriétés sont définies à partir des multiplicateurs de certaines algèbres associés aux groupes. La première partie est consacrée à l'étude de la propriété p-AP, qui est une extension de la AP de Haagerup et Kraus au cadre des opérateurs sur les espaces L_p. Le résultat principal dit que les groupes de Lie simples de rang supérieur et de centre fini ne satisfont p-AP pour aucun p entre 1 et l'infini. La deuxième partie se concentre sur les multiplicateurs de Schur radiaux sur les graphes. L'étude de ces objets est motivée par les liens avec les actions de groupes discrets et la moyennabilité faible. Les trois résultats principaux donnent des conditions nécessaires et suffisantes pour qu'une fonction sur les nombres naturels définisse un multiplicateur radial sur des différentes classes de graphes généralisant les arbres. Plus précisément, les classes de graphes étudiées sont les produits d'arbres, les produits de graphes hyperboliques et les complexes cubiques CAT(0) de dimension finie.

Contents

Introduction en français

Cette thèse porte sur deux sujets principaux, reliés par un concept central : la notion de multiplicateur complètement borné. Les applications complètement bornées sont les morphismes naturels des espaces d'opérateurs, qui peuvent être vus comme des analogues non commutatifs des espaces de Banach. Plus concrètement, tout espace d'opérateurs s'identifie à un sous-espace fermé de l'algèbre des opérateurs bornés $\mathcal{B}(\mathcal{H})$ sur un espace de Hilbert $\mathcal{H}$. Les références principales pour une introduction à cette théorie sont [ER00] et [Pis03].

Soit X un ensemble et soit E un sous-espace de l'espace $\mathbb{C}^X$ de toutes les fonctions de X dans $\mathbb{C}$. On peut donc considérer les fonctions $\varphi : X \to \mathbb{C}$ vérifiant

$$\varphi f \in E, \quad \forall f \in E,$$

où φf est définie par multiplication point par point. Dans ce cas, on dit que φ est un multiplicateur de E et on observe qu'il définit une application linéaire sur E donnée par $f \mapsto \varphi f$. Selon la structure de l'espace E, on peut se demander si cette application est bornée ou même complètement bornée.

Dans cette thèse, nous nous intéressons à deux cas particuliers. Dans un premier temps, nous étudions les multiplicateurs de l'algèbre de Fourier $A(G)$ associée à un groupe localement compact G, ainsi que sa généralisation au cadre des espaces L_p, appelée algèbre de Figà-Talamanca–Herz. L'une des nombreuses caractérisations de la moyennabilité est donnée par l'existence d'une approximation de l'unité dans $A(G)$, bornée en norme, c'est-à-dire une suite généralisée (φ_i) dans $A(G)$ telle que

$$\|a - \varphi_i a\|_{A(G)} \to 0, \quad \forall a \in A(G),$$

et

$$\sup_i \|\varphi_i\|_{A(G)} < \infty.$$

Le fait que les fonctions φ_i définissent des multiplicateurs complètement bornées (c.b.) de $A(G)$ avec $\|\varphi_i\|_{cb} \leq \|\varphi_i\|_{A(G)}$ permet de généraliser cette propriété et considérer ainsi des formes plus faibles de moyennabilité.

La moyennabilité faible est définie de façon similaire, mais en imposant juste que les normes $\|\varphi_i\|_{cb}$ soient uniformément bornées. Une propriété encore plus faible , appelée AP, est définie en remplaçant les deux conditions par une convergence préfaible. Ces propriétés plus faibles ont été introduites dans [CH89] et [HK94] respectivement, mais l'idée derrière la moyenabilité faible était déjà présente dans [Haa79], et plusieurs propriétés de l'espace des multiplicateurs complètement bornés de l'algèbre de Fourier ont été démontrées dans [dCH85].

Le deuxième type de multiplicateurs étudié dans cette thèse est relié à l'algèbre des opérateurs bornés sur un espace de Hilbert. Si X est un ensemble non vide, chaque opérateur T sur $\ell_2(X)$ peut être vu comme une fonction de deux variables sur X donnée par

$$(x, y) \in X \times X \longmapsto T_{x,y} = \langle T\delta_y, \delta_x \rangle \in \mathbb{C}.$$

Une fonction $\phi : X \times X \to \mathbb{C}$ est appelée multiplicateur de Schur sur X si l'application

$$(T_{x,y})_{x,y \in X} \mapsto (\phi(x, y)T_{x,y})_{x,y \in X}$$

est bien définie sur l'algèbre des opérateurs bornés $\mathcal{B}(\ell_2(X))$. Un résultat important de Grothendieck [Gro53] assure que tels multiplicateurs sont toujours complètement bornés. Une très bonne introduction aux multiplicateurs de Schur se trouve dans [Pis01, Chap. 5].

Ces objets peuvent être définis pour n'importe quel ensemble X, mais plusieurs questions intéressantes émergent lorsque cet ensemble est muni d'une structure supplémentaire. Si X est l'ensemble des sommets d'un graphe connexe infini, alors on peut le munir de la distance combinatoire, qui est définie comme la longueur du chemin le plus court entre chaque paire de sommets. On dit qu'une fonction $\phi : X \times X \to \mathbb{C}$ est radiale s'il existe une autre fonction $\dot{\phi} : \mathbb{N} \to \mathbb{C}$ telle que

$$\phi(x, y) = \dot{\phi}(d(x, y)), \quad \forall x, y \in X.$$

Ceci permet de parler de multiplicateurs de Schur radiaux sur des graphes. Dans un manuscrit de 1987 qui est resté non publié pendant des années, Haagerup et Szwarc ont donné une caractérisation de ces multiplicateurs pour le graphe de Cayley du groupe libre $\mathbb{F}_n$. Ce résultat a été étendu par Wysoczański [Wys95] aux produits libres de groupes de même cardinalité. Plus tard, Haagerup, Steenstrup et Szwarc [HSS10] ont amélioré les résultats du manuscrit en incluant les arbres homogènes de degré arbitraire, et quelques applications aux groupes p-adiques $\mathrm{PGL}(2, \mathbb{Q}_p)$. Plus précisément, ils ont montré qu'une fonction $\dot{\phi} : \mathbb{N} \to \mathbb{C}$ définit un multiplicateur de Schur radial sur un arbre homogène de degré q ($3 \leq q \leq \infty$) si et seulement si la matrice de Hankel

$$(\dot{\phi}(i + j) - \dot{\phi}(i + j + 2))_{i,j \in \mathbb{N}}$$

est un opérateur à trace sur $\ell_2(\mathbb{N})$. Dans le même esprit, Mei et de la Salle [MdlS17] ont démontré que si la matrice

$$(\dot{\phi}(i + j) - \dot{\phi}(i + j + 1))_{i,j \in \mathbb{N}}$$

est à trace, alors $\dot{\phi}$ définit un multiplicateur de Schur radial sur tout graphe hyperbolique de degré borné. De plus, l'un des résultat dans [Wys95] affirme que cette condition caractérise les multiplicateurs de Schur radiaux sur le graphe de Cayley de $(\mathbb{Z}/3Z) * (\mathbb{Z}/3Z) * (\mathbb{Z}/3Z)$, qui est hyperbolique. Cela donne donc une caractérisation des fonctions définissant des multiplicateurs de Schur radiaux sur tout graphe hyperbolique à géométrie bornée.

Même si les deux types de multiplicateurs décrits ci-dessus s'originent dans des contextes très différents, ils sont intimement liés, comme il sera expliqué dans la section 1.4.1. Dans la suite, nous décrivons les deux problèmes principaux étudiés dans cette thèse.

La p-propriété d'approximation pour les groupes de Lie simples de centre fini

Dans [ALR10], An, Lee et Ruan ont étendu plusieurs définitions et résultats concernant les espaces d'opérateurs et les groupes au contexte des p-espaces d'opérateurs. Nous donnerons une définition précise de p-espace d'opérateurs dans la section 1.3.3, mais on peut déjà donner les exemples concrets des sous-espaces fermés de $\mathcal{B}(L_p)$, de façon analogue au cas des espaces d'opérateurs ($p = 2$). Cette généralisation trouve son origine dans [Pis90].

En utilisant le fait que l'algèbre de Figà-Talamanca–Herz $A_p(G)$ admet une structure de p-espace d'opérateurs, ils ont donné une définition de p-moyennabilité faible et de p-propriété d'approximation (p-AP) de façon très similaire au cas $p = 2$, en utilisant les multiplicateurs p-complètement bornés de $A_p(G)$.

La première partie de cette thèse est consacrée à l'étude de la p-AP pour les groupes de Lie simples de centre fini. La (2-)AP a été introduite par Haagerup et Kraus [HK94] sans donner d'exemple concret de groupe localement compact ne possédant pas cette propriété, mais ils ont conjecturé que $\mathrm{SL}(3,\mathbb{R})$ en serait un. Ceci a été confirmé par Lafforgue et de la Salle [LdlS11]. Plus tard, Haagerup et de Laat [HdL13] ont étendu ce résultat à tous les groupes de Lie simples de centre fini et de rang réel supérieur à 1. Par la suite, ils sont parvenus à enlever l'hypothèse de centre fini [HdL16]. Finalement, Haagerup, Knudby et de Laat [HKdL16] ont donné une caractérisation complète des groupes de Lie connexes possédant la AP. Le théorème suivant étend le résultat principal de [HdL13] à tout $p \in (1, \infty)$.

Théorème 1. *Soit G un groupe de Lie simple, connexe, de centre fini et rang réel supérieur à 1. Alors G n'a pas p-AP pour tout $p \in (1, \infty)$.*

Le théorème 1 est démontré dans le chapitre 3 (voir Thm. 3.1). C'est aussi le résultat principal de l'article [Ver17].

De même que dans [HdL13], la preuve du théorème 1 se concentre sur deux groupes particuliers : le groupe spécial linéaire $\mathrm{SL}(3,\mathbb{R})$ et le groupe symplectique $\mathrm{Sp}(2,\mathbb{R})$. Puisque tout groupe de Lie vérifiant les hypothèses du théorème 1 possède un sous-groupe fermé localement isomorphe soit à $\mathrm{SL}(3,\mathbb{R})$, soit à $\mathrm{Sp}(2,\mathbb{R})$, les propriétés de stabilité de la p-AP, démontrées dans le chapitre 2, permettent de se ramener à ces deux cas. L'idée de la preuve est de construire une suite de Cauchy de mesures sur $\mathrm{SL}(3,\mathbb{R})$ (resp. $\mathrm{Sp}(2,\mathbb{R})$) qui converge vers une forme linéaire qui ne peut pas exister lorsque le groupe a la p-AP. Ces mesures sont construites à l'aide d'un procédé de moyennisation sur un sous-groupe compact maximal. Cette idée a été conçue par Lafforgue [Laf08] pour montrer que $\mathrm{SL}(3,\mathbb{R})$ a la propriété (T) renforcée. Elle a été aussi utilisée plus tard dans [LdlS11] et [HdL13]. La difficulté principale que l'on rencontre en essayant d'adapter ces arguments pour la p-AP est le fait que les outils provenant de la théorie des espaces de Hilbert et des algèbres de von Neumann ne sont plus disponibles. Nous devons donc faire de nouvelles considérations.

Multiplicateurs de Schur radiaux sur quelques généralisations des arbres

La deuxième partie de cette thèse, correspondant au chapitre 4, se concentre sur quelques extensions de la caractérisation des multiplicateurs de Schur radiaux sur les arbres [HSS10] à quelques familles de graphes qui, dans un certain sens, généralisent les arbres. Ces résultats font l'objet de l'article [Ver18]. Soit $\dot{\phi} : \mathbb{N} \to \mathbb{C}$ une fonction. Définissons les dérivées discrètes

$$\begin{aligned} \mathfrak{d}_1\dot{\phi}(n) &= \dot{\phi}(n) - \dot{\phi}(n+1), \\ \mathfrak{d}_2\dot{\phi}(n) &= \dot{\phi}(n) - \dot{\phi}(n+2), \quad \forall n \in \mathbb{N}. \end{aligned}$$

Alors les caractérisations obtenues dans [HSS10], [Wys95] et [MdlS17] s'expriment en termes de $\mathfrak{d}_2$ et $\mathfrak{d}_1$. Nous introduisons cette notation car elle permet de définir des dérivées d'ordre supérieur par récurrence,

$$\mathfrak{d}_j^{m+1}\dot{\phi}(n) = \mathfrak{d}_j(\mathfrak{d}_j^m\dot{\phi})(n),$$

pour $j = 1, 2$ et $m \geq 1$. Avec ces notations, nous pouvons donner des caractérisations similaires pour les produits d'arbres et les produits de graphes hyperboliques.

Théorème 2. *Soit $N \geq 1$ un entier et soit X le produit de N arbres de degré minimal supérieur ou égal à 3. Si $\dot{\phi} : \mathbb{N} \to \mathbb{C}$ est une fonction bornée, alors elle définit un multiplicateur de Schur radial sur X si et seulement si la matrice de Hankel*

$$H = \Big((1+i+j)^{N-1} \mathfrak{d}_2^N \dot{\phi}(i+j) \Big)_{i,j\in\mathbb{N}}$$

est à trace.

Grâce à un théorème de Peller [Pel03], ce résultat admet une formulation équivalente en termes d'espaces de Besov sur $\mathbb{T}$. Plus précisément, $\dot{\phi}$ satisfait les conditions équivalentes du théorème 2 si et seulement si la fonction analytique

$$z \mapsto (1-z^2)^N \sum_{n\geq 0} \dot{\phi}(n) z^n$$

appartient à l'espace de Besov $B_1^N(\mathbb{T})$. Cet espace est défini dans l'annexe C.

Théorème 3. *Soit $N \geq 1$ un entier et soit $\dot{\phi} : \mathbb{N} \to \mathbb{C}$ une fonction bornée. Alors $\dot{\phi}$ définit un multiplicateur de Schur radial sur tout produit de N graphes hyperboliques de degré borné si et seulement si la matrice de Hankel*

$$H = \Big((1+i+j)^{N-1} \mathfrak{d}_1^N \dot{\phi}(i+j) \Big)_{i,j\in\mathbb{N}}$$

est à trace.

À nouveau, cette condition est équivalente à

$$(1-z)^N \sum_{n\geq 0} \dot{\phi}(n) z^n \in B_1^N(\mathbb{T}).$$

Des énoncés plus précis des résultats ci-dessus sont donnés dans la section 4.1 (voir Thm. A et Thm. B), avec des estimés des normes des multiplicateurs de Schur en termes de la norme de la matrice de Hankel H. Leurs preuves utilisent les mêmes idées de [HSS10] et [MdlS17], où la distance entre toute paire de sommets peut être caractérisée en fonction d'une géodésique infinie fixée. Pour les produits, on peut faire pareil en fixant une géodésique sur chaque composante et en utilisant le fait que la distance entre deux points est la somme des distances sur chaque coordonnée. Ainsi, pour montrer que $H \in S_1$ est une condition suffisante pour avoir un multiplicateur de Schur radial, on peut suivre la même stratégie car, dans ce cas, la matrice H est définie en termes de dérivée N-ième, ce qui permet de répéter l'argument sur chacune des N composantes. Le fait que cette condition est aussi nécessaire est plus délicat. Pour les produits d'arbres, la décomposition de Wold–von Neumann joue un rôle crucial (voir la section 4.2.2), permettant d'exprimer les multiplicateurs radiaux en fonction d'un opérateur à trace sur $\ell_2(\mathbb{N}^N)$ et des opérateurs de décalage associés à chacune des coordonnées. Pour les produits de graphes hyperboliques, il suffit d'étudier le groupe hyperbolique $\Gamma = (\mathbb{Z}/3\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z})$, qui se plonge dans l'arbre 3-homogène $\mathcal{T}_3$ en multipliant les distances par 2. Ceci permet de transformer certains multiplicateurs radiaux sur Γ^N en multiplicateurs multi-radiaux sur $\mathcal{T}_3^N$. Ces objets sont définis dans la section 4.2.

Comme conséquence du théorème 3, on déduit que tout groupe discret agissant proprement sur un produit de graphes hyperboliques de degré borné est faiblement moyennable. Nous démontrons cela dans la section 4.4.2. Grâce au théorème 2, la même affirmation est vraie pour les groupes agissant sur des produits d'arbres. Dans ce cas, nous ne faisons aucune hypothèse sur les degrés.

La dernière classe de graphes que nous étudions dans cette thèse correspond aux (1-squelettes des) complexes cubiques CAT(0) de dimension finie. Ces objets sont définis dans la section 4.6. Ce sont des généralisations des arbres car les arbres sont exactement les complexes cubiques CAT(0) de dimension 1. De plus, un produit de N arbres définit un complexe cubique CAT(0) de dimension N. Ainsi, le théorème 2 donne une condition nécessaire pour qu'une fonction définisse un multiplicateur de Schur radial sur tout complexe cubique CAT(0) de dimension N. Le résultat suivant affirme qu'une autre condition (plus forte) est suffisante.

Théorème 4. *Soit $N \geq 1$ un entier et soit $\dot{\phi} : \mathbb{N} \to \mathbb{C}$ une fonction telle que la matrice de Hankel*

$$H = \Big((1+i+j)^{N-1} (\dot{\phi}(i+j) - \dot{\phi}(i+j+2)) \Big)_{i,j\in\mathbb{N}}$$

est à trace. Alors $\dot{\phi}$ définit un multiplicateur de Schur radial sur tout complexe cubique CAT(0) de dimension inférieure ou égale à N.

On a aussi une formulation équivalente en termes d'espaces de Besov donnée par

$$(1 - z^2) \sum_{n \geq 0} \dot{\phi}(n) z^n \in B_1^N(\mathbb{T}).$$

Dans la section 4.1, nous énonçons à nouveau ce résultat avec des estimés pour la norme du multiplicateur (voir Thm. C).

Le lien avec la moyennabilité faible est toujours présent dans ce contexte. La preuve du théorème 4 est inspirée de [Miz08], où Mizuta montre que les groupes agissant proprement sur des complexes cubiques CAT(0) de dimension finie sont faiblement moyennables. De même, le résultat de Mei et de la Salle [MdlS17] sur les groupes hyperboliques est inspiré de [Oza08], où Ozawa montre que les groupes hyperboliques sont faiblement moyennables. Dans le cas des complexes cubiques CAT(0), il est toujours possible de caractériser la distance en fonction d'une géodésique fixée grâce à la structure d'espace médian. De plus, cette structure permet aussi d'obtenir des bornes pour la norme du multiplicateur en utilisant un argument combinatoire très astucieux introduit dans [Miz08].

Organisation du manuscrit

Le premier chapitre se compose essentiellement de définitions et quelques résultats classiques. Entre autres, nous introduisons les espaces d'opérateurs, les multiplicateurs de Schur et les différentes propriétés d'approximation étudiés dans cette thèse.

Le chapitre 2 explore la p-AP. Nous établissons quelques résultats généraux sur cette propriété, sur l'algèbre de Figà-Talamanca–Herz $A_p(G)$ et sur ses multiplicateurs p-complètement bornés. Nous abordons aussi une question très intéressante sur les algèbres des convoluteurs et des pseudo-mesures.

Le troisième chapitre est consacré à la preuve du fait que les groupes de Lie simples de centre fini et de rang supérieur n'ont pas p-AP pour tout $p \in (1, \infty)$ (Thm. 3.1 dans cette thèse et Thm. 1.5 dans [Ver17]). Une grande partie du chapitre se concentre sur les groupes $\mathrm{SL}(3, \mathbb{R})$ et $\mathrm{Sp}(2, \mathbb{R})$ car il suffit d'étudier ces deux groupes grâce aux propriétés de stabilité obtenues dans le chapitre 2.

Finalement, le chapitre 4 est consacré à l'étude des multiplicateurs de Schur radiaux et aux preuves des caractérisations décrites ci-dessus (Thms. A, B et C dans cette thèse et dans [Ver18]). Il s'agit d'un chapitre assez auto-contenu, qui repose uniquement sur la définition des multiplicateurs de Schur donnée dans la section 1.3.2 et sur les résultats des annexes B, C et D.

Introduction

This thesis focusses on two main subjects, which are related by a key concept: the notion of completely bounded multipliers. Completely bounded maps are the natural morphisms between operator spaces, which may be viewed as noncommutative analogues of Banach spaces. More concretely, every operator space corresponds to a closed subspace of the algebra of bounded operators $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$. The books [ER00] and [Pis03] are the main references for an introduction to this theory.

Let X be a set and E be a subspace of the space $\mathbb{C}^X$ of complex-valued functions on X. Then we may consider the functions $\varphi : X \to \mathbb{C}$ which satisfy

$$\varphi f \in E, \quad \forall f \in E,$$

where φf is defined by pointwise multiplication. We say that φ is a multiplier of E. In that case, it defines a linear map on E given by $f \mapsto \varphi f$. Depending on the structure of E, we can ask whether this map is bounded or even more, completely bounded.

Two particular cases are of special interest in the present thesis. The first one involves multipliers of the Fourier algebra $A(G)$ associated to a locally compact group G, and its generalisation to the context of L_p spaces, called the Figà-Talamanca–Herz algebra $A_p(G)$. One of the many characterisations of amenability for a group G is the existence of a bounded approximate identity in $A(G)$, namely, a net (φ_i) in $A(G)$ satisfying

$$\|a - \varphi_i a\|_{A(G)} \to 0, \quad \forall a \in A(G),$$

and

$$\sup_i \|\varphi_i\|_{A(G)} < \infty.$$

The fact that the functions φ_i necessarily define completely bounded (c.b.) multipliers of $A(G)$ with $\|\varphi_i\|_{cb} \leq \|\varphi_i\|_{A(G)}$ allows one to generalise this property, and hence, consider weaker forms of amenability.

Weak amenability is defined analogously, but only requiring the norms $\|\varphi_i\|_{cb}$ to be uniformly bounded. An even weaker property, called AP, is defined by replacing both conditions by a weak* convergence. These weaker approximation properties were first defined in [CH89] and [HK94] respectively, although the idea of weak amenability was already present in [Haa79], and many properties of the space of completely bounded multipliers of the Fourier algebra were obtained in [dCH85].

The other notion of multipliers which this thesis addresses is related to the algebra of bounded operators on a Hilbert space. If X is a nonempty set, a bounded operator

T on $\ell_2(X)$ may be viewed as a two variable function on X given by

$$(x,y) \in X \times X \longmapsto T_{x,y} = \langle T\delta_y, \delta_x \rangle \in \mathbb{C}.$$

A function $\phi : X \times X \to \mathbb{C}$ is called Schur multiplier on X if the map

$$(T_{x,y})_{x,y\in X} \mapsto (\phi(x,y)T_{x,y})_{x,y\in X}$$

is well defined in the algebra of bounded operators $\mathcal{B}(\ell_2(X))$. An important result of Grothendieck [Gro53] implies that such multipliers are always completely bounded. A very good introduction to Schur multipliers can be found in [Pis01, Chapter 5].

Although these objects can be defined for any set X, many interesting questions arise when the set has some additional structure. If X is the set of vertices of a connected infinite graph, it can be endowed with the combinatorial distance, which is defined as the length of the shortest path between each pair of vertices. Let $\mathbb{N}$ denote the natural numbers $\{0,1,2,...\}$. We say that a function $\phi : X \times X \to \mathbb{C}$ is radial if there exists $\dot{\phi} : \mathbb{N} \to \mathbb{C}$ such that

$$\phi(x,y) = \dot{\phi}(d(x,y)), \quad \forall x,y \in X.$$

This allows talking about radial Schur multipliers on graphs. In a manuscript from 1987 that remained unpublished for many years, Haagerup and Szwarc gave a characterisation of these multipliers for the Cayley graph of the free group $\mathbb{F}_n$. This was extended by Wysoczański [Wys95] to free products of groups of same cardinality. Later, Haagerup, Steenstrup and Szwarc [HSS10] improved the results from the unpublished manuscript in order to cover homogeneous trees of arbitrary degree, and some applications to the p-adic groups $\mathrm{PGL}(2,\mathbb{Q}_p)$. More precisely, they proved that a function $\dot{\phi} : \mathbb{N} \to \mathbb{C}$ defines a radial Schur multiplier on any homogeneous tree of degree q ($3 \leq q \leq \infty$) if and only if the Hankel matrix

$$(\dot{\phi}(i+j) - \dot{\phi}(i+j+2))_{i,j\in\mathbb{N}}$$

defines a trace class operator on $\ell_2(\mathbb{N})$. In the same spirit, Mei and de la Salle [MdlS17] proved that if the matrix

$$(\dot{\phi}(i+j) - \dot{\phi}(i+j+1))_{i,j\in\mathbb{N}}$$

is of trace class, then $\dot{\phi}$ defines a radial Schur multiplier on any hyperbolic graph of bounded degree. Furthermore, one of the results in [Wys95] implies that this condition actually characterises radial Schur multipliers on the Cayley graph of $(\mathbb{Z}/3Z) * (\mathbb{Z}/3Z) * (\mathbb{Z}/3Z)$, which is hyperbolic. Hence, this provides a characterisation of functions defining radial Schur multipliers on every hyperbolic graph of bounded geometry.

Although the two kinds of multipliers described above arise from quite different contexts, they are intimately related, as it will be explained in Section 1.4.1. In what follows, the two main problems considered in this thesis are described.

The p-approximation property for simple Lie groups with finite centre

In [ALR10], An, Lee and Ruan extended many definitions and results related to operator spaces and groups to the context of p-operator spaces. The precise definition of p-operator space will be given in Section 1.3.3, but one may have in mind the concrete examples given by closed subspaces of $\mathcal{B}(L_p)$, in analogy with the case of operator spaces ($p = 2$). This generalisation finds its origin in [Pis90].

The fact that the Figà-Talamanca–Herz algebra $A_p(G)$ admits a p-operator space structure allowed them to define p-weak amenability and the p-approximation property (p-AP) in a very similar way to the case $p = 2$, by means of the p-completely bounded multipliers of $A_p(G)$.

The first part of this thesis is devoted to the study of the p-AP for simple Lie groups with finite centre. When the (2-)AP was defined by Haagerup and Kraus [HK94], they did not provide any concrete example of a locally compact group failing this property, but they conjectured that $\mathrm{SL}(3, \mathbb{R})$ would be one. This was confirmed by Lafforgue and de la Salle [LdlS11]. After this, Haagerup and de Laat [HdL13] were able to extend this result, and proved that the AP is not satisfied by any simple Lie group with finite centre and real rank greater than 1. Afterwards, they were able to remove the assumption of finite centre [HdL16]. Finally, Haagerup, Knudby and de Laat [HKdL16] gave a complete characterisation of connected Lie groups with the AP. The following theorem extends the main result of [HdL13] to all $p \in (1, \infty)$.

Theorem 1. *Let G be a connected simple Lie group with finite centre and real rank greater than 1. Then G does not satisfy the p-AP for any $p \in (1, \infty)$. The same holds for any lattice in G.*

Theorem 1 will be proved in Chapter 3, where it is restated as Theorem 3.1. It is also the main result of the article [Ver17].

As in [HdL13], the proof of Theorem 1 concentrates on two particular groups: the special linear group $\mathrm{SL}(3, \mathbb{R})$ and the symplectic group $\mathrm{Sp}(2, \mathbb{R})$. Since every Lie group satisfying the hypotheses of Theorem 1 possesses a closed subgroup which is locally isomorphic to either $\mathrm{SL}(3, \mathbb{R})$ or $\mathrm{Sp}(2, \mathbb{R})$, the stability properties of the p-AP, which are proved in Chapter 2, make it possible to reduce the analysis to these two cases. The idea of the proof is to construct a Cauchy net of measures on $\mathrm{SL}(3, \mathbb{R})$ (resp. $\mathrm{Sp}(2, \mathbb{R})$) converging to a linear form that cannot exist if the group has the p-AP. These measures are constructed by means of an averaging process over a maximal compact subgroup. This idea was originally devised by Lafforgue [Laf08] in order to prove that $\mathrm{SL}(3, \mathbb{R})$ has strong property (T), and it was later exploited in [LdlS11] and [HdL13]. The main difficulty when trying to adapt these arguments for the p-AP is that the tools from Hilbert spaces and von Neumann algebras are no longer available, and hence some new considerations must be made.

Radial Schur multipliers on some generalisations of trees

The second part of the thesis, which corresponds to Chapter 4, focusses on some extensions of Haagerup, Steenstrup and Szwarc's characterisation [HSS10] of radial

Schur multipliers to some other classes of graphs which, in some sense, generalise trees. These results can be also found in [Ver18]. For a function $\dot{\phi} : \mathbb{N} \to \mathbb{C}$, define the discrete derivatives

$$\begin{aligned} \mathfrak{d}_1\dot{\phi}(n) &= \dot{\phi}(n) - \dot{\phi}(n+1), \\ \mathfrak{d}_2\dot{\phi}(n) &= \dot{\phi}(n) - \dot{\phi}(n+2), \quad \forall n \in \mathbb{N}. \end{aligned}$$

Then, as explained above, the characterisations given in [HSS10], [Wys95] and [MdlS17] can be expressed in terms of $\mathfrak{d}_2$ and $\mathfrak{d}_1$. The purpose of introducing this notation is that it allows the definition of higher order derivatives by induction,

$$\mathfrak{d}_j^{m+1}\dot{\phi}(n) = \mathfrak{d}_j(\mathfrak{d}_j^m\dot{\phi})(n),$$

for $j = 1, 2$ and $m \geq 1$. With these notations, similar characterisations can be given for products of trees and products of hyperbolic graphs.

Theorem 2. *Let $N \geq 1$ be an integer, and let X be the product of N trees such that the degree of each vertex is at least 3. Then a bounded function $\dot{\phi} : \mathbb{N} \to \mathbb{C}$ defines a radial Schur multiplier on X if and only if the Hankel matrix*

$$H = \left((1+i+j)^{N-1}\mathfrak{d}_2^N\dot{\phi}(i+j)\right)_{i,j\in\mathbb{N}}$$

is of trace class.

By a theorem of Peller [Pel03], this result admits an equivalent formulation in terms of Besov spaces on the torus $\mathbb{T}$. Namely, $\dot{\phi}$ satisfies the equivalent conditions of Theorem 2 if and only if the analytic function

$$z \mapsto (1-z^2)^N \sum_{n\geq 0} \dot{\phi}(n)z^n$$

belongs to the Besov space $B_1^N(\mathbb{T})$. This space is defined in Appendix C.

Theorem 3. *Let $N \geq 1$ and let $\dot{\phi} : \mathbb{N} \to \mathbb{C}$ be a bounded function. Then $\dot{\phi}$ defines a radial Schur multiplier on every product of N hyperbolic graphs with bounded degree if and only if the Hankel matrix*

$$H = \left((1+i+j)^{N-1}\mathfrak{d}_1^N\dot{\phi}(i+j)\right)_{i,j\in\mathbb{N}}$$

is of trace class.

Again, this condition is equivalent to

$$(1-z)^N \sum_{n\geq 0} \dot{\phi}(n)z^n \in B_1^N(\mathbb{T}).$$

More precise statements of the above results are given in Section 4.1 (see Theorems A and B), including estimates of the norms of the Schur multipliers in terms of the trace class norm of the Hankel matrix H. The proofs follow the same ideas as in [HSS10] and [MdlS17], where the distance between any pair of vertices can be characterised in terms of a fixed infinite geodesic on the graph. For a product, the same can be done by fixing an infinite geodesic on each component, and then using the fact that the distance between two points is the sum of the distances on each coordinate. Then the proof that the condition $H \in S_1$ is sufficient to have a radial Schur multiplier follows analogously, since now H is defined in terms of the N-th derivative, which makes it possible to repeat the argument on each of the N components. The fact that this condition is also necessary requires more work. For products of trees, a crucial role is played by the Wold–von Neumann decomposition for double commuting isometries (see Section 4.2.2), which allows one to express radial multipliers in terms of a trace class operator on $\ell_2(\mathbb{N}^N)$ and the N forward shift operators associated to each coordinate. For products of hyperbolic graphs, it is sufficient to study the hyperbolic group $\Gamma = (\mathbb{Z}/3\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z})$, which embeds in the 3-homogeneous tree $\mathcal{T}_3$ by doubling distances. This provides a way to transform certain radial multipliers on Γ^N into multi-radial multipliers on $\mathcal{T}_3^N$. These objects are defined in Section 4.2.

A consequence of Theorem 3 is that discrete groups acting properly on products of hyperbolic graphs with bounded degrees are weakly amenable. A proof of this fact can be found in Section 4.4.2. The same can be said about groups acting properly on finite products of trees, using Theorem 2 instead. Here there is no assumption on the degrees.

The last class of graphs considered in this thesis corresponds to (1-skeletons of) finite dimensional CAT(0) cube complexes, or equivalently, median graphs [Che00]. A precise definition will be given in Section 4.6. They generalise trees, in the sense that trees are exactly the 1-dimensional CAT(0) cube complexes. Furthermore, a product of N trees defines an N-dimensional CAT(0) cube complex. Hence, Theorem 2 provides a necessary condition for a function to define a radial Schur multiplier on every N-dimensional CAT(0) cube complex. The following result asserts that another (stronger) condition is sufficient.

Theorem 4. *Let $N \geq 1$ be an integer, and let $\dot{\phi} : \mathbb{N} \to \mathbb{C}$ be a function such that the Hankel matrix*

$$H = \Big((1+i+j)^{N-1}(\dot{\phi}(i+j) - \dot{\phi}(i+j+2))\Big)_{i,j\in\mathbb{N}}$$

is of trace class. Then $\dot{\phi}$ defines a radial Schur multiplier on every CAT(0) cube complex of dimension at most N.

Once again, there is an equivalent formulation in terms of Besov spaces given by

$$(1-z^2)\sum_{n\geq 0}\dot{\phi}(n)z^n \in B_1^N(\mathbb{T}).$$

As before, this result is stated again in Section 4.1 (see Theorem C), including an estimate of the norm of the Schur multiplier.

The link with weak amenability is still present in this context. In fact, the proof of Theorem 4 is inspired by Mizuta's proof [Miz08] that groups acting properly on finite dimensional CAT(0) cube complexes are weakly amenable; in the same way as Mei and de la Salle's result [MdlS17] for hyperbolic graphs was inspired by Ozawa's proof [Oza08] that hyperbolic groups are weakly amenable. In this case, it is still possible to characterise the distance by a single fixed infinite geodesic, thanks to the median structure of the graph. Moreover, this structure also allows one to obtain similar bounds for the norms by means of a very clever combinatorial argument devised in [Miz08].

Organisation of the manuscript

The first chapter consists mainly of definitions and some classical results. In particular, we define operator spaces, Schur multipliers and the different approximation properties of groups considered in this thesis.

Chapter 2 explores the p-approximation property, providing some general results concerning this property, the Figà-Talamanca–Herz algebra $A_p(G)$ and its p-completely bounded multipliers. We also discuss a very interesting question regarding the algebras of convoluters and pseudo-measures.

The third chapter is devoted to the proof that higher rank simple Lie groups with finite centre do not satisfy p-AP for any $p \in (1, \infty)$ (Theorem 3.1 in this thesis and Theorem 1.5 in [Ver17]). Most of the chapter deals with the groups $\mathrm{SL}(3, \mathbb{R})$ and $\mathrm{Sp}(2, \mathbb{R})$, since obtaining the result for these two groups is sufficient to prove it in the general case thanks to the stability properties of the p-AP proven in Chapter 2.

Finally, the fourth chapter concentrates on radial Schur multipliers and the proofs of the characterisations described above (Theorems A, B and C in this thesis, as well as in [Ver18]). It is a somewhat self-contained chapter, which relies only on the definition of Schur multipliers given in Section 1.3.2 and the results from Appendices B, C and D.

Chapter 1

Preliminaries

1.1 Basic definitions and notations

Unless otherwise stated, all normed spaces considered in this thesis are assumed to be over the field of complex numbers $\mathbb{C}$. Given a normed space E, we denote by E^* its dual space, and by $\mathcal{B}(E)$ the algebra of bounded operators on E. The space of $n \times n$ matrices with coefficients in E is denoted by $M_n(E)$, and M_n stands for the particular case $E = \mathbb{C}$. Given $x \in M_n(E)$ and $y \in M_m(E)$, we write $x \oplus y$ for the element of $M_{n+m}(E)$ defined by

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

Let E and F be two Banach spaces. We denote by $E \hat{\otimes} F$ their projective tensor product, namely, the completion of the algebraic tensor product for the norm

$$\|x\| = \inf \left\{ \sum_i \|u_i\| \|v_i\| \ : \ x = \sum_i u_i \otimes v_i \right\}.$$

For two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, we simply write $\mathcal{H} \otimes \mathcal{K}$ for the tensor product Hilbert space, and $\mathcal{H} \oplus \mathcal{K}$ for the ℓ_2-direct sum. We shall also write $\mathcal{H}^{\oplus n} = \underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{n \text{ times}}$.

1.2 Operator algebras and groups

A C*-algebra is a Banach $*$-algebra $\mathcal{A}$ satisfying

$$\|x^* x\| = \|x\|^2, \quad \forall x \in \mathcal{A}.$$

The Gelfand-Naimark theorem [GN43] states that every C*-algebra can be identified with a closed sub-$*$-algebra of $\mathcal{B}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$. A von Neumann algebra is a C*-algebra which is closed for the weak operator topology (WOT) of $\mathcal{B}(\mathcal{H})$ and contains the identity operator. A very important characterisation of these algebras is given by von Neumann's bicommutant theorem [vN30]. Given a subset S of $\mathcal{B}(\mathcal{H})$, we define its commutant S' as

$$S' = \{u \in \mathcal{B}(\mathcal{H}) \,:\, uv = vu, \ \forall v \in S\}.$$

The bicommutant S'' is the commutant of S'.

Theorem 1.1 (von Neumann [vN30])**.** *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{A}$ a sub-$*$-algebra of $\mathcal{B}(\mathcal{H})$ containing the identity. Then $\mathcal{A}$ is a von Neumann algebra if and only if $\mathcal{A}'' = \mathcal{A}$.*

Let G be a locally compact group endowed with a left Haar measure. The left regular representation $\lambda : G \to \mathcal{B}(L_2(G))$ is defined by

$$\lambda(s)f = f(s^{-1}\,\cdot\,), \quad \forall f \in L_2(G), \ \forall s \in G. \tag{1.1}$$

This representation extends to $L_1(G)$ in the following way. For $h \in L_1(G)$, we put

$$\lambda(h)f = \int_G h(s)\lambda(s)f(\,\cdot\,)\,ds = \int_G h(s)f(s^{-1}\,\cdot\,)\,ds, \quad \forall f \in L_2(G).$$

The reduced C*-algebra $C_r^*(G)$ is defined as the norm closure of $\lambda(L_1(G))$ in $\mathcal{B}(L_2(G))$. The group von Neumann algebra $\mathcal{L}(G)$ is defined analogously by taking the WOT-closure instead. Thanks to Theorem 1.1, $\mathcal{L}(G)$ admits the following description:

$$\mathcal{L}(G) = \{\lambda(s) \,:\, s \in G\}''. \tag{1.2}$$

1.3 Operator spaces

1.3.1 Operator spaces and completely bounded maps

An (abstract) operator space is a Banach space E together with a family of norms α_n on $M_n(E)$ $(n \geq 1)$ satisfying

(R1) For all $m, n \geq 1$, $x \in M_n(E)$ and $y \in M_m(E)$,

$$\alpha_{n+m}(x \oplus y) = \max\{\alpha_n(x), \alpha_m(y)\}.$$

(R2) For all $n \geq 1$, $a, b \in M_n = \mathcal{B}(\ell_2^n)$ and $x \in M_n(E)$,

$$\alpha_n(axb) \leq \|a\|\|b\|\alpha_n(x).$$

If $\mathcal{H}$ is a Hilbert space and E is a closed subspace of $\mathcal{B}(\mathcal{H})$, then E possesses a canonical operator space structure given by the identification $M_n(\mathcal{B}(\mathcal{H})) \cong \mathcal{B}(\mathcal{H}^{\oplus n})$. In other words, α_n is simply the operator norm on $\mathcal{B}(\mathcal{H}^{\oplus n})$. In this case, we say that E is a concrete operator space. Ruan's theorem (Theorem 1.2 below) asserts that the two notions are actually equivalent.

Let E and F be two operator spaces and $T : E \to F$ a linear map. Define, for each $n \geq 1$, a new linear map $T_n : M_n(E) \to M_n(F)$ by

$$(T_n(x))_{ij} = Tx_{ij}, \quad \forall i, j \in \{1, ..., n\}. \tag{1.3}$$

We say that T is completely bounded (c.b.) if the operators T_n are uniformly bounded. In that case, we write

$$\|T\|_{cb} = \sup_{n\geq 1} \|T_n\|_{M_n(E)\to M_n(F)}.$$

We say that T is a complete isometry if all the maps T_n are isometries.

Theorem 1.2 (Ruan [Rua88]). *Let E be an abstract operator space, then there exists a Hilbert space $\mathcal{H}$ and a completely isometric embedding $E \to \mathcal{B}(\mathcal{H})$.*

1.3.2 Schur multipliers

For any nonempty set X, let $\ell_2(X)$ be the Hilbert space of square-summable complex-valued functions on X. For each bounded operator $T \in \mathcal{B}(\ell_2(X))$, we may define its matrix coefficients by

$$T_{x,y} = \langle T\delta_y, \delta_x \rangle, \quad \forall x, y \in X. \tag{1.4}$$

Observe that these coefficients completely determine the operator T. We say that a function $\phi : X \times X \to \mathbb{C}$ is a Schur multiplier on X if the map $M_\phi : \mathcal{B}(\ell_2(X)) \to \mathcal{B}(\ell_2(X))$ given by

$$(M_\phi T)_{x,y} = \phi(x, y)T_{x,y}$$

is well defined. In that case, the closed graph theorem implies that M_ϕ is a bounded operator on $\mathcal{B}(\ell_2(X))$. The following result, due essentially to Grothendieck [Gro53], gives a very useful characterisation of Schur multipliers. A proof can be found in [Pis01, Theorem 5.1].

Theorem 1.3 (Grothendieck). *Let X be a nonempty set, $\phi : X \times X \to \mathbb{C}$ a function, and $C \geq 0$ a constant. The following are equivalent:*

(i) The function ϕ is a Schur multiplier and $\|M_\phi\| \leq C$.

(ii) The function ϕ is a Schur multiplier and the operator M_ϕ is completely bounded, with $\|M_\phi\|_{cb} \leq C$.

(iii) There exist a Hilbert space $\mathcal{H}$ and bounded functions $P, Q : X \to \mathcal{H}$ such that

$$\phi(x, y) = \langle P(x), Q(y) \rangle, \quad \forall x, y \in X,$$

and

$$\left(\sup_{x\in X} \|P(x)\|\right)\left(\sup_{y\in X} \|Q(y)\|\right) \leq C.$$

Motivated by the equivalence (i) $\iff$ (ii) in the previous theorem, we shall write

$$\|\phi\|_{cb} = \|M_\phi\|.$$

Let G be a (discrete) group. We say that a function $\varphi : G \to \mathbb{C}$ is a Herz-Schur multiplier if the function $(s,t) \in G \times G \mapsto \varphi(st^{-1}) \in \mathbb{C}$ is a Schur multiplier on G.

1.3.3 p-Operator spaces

For $1 < p < \infty$, a p-operator space is defined as a Banach space E together with a family of norms α_n on $M_n(E)$ $(n \geq 1)$ satisfying

(R1) For all $m, n \geq 1$, $x \in M_n(E)$ and $y \in M_m(E)$,

$$\alpha_{n+m}(x \oplus y) = \max\{\alpha_n(x), \alpha_m(y)\}.$$

(R2$_p$) For all $n \geq 1$, $a, b \in M_n = \mathcal{B}(\ell_p)$ and $x \in M_n(E)$,

$$\alpha_n(axb) \leq \|a\|\|b\|\alpha_n(x).$$

A linear map $T : E \to F$ between two p-operator spaces is said to be p-completely bounded if

$$\|T\|_{p\text{-}cb} = \sup_{n \geq 1} \|T_n\|_{M_n(E) \to M_n(F)} < \infty,$$

where T_n is defined as in (1.3). It is a p-complete isometry if T_n is an isometry for all n.

Let SQ_p the class of Banach spaces which are quotients of subspaces (or equivalently subspaces of quotients) of L_p spaces. For $X \in SQ_p$, a closed subspace of $\mathcal{B}(X)$ admits a canonical structure of p-operator space, just like in the case of operator spaces. Moreover, Le Merdy [LM96] proved an analogue of Ruan's theorem in this more general context.

Theorem 1.4 (Le Merdy [LM96]). *Let E be a p-operator space. Then there exists $X \in SQ_p$ and a p-completely isometric embedding $E \to \mathcal{B}(X)$.*

For a locally compact group G, there exist analogues of $C_r^*(G)$ and $\mathcal{L}(G)$ in this setting. Let $\lambda_p : G \to \mathcal{B}(L_p(G))$ be the left regular representation, which is defined in the same way as before, by replacing L_2 by L_p. The algebra of p-pseudo-functions $PF_p(G)$ is the norm closure of $\lambda_p(L_1(G))$ in $\mathcal{B}(L_p(G))$. Let p' be the Hölder conjugate of p $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$. Then the dual of the projective tensor product $L_{p'}(G)\hat{\otimes}L_p(G)$ may be identified with $\mathcal{B}(L_p(G))$ by

$$\langle T, g \otimes f\rangle = \langle g, T(f)\rangle, \quad \forall T \in \mathcal{B}(L_p(G)),\ \forall g \in L_{p'}(G),\ \forall f \in L_p(G).$$

The algebra of p-pseudo-measures $PM_p(G)$ is defined as the weak*-closed linear span of $\lambda_p(G)$ in $\mathcal{B}(L_p(G))$. For $p = 2$, we have $PF_2(G) = C_r^*(G)$ and $PM_p(G) = \mathcal{L}(G)$.

There is another possible generalisation of the group von Neumann algebra $\mathcal{L}(G)$ in this context, which is motivated by the characterisation (1.2). The algebra of p-convoluters $CV_p(G)$ is defined as the commutant of $\rho_p(G)$, where $\rho_p : G \to \mathcal{B}(L_p(G))$ stands for the right regular representation

$$\rho_p(s)f = \Delta(s)^{\frac{1}{p}} f(\,\cdot\, s),$$

and $\Delta : G \to (0,\infty)$ is the modular function. This is the classical definition of $CV_p(G)$, but it was proven by Daws and Spronk [DS19] that

$$CV_p(G) = PM_p(G)'' = \lambda_p(G)''.$$

Hence, it is always true that $PM_p(G) \subseteq CV_p(G)$. It is not known whether the other inclusion holds in general, but it has been proved for a very large class of groups, as it is explained in Section 2.4.

1.4 Approximation properties of groups

1.4.1 Locally compact groups and the Fourier algebra

A locally compact group G is said to be amenable if it admits a left-invariant mean, that is, a linear functional μ on $L_\infty(G)$ of norm 1 such that

(i) $\mu(f) \geq 0$ for all $f \in L_\infty(G)$ such that $f \geq 0$ a.e.

(ii) $\mu(s \cdot f) = \mu(f)$ for all $s \in G$ and all $f \in L_\infty(G)$, where $(s \cdot f)(t) = f(s^{-1}t)$.

The Fourier algebra of G is the subalgebra of $C_0(G)$ given by functions of the form

$$a(s) = \langle g, \lambda(s) f \rangle = g * \check{f}(s), \quad \forall s \in G,$$

where $f, g \in L_2(G)$, $\check{f}(t) = f(t^{-1})$, and λ is the left regular representation (1.1). We denote it by $A(G)$, and endow it with the norm

$$\|a\|_{A(G)} = \inf \left\{ \|g\|_2 \|f\|_2 \mid a = g * \check{f},\ f, g \in L_2(G) \right\}.$$

It was originally defined by Eymard in [Eym64], where it is also proved that it is a Banach algebra. The group von Neumann algebra $\mathcal{L}(G)$ can be identified with the dual of $A(G)$ by

$$\langle T, g * \check{f} \rangle = \langle g, T(f) \rangle, \quad \forall T \in \mathcal{L}(G),\ \forall f, g \in L_2(G).$$

If $\mathcal{A}$ is a commutative Banach algebra, we say that a net (φ_i) in $\mathcal{A}$ is an approximate identity (a.i.) if

$$\|\varphi_i a - a\| \to 0, \quad \forall a \in \mathcal{A}.$$

Theorem 1.5 (Leptin [Lep68]). *Let G be a locally compact group. Then G is amenable if and only if $A(G)$ possesses a (norm) bounded approximate identity.*

A function $\varphi : G \to \mathbb{C}$ is said to be a multiplier of $A(G)$ if the linear map $m_\varphi : A(G) \to A(G)$ given by

$$(m_\varphi a)(s) = \varphi(s)a(s), \quad \forall s \in G,$$

is well defined. In that case, an application of the closed graph theorem shows that m_φ is bounded. Moreover, by duality, m_φ defines a bounded operator $M_\varphi = m_\varphi^* \in \mathcal{B}(\mathcal{L}(G))$ satisfying

$$M_\varphi \lambda(s) = \varphi(s)\lambda(s), \quad \forall s \in G.$$

We denote by $M(G)$ the space of multipliers of $A(G)$, also called Fourier multipliers, endowed with the norm

$$\|\varphi\|_{M(G)} = \|m_\varphi\|_{A(G)\to A(G)} = \|M_\varphi\|_{\mathcal{L}(G)\to\mathcal{L}(G)}.$$

Since $\mathcal{L}(G)$ is a von Neumann algebra, it is in particular a concrete operator space. We say that $\varphi \in M(G)$ is a completely bounded multiplier of $A(G)$ if the operator M_φ is completely bounded. We denote by $M_0(G)$ the space of such multipliers and we endow it with the norm

$$\|\varphi\|_{M_0(G)} = \|M_\varphi\|_{cb}.$$

This space can also be defined using the fact that $A(G)$ admits an operator space structure; however, this is less straightforward.

Suppose now that Γ is a discrete group and take $x \in \lambda(\mathbb{C}[\Gamma]) \subset \mathcal{L}(\Gamma) \subset \mathcal{B}(\ell_2(\Gamma))$. This means that x can be written as

$$x = \sum_{s\in G} x_s \lambda(s),$$

where all but finitely many of the coefficients $x_s \in \mathbb{C}$ are zero. Moreover, $M_\varphi x = \sum \varphi(s) x_s \lambda(s)$. Then, using the matrix notation from (1.4), for all $s, t \in \Gamma$,

$$(M_\varphi x)_{s,t} = \sum_{r\in\Gamma} \varphi(r) x_r \langle \delta_{rt}, \delta_s \rangle = \varphi(st^{-1}) x_{s,t}.$$

This suggest that completely bounded Fourier multipliers and Herz-Schur multipliers are related. In fact, Bożejko and Fendler [BF84] proved that they are the same in the more general context of locally compact groups. Their proof relies on a result proved by J. E. Gilbert in an unpublished manuscript, which extends the decomposition of multipliers given by Theorem 1.3 to this setting. Later, Jolissaint [Jol92] gave a shorter proof of this fact.

Theorem 1.6 (Bożejko–Fendler [BF84])**.** *Let G be a locally compact group, $\varphi : G \to \mathbb{C}$ a function, and $C > 0$ a constant. The following are equivalent.*

a) The function φ belongs to $M_0(G)$ and $\|\varphi\|_{M_0(G)} \le C$.

b) There is a Hilbert space $\mathcal{H}$ and continuous bounded functions $\alpha, \beta : G \to \mathcal{H}$ such that

$$\varphi(st^{-1}) = \langle \beta(s), \alpha(t) \rangle, \quad \forall s, t \in G,$$

and

$$\left(\sup_{s \in G} \|\beta(s)\| \right) \left(\sup_{t \in G} \|\alpha(t)\| \right) \le C.$$

We have the following contractive inclusions.

$$A(G) \hookrightarrow M_0(G) \hookrightarrow M(G) \hookrightarrow C_b(G),$$

where $C_b(G)$ stands for the space of continuous bounded functions on G with the supremum norm. In particular, a bounded approximate identity for $A(G)$ is bounded as a net in $M_0(G)$. This motivates the following definition.

Definition 1.7. *A locally compact group G is said to be weakly amenable if $A(G)$ possesses an approximate identity (φ_i) such that*

$$\sup_i \|\varphi_i\|_{M_0(G)} \le C,$$

for some constant $C > 0$. The Cowling-Haagerup constant $\Lambda(G)$ is defined as the infimum of all such C over all possible approximate identities.

Now consider the injective contraction $L_1(G) \to M_0(G)^*$ given by

$$\langle f, \varphi \rangle = \int_G f(t)\varphi(t)\, dt, \quad \forall \varphi \in M_0(G),\ \forall f \in L_1(G). \tag{1.5}$$

Let $Q(G)$ be the norm closure of $L_1(G)$ in $M_0(G)^*$. Then every $\varphi \in M_0(G)$ defines a bounded linear form on $Q(G)$. Moreover, we have the following.

Theorem 1.8 (De Cannière–Haagerup [dCH85])**.** *The space $M_0(G)$ is isometrically isomorphic to the dual space of $Q(G)$.*

Theorem 1.9 (Haagerup–Kraus [HK94])**.** *Let G be a locally compact group and $C > 0$. The following are equivalent*

a) G is weakly amenable with $\Lambda(G) \le C$.

b) The constant function 1 is in the $\sigma(M_0(G), Q(G))$-closure of the set

$$\{a \in A(G) \,:\, \|a\|_{M_0} \le C\}.$$

This motivates the following definition.

Definition 1.10. *A locally compact group G is said to have the approximation property (AP) if there is a net (φ_i) in $A(G)$ which converges to the constant function 1 in the weak* topology $\sigma(M_0(G), Q(G))$.*

Thanks to Theorems 1.5 and 1.9, we have the following implications.

$$\text{Amenability} \implies \text{Weak amenability} \implies \text{AP}.$$

Moreover, the converses are not true. Haagerup [Haa79] proved that the free group $\mathbb{F}_2$ is weakly amenable, and it is well known that this group is not amenable. He also proved [Haa16] that the semidirect product $\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$ is not weakly amenable, and since the AP is stable by group extensions [HK94, Theorem 1.15], this group has the AP.

1.4.2 p-Approximation properties

In [ALR10], An, Lee and Ruan generalised the approximation properties described in Section 1.4.1 to the context of p-operator spaces. The main idea is to replace the Fourier algebra by what is called the Figà-Talamanca–Herz algebra, which admits a p-operator space structure, and therefore we can look at its p-completely bounded multipliers. The fact that the Fourier algebra admits such a simple description ($A(G) = L_2(G) * \check{L_2(G)}$) is very particular to the case $p = 2$, and its proof makes extensive use of the theory of von Neumann algebras. The definition of $A_p(G)$ is a little more complicated.

Take $1 < p < \infty$, and recall that λ_p stands for the left regular representation on $\mathcal{B}(L_p(G))$. If $\frac{1}{p} + \frac{1}{p'} = 1$, we may define a map $\Lambda_p : L_{p'}(G)\hat{\otimes}L_p(G) \to C_0(G)$, where $C_0(G)$ is the space of continuous functions on G that vanish at infinity, by

$$\Lambda_p(g \otimes f)(s) = \langle g, \lambda_p(s)f\rangle = g * \check{f}(s), \tag{1.6}$$

where $\check{f}(t) = f(t^{-1})$. The Figà-Talamanca–Herz algebra $A_p(G)$ is defined as the image of Λ_p endowed with the norm given by $L_{p'}(G)\hat{\otimes}L_p(G)/Ker(\Lambda_p)$, namely

$$\|a\|_{A_p(G)} = \inf\left\{ \sum_n \|g_n\|_{p'}\|f_n\|_p \;\middle|\; a = \sum_n g_n * \check{f}_n,\ g_n \in L_{p'}(G),\ f_n \in L_p(G) \right\}.$$

This is again a commutative Banach algebra, which coincides with the Fourier algebra $A(G)$ when $p = 2$. This algebra was originally defined by Figà-Talamanca [FT65] in some particular cases, and the general definition was given by Herz [Her71].

We could define p-amenability by the existence of a bounded approximate identity in $A_p(G)$; however, the following theorem of Herz [Her73, Theorem 6] says that we do not obtain anything new.

Theorem 1.11 (Herz [Her73])**.** *Let G be a locally compact group and let $p \in (1,\infty)$. Then G is amenable if and only if $A_p(G)$ possesses a bounded approximate identity.*

Therefore, we just talk about amenability. In what follows, we will define p-weak amenability and the p-approximation property. It is not known whether these notions are different from the original ones ($p = 2$).

The algebra of p-pseudo-measures $PM_p(G)$ can be identified with $A_p(G)^*$ by

$$\langle T, \Lambda_p(g \otimes f)\rangle = \langle g, T(f)\rangle, \quad \forall T \in PM_p(G),\ \forall g \in L_{p'}(G),\ \forall f \in L_p(G).$$

Since $PM_p(G)$ is a concrete p-operator space, we can define the space of p-completely bounded multipliers of $A_p(G)$ in the same way as we did for $A(G)$. We denote this space by $M_{p\text{-}cb}(G)$. An analogue of Theorem 1.6 was proven by Daws [Daw10, Theorem 8.3].

Theorem 1.12 (Daws [Daw10]). *Let G be a locally compact group, $\varphi : G \to \mathbb{C}$ a function, and $C > 0$ a constant. The following are equivalent.*

a) The function φ belongs to $M_{p\text{-}cb}(G)$ and $\|\varphi\|_{M_{p\text{-}cb}(G)} \leq C$.

b) There exists $E \in SQ_p$ and continuous bounded maps $\alpha : G \to E$, $\beta : G \to E^*$ such that

$$\varphi(st^{-1}) = \langle \beta(s), \alpha(t) \rangle, \quad \forall s, t \in G, \tag{1.7}$$

and

$$\left(\sup_{s \in G} \|\beta(s)\| \right) \left(\sup_{t \in G} \|\alpha(t)\| \right) \leq C.$$

Remark 1.13. *Suppose that $\varphi \in M_{p\text{-}cb}(G)$ and take E, α and β such that φ decomposes as in (1.7). Then there exist a measure space Ω and a closed subspace $F \subset L_p(\Omega)$ such that E is a subspace of $L_p(\Omega)/F$. Then, for every $t \in G$, $\alpha(t) \in L_p(\Omega)/F$, and this space may be identified with the dual of $F^\perp \subset L_q(\Omega)$. By the Hahn-Banach theorem, $\alpha(t)$ extends to an element $\tilde{\alpha}(t) \in L_q(\Omega)^* = L_p(\Omega)$ of same norm. Likewise, for all $s \in G$, $\beta(s) \in E^*$ extends to $\tilde{\beta}(s) \in (L_p(\Omega)/F)^*$, and this space may be identified with $F^\perp \subset L_q(\Omega)$. Hence, we get functions $\tilde{\alpha} : G \to L_p(\Omega)$ and $\tilde{\beta} : G \to L_q(\Omega)$ such that $\varphi(st^{-1}) = \langle \tilde{\beta}(s), \tilde{\alpha}(t) \rangle$ for all $s, t \in G$. However, we do not know whether these functions are continuous.*

Again, we have contractive inclusions.

$$A_p(G) \hookrightarrow M_{p\text{-}cb}(G) \hookrightarrow C_b(G).$$

Definition 1.14. *A locally compact group G is said to be p-weakly amenable if $A_p(G)$ possesses an approximate identity (φ_i) which is bounded in the $M_{p\text{-}cb}(G)$-norm.*

This allows us to define an analogue of the Cowling–Haagerup constant for p-weak amenability, which we denote by $\Lambda_p(G)$.

Let $Q_{p\text{-}cb}(G)$ be the norm closure of $L_1(G)$ in $M_{p\text{-}cb}(G)^*$. Miao proved in an unpublished manuscript that $M_{p\text{-}cb}(G) = Q_{p\text{-}cb}(G)^*$, in analogy with the case $p = 2$.

Definition 1.15. *A locally compact group G is said to have the p-approximation property (p-AP) if there is a net (φ_i) in $A_p(G)$ which converges to the constant function 1 in the weak* topology $\sigma(M_{p\text{-}cb}(G), Q_{p\text{-}cb}(G))$.*

Remark 1.16. *Define $A_{p,c}(G) = A_p(G) \cap C_c(G)$, where $C_c(G)$ is the space of continuous compactly supported functions on G. Then $A_{p,c}(G)$ is dense in $A_p(G)$, and since the inclusion $A_p(G) \hookrightarrow M_{p\text{-}cb}(G)$ is contractive, the net (φ_i) can be taken in $A_{p,c}(G)$.*

1.4.3 Characterisations in terms of operator algebras

For a discrete group, the approximation properties described in Section 1.4.1 admit equivalent formulations in terms of approximation properties of the operator algebras associated to it. Although the following characterisations will not be used in this thesis, they are very important, as they are in the heart of the development of this subject. As a matter of fact, the main result of [Haa79] says that the reduced C*-algebra of the free group on two generators $C_r^*(\mathbb{F}_2)$ has the metric approximation property (MAP), providing the first example of a non nuclear C*-algebra verifying this. It was later observed in [dCH85] that his construction actually shows that $C_r^*(\mathbb{F}_2)$ satisfies the completely bounded approximation property (CBAP), extending this to a larger class of groups. Afterwards, this led to the definition of weak amenability [CH89].

Let Γ be a discrete group. We shall only state the aforementioned characterisations. See Appendix A for more details on the approximation properties of operator algebras.

Theorem 1.17 (Lance [Lan73], Effros-Lance [EL77])**.** *The following are equivalent.*

a) Γ *is amenable.*

b) $C_r^*(\Gamma)$ *is nuclear (* $\iff$ *has the CPAP).*

c) $\mathcal{L}(\Gamma)$ *is semidiscrete.*

Theorem 1.18 (Haagerup [Haa16])**.** *The following are equivalent.*

a) Γ *is weakly amenable.*

b) $C_r^*(\Gamma)$ *has the CBAP.*

c) $\mathcal{L}(\Gamma)$ *has the weak* CBAP.*

In that case, $\Lambda(\Gamma) = \Lambda_{cb}(C_r^*(\Gamma)) = \Lambda_{cb*}(\mathcal{L}(\Gamma))$.

Theorem 1.19 (Haagerup–Kraus [HK94])**.** *The following are equivalent.*

a) Γ *has the AP.*

b) $C_r^*(\Gamma)$ *has the OAP.*

c) $\mathcal{L}(\Gamma)$ *has the weak* OAP.*

It was proven in [ALR10] that the p-approximation properties defined in Section 1.4.2 admit similar characterisations in terms approximation properties of the p-operator algebras $PF_p(\Gamma)$ and $PM_p(\Gamma)$. See Appendix A for definitions.

Theorem 1.20 (An–Lee–Ruan [ALR10])**.** *Let* $1 < p < \infty$*. The following are equivalent.*

a) Γ *is* p*-weakly amenable.*

b) $PF_p(\Gamma)$ *has the* p*-CBAP.*

c) $PM_p(\Gamma)$ *has the weak** p*-CBAP.*

In that case, $\Lambda_p(\Gamma) = \Lambda_{p\text{-}cb}(C_r^*(\Gamma)) = \Lambda_{p\text{-}cb*}(\mathcal{L}(\Gamma))$.

Theorem 1.21 (An–Lee–Ruan [ALR10])**.** *Let* $1 < p < \infty$*. The following are equivalent.*

a) Γ *has the* p*-AP.*

b) $PF_p(\Gamma)$ *has the* p*-OAP.*

c) $PM_p(\Gamma)$ *has the weak** p*-OAP.*

Chapter 2

The p-approximation property

The aim of this chapter is to study the p-approximation property (p-AP). In some sense, it is summary of the general results that will be used in Chapter 3 in order to show that higher rank simple Lie groups with finite centre do not satisfy this property. We end it with the proof that p-AP implies that $CV_p(G) = PM_p(G)$.

2.1 Some useful results

Recall that the Figà-Talamanca–Herz algebra $A_p(G)$ is defined as the image of the map $\Lambda_p : L_{p'}(G)\hat{\otimes}L_p(G) \to C_0(G)$ given by (1.6). This is a particular case of what Herz calls representative functions (see [Her71]). As explained in [Daw10, §8] in a more modern language, if E is Banach space and $\pi : G \to \mathcal{B}(E)$ is a strongly-continuous representation such that $\pi(s)$ is an isometry for all $s \in G$, then we can define a map $\Pi : E^*\hat{\otimes}E \to C(G)$ by

$$\Pi(\mu \otimes x)(s) = \langle \mu, \pi(s)x \rangle.$$

The space $A(\pi)$ is defined as the image of Π endowed with the norm of $(E^*\hat{\otimes}E)/Ker(\Pi)$. Thus, by definition, $A_p(G) = A(\lambda_p)$. Moreover, consider $L_p(G; E)$ the Banach space of E-valued L_p functions on G. The algebraic tensor product $L_p(G) \otimes E$ defines a dense subspace of $L_p(G; E)$ by identifying

$$f \otimes x \mapsto f(\cdot)x, \quad \forall f \in L_p(G),\ \forall x \in E.$$

See [DF93, Chapter 7] for more details. Then, we may define the representation $\lambda_p \otimes I_E : G \to \mathcal{B}(L_p(G; E))$ by

$$[(\lambda_p \otimes I_E)(s)]\,(f \otimes x) = (\lambda_p(s)f) \otimes x, \quad \forall s \in G,\ \forall f \in L_p(G),\ \forall x \in E.$$

Herz [Her71, Lemma 0] proved the following.

Theorem 2.1 (Herz [Her71])**.** *If $E \in SQ_p$, then*

$$A(\lambda_p \otimes I_E) = A(\lambda_p) = A_p(G).$$

This theorem allows us to obtain the following characterisation.

Proposition 2.2. *Let $1 < p < \infty$. If K is a compact group, then $M_{p\text{-}cb}(K) = A_p(K)$ with equality of norms.*

Proof. Recall that the inclusion $A_p(K) \hookrightarrow M_{p\text{-}cb}(K)$ is a contraction. Now let $\varphi \in M_{p\text{-}cb}(K)$ and take E, α and β as in (1.7). Since K is compact, we can endow it with its normalised Haar measure. Define $X = L_p(K; E)$ and observe that if p' is the Hölder conjugate of p, then every $g \in L_{p'}(K; E^*)$ defines an element of X^* by

$$\langle g, f \rangle_{X^*,X} = \int_K \langle g(s), f(s) \rangle \, ds, \qquad \forall f \in X.$$

Thus, for all $s, t \in K$,

$$\begin{aligned}
\varphi(s^{-1}t) &= \int_K \varphi(s^{-1}kk^{-1}t) \, dk \\
&= \int_K \langle \beta(s^{-1}k), \alpha(t^{-1}k) \rangle \, dk \\
&= \int_K \langle \beta(k), \alpha(t^{-1}sk) \rangle \, dk \\
&= \left\langle \beta, \left[(\lambda_p \otimes I_E)(s^{-1}t) \right] \alpha \right\rangle_{X^*,X}.
\end{aligned}$$

Therefore, for all $k \in K$,

$$\varphi(k) = \langle \beta, [(\lambda_p \otimes I_E)(k)] \, \alpha \rangle_{X^*,X}.$$

Hence, Theorem 2.1 implies that $\varphi \in A_p(K)$ with

$$\|\varphi\|_{A_p(K)} \leq \|\beta\|_{L_{p'}(K;E^*)} \|\alpha\|_{L_p(K;E)} \leq \sup_{s \in K} \|\beta(s)\| \sup_{t \in K} \|\alpha(t)\|.$$

Taking the infimum over all E, α and β, we get $\|\varphi\|_{A_p(K)} \leq \|\varphi\|_{M_{p\text{-}cb}(K)}$. ☐

The following lemma will be useful when defining new multipliers as averages over a compact subgroup.

Lemma 2.3. *Let K be a compact subgroup of G endowed with its normalised Haar measure. Let E be a Banach space and $f : G \to E$ a continuous function. Then the function $\tilde{f} : G \to E$ given by*

$$\tilde{f}(t) = \int_K f(k^{-1}t) \, dk$$

is well defined and continuous.

Proof. This function is well defined because f is continuous and K is compact. For the continuity, we will use the fact that, if $\eta : G \to E$ is continuous and compactly supported, then it is left uniformly continuous (see e.g. the proof of [Fol95, Proposition 2.6]). This means that

$$\sup_{t\in G} \left\| \eta(s^{-1}t) - \eta(t) \right\| \underset{s\to 1}{\longrightarrow} 0.$$

Let V be a compact symmetric neighbourhood of $1 \in G$ and define $\tilde{K} = VK$. Observe that this is a compact subset of G and $K \subset \tilde{K}$. Take a function $\psi \in C_c(G)$ such that $\psi = 1$ on $\tilde{K}$. Now fix $t \in G$ and define $\eta_t(s) = \psi(s) f(s^{-1}t)$. We know that

$$\sup_{k\in K} \left\| \eta_t(s^{-1}k) - \eta_t(k) \right\| \underset{s\to 1}{\longrightarrow} 0.$$

Moreover, for all $s \in V$ and $k \in K$,

$$\eta_t(s^{-1}k) = \psi(s^{-1}k) f(k^{-1}st) = f(k^{-1}st).$$

Therefore, for all $s \in V$,

$$\begin{aligned} \left\| \tilde{f}(st) - \tilde{f}(t) \right\| &= \left\| \int_K \left(\eta_t(s^{-1}k) - \eta_t(k) \right) \, dk \right\| \\ &\leq \sup_{k\in K} \left\| \eta_t(s^{-1}k) - \eta_t(k) \right\| \underset{s\to 1}{\longrightarrow} 0. \end{aligned}$$

□

2.2 Relations between p-AP and q-AP

The following proposition was proven in [ALR10, §6] for G discrete. We include the proof of the general case for the sake of completeness.

Proposition 2.4. *Let $1 < p \leq q \leq 2$ or $2 \leq q \leq p < \infty$. If G has the q-AP, then it has the p-AP.*

Proof. By [ALR10, Proposition 6.1], the inclusion $\iota : M_{q\text{-}cb}(G) \hookrightarrow M_{p\text{-}cb}(G)$ is a contraction. Thus, the adjoint map $\iota^* : M_{p\text{-}cb}(G)^* \hookrightarrow M_{q\text{-}cb}(G)^*$ is also a contraction and it maps $Q_{p\text{-}cb}(G)$ to $Q_{q\text{-}cb}(G)$ because

$$|\langle f, \varphi \rangle| \leq \|f\|_{M_{p\text{-}cb}(G)^*} \|\varphi\|_{M_{p\text{-}cb}(G)} \leq \|f\|_{M_{p\text{-}cb}(G)^*} \|\varphi\|_{M_{q\text{-}cb}(G)},$$

for all $f \in L_1(G)$ and $\varphi \in M_{q\text{-}cb}(G)$. Since G has the q-AP, there is a net (φ_i) in $A_q(G)$ such that $\varphi_i \to 1$ in $\sigma(M_{q\text{-}cb}(G), Q_{q\text{-}cb}(G))$. Observe that since $C_c(G) \otimes C_c(G)$ is dense in $L_{q'}(G) \hat{\otimes} L_q(G)$ and the map $\Lambda_q : L_{q'}(G) \hat{\otimes} L_q(G) \to A_q(G)$ is contractive, we may take the net in $\Lambda_q(C_c(G) \otimes C_c(G))$, which is equal to $\Lambda_p(C_c(G) \otimes C_c(G))$ as sets, so $\varphi_i = \iota(\varphi_i) \in A_p(G)$. Therefore, for all $\mu \in Q_{p\text{-}cb}$,

$$\langle \iota(\varphi_i), \mu \rangle = \langle \varphi_i, \iota^*(\mu) \rangle \to \langle 1, \iota^*(\mu) \rangle = \langle 1, \mu \rangle.$$

Hence, G has the p-AP. □

Now let $1 < p, p' < \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Recall the notation $\check{\varphi}(s) = \varphi(s^{-1})$ for a function $\varphi : G \to \mathbb{C}$.

Lemma 2.5. *The map $\varphi \mapsto \check{\varphi}$ defines an isometric isomorphism from $M_{p\text{-}cb}(G)$ to $M_{p'\text{-}cb}(G)$.*

Proof. Let $\varphi \in M_{p\text{-}cb}(G)$. Let $E \in SQ_p$ and $\alpha : G \to E$, $\beta : G \to E^*$ be continuous bounded functions such that

$$\varphi(st^{-1}) = \langle \beta(s), \alpha(t) \rangle, \quad \forall s, t \in G.$$

Then $\check{\varphi}(st^{-1}) = \langle \beta(t), \alpha(s) \rangle$. Now observe that $E^* \in SQ_{p'}$. Indeed, since $E \in SQ_p$, there exist a measure space (Ω, μ) and a closed subspace $F \subset L_p(\mu)$ such that E is a subspace of $L_p(\mu)/F$. So $E^* \cong (L_p(\mu)/F)^* / E^\perp$ and $(L_p(\mu)/F)^* \cong F^\perp \subset L_{p'}(\mu)$. Since $\alpha(t)$ defines an element of E^{**} of equal norm, we get that $\check{\varphi} \in M_{p'\text{-}cb}(G)$ and $\|\check{\varphi}\|_{M_{p'\text{-}cb}(G)} \leq \|\varphi\|_{M_{p\text{-}cb}(G)}$. The other inequality follows analogously. □

For $\mu \in M_{p'\text{-}cb}(G)^*$, we may define $\check{\mu} \in M_{p\text{-}cb}(G)^*$ by duality

$$\langle \check{\mu}, \varphi \rangle = \langle \mu, \check{\varphi} \rangle, \quad \forall \varphi \in M_{p\text{-}cb}(G),$$

which again defines an isometric isomorphism between $M_{p'\text{-}cb}(G)^*$ and $M_{p\text{-}cb}(G)^*$.

Proposition 2.6. *If G has the p-AP, then G has the p'-AP.*

Proof. Let (φ_i) be a net in $A_p(G)$ such that $\varphi_i \to 1$ in $\sigma(M_{p\text{-}cb}(G), Q_{p\text{-}cb}(G))$. The identity $(g * \check{f})^\vee = f * \check{g}$ shows that $\check{\varphi}_i \in A_{p'}(G)$ for all i. On the other hand, if $\mu \in Q_{p'\text{-}cb}(G)$ is given by a function $g \in L_1(G)$, a change of variable shows that $\check{\mu}$ is given by the function $\tilde{g}(s) = \Delta(s^{-1}) g(s^{-1})$ which satisfies $\|\tilde{g}\|_1 = \|g\|_1$. Thus, the isometry $\mu \mapsto \check{\mu}$ maps $Q_{p'\text{-}cb}(G)$ to $Q_{p\text{-}cb}(G)$. Therefore

$$\langle \check{\varphi}_i, \mu \rangle = \langle \varphi_i, \check{\mu} \rangle \to \langle 1, \check{\mu} \rangle = \langle 1, \mu \rangle, \quad \forall \mu \in Q_{p'\text{-}cb}(G).$$

□

Thanks to Proposition 2.6 we may restrict all the analysis to the case $2 \leq p < \infty$.

2.3 Stability properties

Let G be a second countable locally compact group and let $1 < p < \infty$.

Proposition 2.7. *If K is a compact normal subgroup of G, then G/K has the p-AP if and only if G has the p-AP.*

Proof. Suppose first that G/K has the p-AP. Define a linear map $\Psi : M_{p\text{-}cb}(G/K) \to M_{p\text{-}cb}(G)$ by $\Psi(\varphi)(s) = \varphi(\dot{s})$. It is well defined because for every $\varphi \in M_{p\text{-}cb}(G/K)$, we can write

$$\Psi(\varphi)(st^{-1}) = \langle \beta \circ \pi(s), \alpha \circ \pi(t) \rangle,$$

where α and β are as in (1.7) and $\pi : G \to G/K$ is the quotient map, which is continuous. Hence, taking the infimum over all α, β, we see that $\|\Psi(\varphi)\|_{M_{p\text{-}cb}(G)} \leq \|\varphi\|_{M_{p\text{-}cb}(G/K)}$. Furthermore, we shall prove that Ψ is $\sigma(M_{p\text{-}cb}(G/K), Q_{p\text{-}cb}(G/K)) - \sigma(M_{p\text{-}cb}(G), Q_{p\text{-}cb}(G))$ continuous. Consider the linear map $T : C_c(G) \to C_c(G/K)$ given by

$$T(f)(\dot{s}) = \int_K f(sk)\,dk, \quad \forall f \in C_c(G),$$

where $\dot{s} = \pi(s)$. This map is well defined and does not depend on the choice of the representative of $\dot{s}$. As shown in [RS00, §3.4], we may fix the Haar measure on G/K so that T extends to a contraction from $L_1(G)$ onto $L_1(G/K)$, and

$$\int_{G/K} T(f)(\dot{s})\,d\dot{s} = \int_G f(s)\,ds, \tag{2.1}$$

for every $f \in L_1(G)$. We wish now to extend T to $Q_{p\text{-}cb}(G)$. Take $\varphi \in M_{p\text{-}cb}(G/K)$ and $f \in L_1(G)$. Observe that $\Psi(\varphi)(sk) = \Psi(\varphi)(s)$ for all $s \in G$, $k \in K$. Then, regarding $T(f)$ as an element of $M_{p\text{-}cb}(G/K)^*$ as in (1.5), we get

$$\begin{aligned}
\langle T(f), \varphi\rangle &= \int_{G/K} T(f)(\dot{s})\varphi(\dot{s})\,d\dot{s}\\
&= \int_{G/K} \left(\int_K f(sk)\,dk\right) \Psi(\varphi)(s)\,d\dot{s}\\
&= \int_{G/K}\int_K f(sk)\Psi(\varphi)(sk)\,dk\,d\dot{s}\\
&= \int_{G/K} T\left(f\Psi(\varphi)\right)(\dot{s})\,d\dot{s}\\
&= \int_G f(s)\Psi(\varphi)(s)\,ds\\
&= \langle f, \Psi(\varphi)\rangle.
\end{aligned}$$

Here we have used the fact that $\Psi(\varphi) \in C_b(G)$, which implies that $f\Psi(\varphi) \in L_1(G)$. The previous computation shows two things. First, it implies that T may be extended to a map from $Q_{p\text{-}cb}(G)$ onto $Q_{p\text{-}cb}(G/K)$, since

$$|\langle T(f), \varphi\rangle| \leq \|f\|_{M_{p\text{-}cb}(G)^*}\|\Psi(\varphi)\|_{M_{p\text{-}cb}(G)} \leq \|f\|_{M_{p\text{-}cb}(G)^*}\|\varphi\|_{M_{p\text{-}cb}(G/K)}.$$

Second, it shows that $\Psi = T^*$. Hence, Ψ is weak*-weak* continuous. Now take a net (φ_i) in $A_p(G/K)$ such that $\varphi_i \to 1$ in $\sigma(M_{p\text{-}cb}(G/K), Q_{p\text{-}cb}(G/K))$. We have that $\Psi(\varphi_i) \to \Psi(1)$ in $\sigma(M_{p\text{-}cb}(G), Q_{p\text{-}cb}(G))$. Moreover, since K is compact, by [Her73, Proposition 6], $\Psi(\varphi_i) \in A_p(G)$ for all i. Since $\Psi(1)$ is the constant function 1 on G, we conclude that G has the p-AP.

The proof of the other direction follows the same idea. We shall define operators $\tilde{T}$ and $\tilde{\Psi}$ by the same formulas as T and Ψ, but in different spaces. Consider $\tilde{T} : M_{p\text{-}cb}(G) \to M_{p\text{-}cb}(G/K)$ given by

$$\tilde{T}(\varphi)(\dot{s}) = \int_K \varphi(sk)\,dk, \quad \forall \varphi \in M_{p\text{-}cb}(G).$$

Let us prove that this map is well defined and continuous. Take $\varphi \in M_{p\text{-}cb}(G)$ and functions $\alpha : G \to E$, $\beta : G \to E^*$ as in (1.7). Thus

$$\tilde{T}(\varphi)(\dot{s}\dot{t}^{-1}) = \int_K \langle \beta(s), \alpha(k^{-1}t) \rangle \, dk = \left\langle \beta(s), \int_K \alpha(k^{-1}t) \, dk \right\rangle, \quad \forall s, t \in G.$$

Define $\tilde{\alpha}(\dot{t}) = \int_K \alpha(k^{-1}t) \, dk$. To see that it is well defined, take $t \in G$, $k' \in K$ and recall that $tk't^{-1} \in K$ because K is normal. Then

$$\int_K \alpha(k^{-1}tk') \, dk = \int_K \alpha(k^{-1}tk't^{-t}t) \, dk = \int_K \alpha(k^{-1}t) \, dk.$$

Now let F be the closed subspace of E generated by $\{\tilde{\alpha}(\dot{t}) \ : \ t \in G\}$. Observe that $F \in SQ_p$ and that $\tilde{\alpha} : G/K \to F$ is a continuous function because $t \mapsto \int_K \alpha(k^{-1}t) \, dk$ is continuous thanks to Lemma 2.3. Now define a function $\tilde{\beta} : G/K \to F^*$ by

$$\langle \beta(\dot{s}), x \rangle = \langle \beta(s), x \rangle, \quad \forall x \in F.$$

Again, it is well defined because

$$\begin{aligned} \left\langle \beta(sk'), \int_K \alpha(k^{-1}t) \, dk \right\rangle &= \int_K \varphi(sk't^{-1}k) \, dk \\ &= \int_K \varphi(st^{-1}tk't^{-1}k) \, dk \\ &= \int_K \varphi(st^{-1}k) \, dk \\ &= \left\langle \beta(s), \int_K \alpha(k^{-1}t) \, dk \right\rangle, \end{aligned}$$

for all $s, t \in G$ and $k' \in K$. Moreover, it is continuous since, by definition,

$$\|\tilde{\beta}(\dot{s}_1) - \tilde{\beta}(\dot{s}_2)\|_{F^*} \leq \|\beta(s_1) - \beta(s_2)\|_{E^*}, \quad \forall s_1, s_2 \in G.$$

Hence

$$\tilde{T}(\varphi)(\dot{s}\dot{t}^{-1}) = \langle \beta(\dot{s}), \tilde{\alpha}(\dot{t}) \rangle, \quad \forall \dot{s}, \dot{t} \in G/K,$$

and $\|\tilde{T}(\varphi)\|_{M_{p\text{-}cb}(G/K)} \leq \|\tilde{\alpha}\|_\infty \|\tilde{\beta}\|_\infty \leq \|\alpha\|_\infty \|\beta\|_\infty$. Thus $\|\tilde{T}(\varphi)\|_{M_{p\text{-}cb}(G/K)} \leq \|\varphi\|_{M_{p\text{-}cb}(G)}$. Now define $\tilde{\Psi} : C_c(G/K) \to C_c(G)$ by $\tilde{\Psi}(f)(s) = f(\dot{s})$. We wish to extend this map to $Q_{p\text{-}cb}(G/K)$ and prove that $\tilde{\Psi}^* = \tilde{T}$. Observe that by (2.1),

$$\begin{aligned} \int_G |\tilde{\Psi}(f)(s)| \, ds &= \int_{G/K} T\left(|\tilde{\Psi}(f)|\right)(\dot{s}) \, d\dot{s} \\ &= \int_{G/K} \int_K |f(\dot{s})| \, dk \, d\dot{s} \\ &= \int_{G/K} |f(\dot{s})| \, d\dot{s}, \end{aligned}$$

for every $f \in C_c(G/K)$. Thus, $\tilde{\Psi}$ extends to an isometry from $L_1(G/K)$ into $L_1(G)$. Furthermore, for all $\varphi \in M_{p\text{-}cb}(G)$ and $f \in C_c(G/K)$,

$$\begin{aligned}\langle \tilde{T}(\varphi), f\rangle &= \int_{G/K} \tilde{T}(\varphi)(\dot{s})\tilde{\Psi}(f)(s)\, d\dot{s}\\ &= \int_{G/K}\int_K \varphi(sk)\tilde{\Psi}(f)(sk)\, dk\, d\dot{s}\\ &= \int_G \varphi(s)\tilde{\Psi}(f)(s)\, ds\\ &= \langle \varphi, \tilde{\Psi}(f)\rangle.\end{aligned}$$

Hence,

$$|\langle \varphi, \tilde{\Psi}(f)\rangle| \leq \|\tilde{T}(\varphi)\|_{M_{p\text{-}cb}(G/K)}\|f\|_{M_{p\text{-}cb}(G/K)^*} \leq \|\varphi\|_{M_{p\text{-}cb}(G)}\|f\|_{M_{p\text{-}cb}(G/K)^*}.$$

Thus, $\tilde{\Psi}$ extends to a contraction from $Q_{p\text{-}cb}(G/K)$ to $Q_{p\text{-}cb}(G)$, and $\tilde{\Psi}^* = \tilde{T}$. Finally, by [Her73, Proposition 6], $\tilde{T}$ maps $A_p(G)$ to $A_p(G/K)$, so we conclude as before. □

Remark 2.8. *A shorter, more conceptual proof of Proposition 2.7 can be given as follows. If ν_K stands for the normalised Haar measure on K, viewed as a measure on G, then it satifies $\nu_K = \nu_K * \nu_K = \check{\nu}_K = \delta_s * \nu_K * \delta_{s^{-1}}$, and there is a standard identification $\nu_K * C_c(G) = C_c(G/K)$ which yields an identification $\nu_K * L_p(G) = L_p(G/K)$ and thus $\nu_K * A_p(G) = A_p(G/K)$. By similar computations as those of the previous proof, we also find $\nu_K * M_{p\text{-}cb}(G) = M_{p\text{-}cb}(G/K)$ and $\nu_K * Q_{p\text{-}cb}(G) = Q_{p\text{-}cb}(G/K)$. Therefore, the equivalence of the p-AP for G and G/K follows from all these identifications.*

Proposition 2.9. *If G has the p-AP, then every closed subgroup of G has the p-AP.*

Proof. Let H be a closed subgroup of G endowed with a left Haar measure. As in the proof of Proposition 2.7, we shall define a map from $Q_{p\text{-}cb}(H)$ to $Q_{p\text{-}cb}(G)$ and then consider its adjoint. Take $\psi \in C_c(G)$ such that $\psi \geq 0$ and $\int_G \psi = 1$, and consider the convolution map $\Phi : L_1(H) \to L_1(G)$ given by

$$\Phi(f)(s) = \int_H f(t)\psi(t^{-1}s)\, dt.$$

This map is well defined since $\Phi(f) = \mu_f * \psi$, where μ_f is the measure on G given by

$$\int_G g(s)\, d\mu_f(s) = \int_H g(t)f(t)dt, \quad \forall g \in C_c(G).$$

Moreover, it is a contraction. Indeed, for every $f \in L_1(H)$,

$$\|\Phi(f)\|_{L_1(G)} \leq \int_H |f(t)| \int_G |\psi(t^{-1}s)|\, ds\, dt \leq \|\psi\|_{L_1(G)}\|f\|_{L_1(H)} = \|f\|_{L_1(H)}.$$

Take now $\varphi \in M_{p\text{-}cb}(G)$.

$$\langle \Phi(f), \varphi \rangle = \int_G \left(\int_H f(t)\psi(t^{-1}s)\, dt \right) \varphi(s)\, ds.$$

Observe that

$$\int_G \int_H |f(t)||\psi(t^{-1}s)||\varphi(s)|\, dt\, ds \leq \|f\|_{L_1(H)} \|\varphi\|_\infty.$$

So by Fubini's theorem,

$$\langle \Phi(f), \varphi \rangle = \int_H f(t) \int_G \psi(t^{-1}s)\varphi(s)\, ds\, dt = \int_H f(t)\ \varphi * \check{\psi}(t)\, dt. \tag{2.2}$$

Let us prove now that $\varphi * \check{\psi}$ defines an element of $M_{p\text{-}cb}(H)$. Take α and β like in (1.7), so

$$\varphi(st^{-1}) = \langle \beta(s), \alpha(t) \rangle, \quad \forall s, t \in G.$$

Then

$$\begin{aligned} \varphi * \check{\psi}(st^{-1}) &= \int_G \psi(ts^{-1}r)\varphi(r)\, dr \\ &= \int_G \psi(r)\varphi(st^{-1}r)\, dr \\ &= \int_G \psi(r)\langle \beta(s), \alpha(r^{-1}t) \rangle\, dr \\ &= \langle \beta(s), \tilde{\alpha}(t) \rangle, \end{aligned} \tag{2.3}$$

where

$$\tilde{\alpha}(t) = \int_G \psi(r)\alpha(r^{-1}t)\, dr.$$

Recall that the function ψ has compact support, so $\tilde{\alpha}$ is well defined and continuous, and the equality (2.3) is justified. Moreover,

$$\|\tilde{\alpha}(t)\| \leq \int_G |\psi(r)| \|\alpha(r^{-1}t)\|\, dr \leq \sup_{t\in G} \|\alpha(t)\|, \quad \forall t \in G.$$

So, in particular,

$$\left(\sup_{s\in H} \|\beta(t)\| \right) \left(\sup_{t\in H} \|\tilde{\alpha}(t)\| \right) \leq \left(\sup_{s\in G} \|\beta(t)\| \right) \left(\sup_{t\in G} \|\alpha(t)\| \right).$$

Thus $\big(\varphi * \check{\psi}\big)|_H \in M_{p\text{-}cb}(H)$ and $\big\| \big(\varphi * \check{\psi}\big)|_H \big\|_{M_{p\text{-}cb}(H)} \leq \|\varphi\|_{M_{p\text{-}cb}(G)}$. Using (2.2) we get

$$|\langle \Phi(f), \varphi \rangle| \leq \|f\|_{M_{p\text{-}cb}(H)^*} \big\| \big(\varphi * \check{\psi}\big)|_H \big\|_{M_{p\text{-}cb}(H)} \leq \|f\|_{M_{p\text{-}cb}(H)^*} \|\varphi\|_{M_{p\text{-}cb}(G)}.$$

Hence, Φ extends to a contraction from $Q_{p\text{-}cb}(H)$ to $Q_{p\text{-}cb}(G)$, and its adjoint $\Phi^* : M_{p\text{-}cb}(G) \to M_{p\text{-}cb}(H)$ is given by $\Phi^*(\varphi) = \big(\varphi * \check{\psi}\big)|_H$. Observe that $\Phi^*(1) = 1$, so if we prove that Φ^* maps $A_p(G)$ to $A_p(H)$, we can conclude that H has the p-AP as in Proposition 2.7. Take $f \in L_p(G)$ and $g \in L_{p'}(G)$, where p' is the Hölder conjugate of p, and observe that

$$\left(g * \check{f}\right) * \check{\psi} = g * (\psi * f)^{\vee}.$$

Since $\|\psi * f\|_p \leq \|f\|_p$, we get

$$\left\| \left(\sum_{n=1}^{N} g_n * \check{f}_n \right) * \check{\psi} \right\|_{A_p(G)} \leq \sum_{n=1}^{N} \|g_n\|_{p'} \|\psi * f_n\|_p \leq \sum_{n=1}^{N} \|g_n\|_{p'} \|f_n\|_p,$$

for all finite families $f_1, ..., f_N \in L_p(G)$, $g_1, ..., g_N \in L_{p'}(G)$. By density, this implies that $\|a * \check{\psi}\|_{A_p(G)} \leq \|a\|_{A_p(G)}$ for all $a \in A_p(G)$. Since H is a closed subgroup of G, then the restriction $\left(a * \check{\psi}\right)|_H$ is an element of $A_p(H)$ and $\left\| \left(a * \check{\psi}\right)|_H \right\|_{A_p(H)} \leq \|a * \check{\psi}\|_{A_p(G)}$ (see e.g. [Der11, Theorem 7.8.2]). Therefore, Φ^* maps $A_p(G)$ to $A_p(H)$. Thus, if (φ_i) is a net in $A_p(G)$ such that $\varphi_i \to 1$ in $\sigma(M_{p\text{-}cb}(G), Q_{p\text{-}cb}(G))$, then $(\Phi^*(\varphi_i))$ is a net in $A_p(H)$ such that $\Phi^*(\varphi_i) \to 1$ in $\sigma(M_{p\text{-}cb}(H), Q_{p\text{-}cb}(H))$. □

Remark 2.10. *For $p = 2$, Proposition 2.9 corresponds to [HK94, Proposition 1.14], whose proof is very different. We are not able to adapt that proof since there is no known analogue of [HK94, Theorem 1.11] for $p \neq 2$. This is related to the fact that, as observed by Daws, we do not have a simple description of what $PM_p(SU(2))$ is, unless $p = 2$ (see the remark right after [Daw10, Proposition 8.7]).*

Now let Γ be lattice in G. This means that Γ is a discrete subgroup of G such that G/Γ has a finite G-invariant measure. This implies the existence of a Borel subset $\Omega \subset G$ such that the restriction of the quotient map $G \to G/\Gamma$ to Ω is bijective. Observe that this allows us to define maps $\omega : G \to \Omega$, $\gamma : G \to \Gamma$ such that

$$s = \omega(s)\gamma(s), \quad \forall s \in G, \tag{2.4}$$

and this decomposition is unique. Let μ_G be the Haar measure on G. Since Γ is a lattice, $\mu_G(\Omega)$ is finite and strictly positive, so we may normalise μ_G in such a way that $\mu_G(\Omega) = 1$. The following proposition corresponds to [HK94, Theorem 2.4] for $p = 2$ and the proof uses the same ideas.

Proposition 2.11. *Let $1 < p < \infty$. If Γ has the p-AP, then so does G.*

Proof. Again the strategy is to define a suitable map $\Phi : M_{p\text{-}cb}(\Gamma) \to M_{p\text{-}cb}(G)$ and consider the image of an approximating net in $A_p(\Gamma)$. For $\varphi \in M_{p\text{-}cb}(\Gamma)$, define $\tilde{\Phi}(\varphi) \in L_\infty(G)$ by

$$\tilde{\Phi}(\varphi)(s) = \varphi(\gamma(s)), \qquad \forall\, s \in G,$$

and put $\Phi(\varphi) = \tilde{\Phi}(\varphi) * \check{\mathbb{1}}_\Omega$, where $\check{\mathbb{1}}_\Omega$ stands for the indicator function of the set Ω^{-1}. Thus,

$$\Phi(\varphi)(s) = \int_\Omega \varphi(\gamma(su))\, du, \quad \forall s \in G.$$

This map satisfies $\Phi(1)=1$. Let us show that $\Phi(\varphi)\in M_{p\text{-}cb}(G)$. Take E, α and β as in (1.7). We have

$$\Phi(\varphi)(st^{-1})=\int_\Omega \varphi(\gamma(st^{-1}u))\,du,\quad \forall s,t\in G.$$

As was shown in the proof of [Haa79, Lemma 2.1], the map $u\in\Omega\mapsto\omega(tu)\in\Omega$ preserves μ_G for all $t\in G$. Hence we get

$$\Phi(\varphi)(st^{-1})=\int_\Omega \varphi(\gamma(st^{-1}\omega(tu)))\,du,\quad \forall s,t\in G.$$

Now observe that

$$st^{-1}\omega(tu)=su(tu)^{-1}\omega(tu)=\omega(su)\gamma(su)\gamma(tu)^{-1}.$$

Since the decomposition (2.4) is unique, we get $\gamma(st^{-1}\omega(tu))=\gamma(su)\gamma(tu)^{-1}$. Thus

$$\Phi(\varphi)(st^{-1})=\int_\Omega \varphi(\gamma(su)\gamma(tu)^{-1})\,du=\int_\Omega\langle\beta(\gamma(su)),\alpha(\gamma(tu))\rangle\,du,\quad \forall s,t\in G.$$

Therefore, if we define $\tilde{\alpha}:G\to L_p(\Omega;E)$ by $\tilde{\alpha}(t)=\alpha(\gamma(t\,\cdot))$, and $\tilde{\beta}:G\to L_p(\Omega;E)^*$ by

$$\langle\tilde{\beta}(s),f\rangle=\int_\Omega\langle\beta(\gamma(su)),f(u)\rangle\,du,\quad \forall s\in G,\ \forall f\in L_p(\Omega;E),$$

we get $\Phi(\varphi)(st^{-1})=\langle\tilde{\beta}(s),\tilde{\alpha}(t)\rangle$ for all $s,t\in G$. Moreover,

$$\|\tilde{\alpha}(t)\|_{L_p(\Omega;E)}=\left(\int_\Omega\|\alpha(tu)\|^p\,du\right)^{\frac{1}{p}}\leq\|\alpha\|_\infty,\quad\forall t\in G. \tag{2.5}$$

And putting $p'=\frac{p}{p-1}$, for all $s\in G$ and $f\in L_p(\Omega;E)$,

$$\begin{aligned}|\langle\tilde{\beta}(s),f\rangle|&\leq\int_\Omega\|\beta(\gamma(su))\|\|f(u)\|\,du\\&\leq\left(\int_\Omega\|\beta(\gamma(su))\|^{p'}\,du\right)^{\frac{1}{p'}}\left(\int_\Omega\|f(u)\|^p\,du\right)^{\frac{1}{p}}\\&\leq\|\beta\|_\infty\|f\|_{L_p(\Omega;E)}.\end{aligned}\tag{2.6}$$

Hence, $\|\tilde{\alpha}\|_\infty\leq\|\alpha\|_\infty$ and $\|\tilde{\beta}\|_\infty\leq\|\beta\|_\infty$. Observe that $\Phi(\varphi)$ is continuous because it is the convolution of $\tilde{\Phi}(\varphi)\in L_\infty(G)$ and $\check{1\!\!1}_\Omega\in L_1(G)$ (see e.g. [Fol95, Proposition 2.39]). In terms of [Daw10, Theorem 8.6], this, together with Remark 1.13, is enough to conclude that $\Phi(\varphi)\in M_{p\text{-}cb}(G)$ and $\|\Phi(\varphi)\|_{M_{p\text{-}cb}(G)}\leq\|\alpha\|_\infty\|\beta\|_\infty$. Taking the infimum over all the decompositions, we obtain

$$\|\Phi(\varphi)\|_{M_{p\text{-}cb}(G)}\leq\|\varphi\|_{M_{p\text{-}cb}(\Gamma)}.$$

Let us show now that Φ is an adjoint operator. Let $\varphi \in M_{p\text{-}cb}(\Gamma)$ and $f \in L_1(G)$. We have

$$\begin{aligned}
\langle f, \Phi(\varphi)\rangle &= \int_G f(s) \int_\Omega \varphi(\gamma(su))\, du\, ds \\
&= \int_\Omega \int_G f(s)\varphi(\gamma(su))\, ds\, du \\
&= \int_\Omega \int_G f(su^{-1})\varphi(\gamma(s))\, ds\, du \\
&= \int_\Omega \left(\sum_{r\in\Gamma} \int_{\Omega r} f(su^{-1})\varphi(\gamma(s))\, ds \right) du \\
&= \sum_{r\in\Gamma} \varphi(r) \int_\Omega \int_{\Omega r} f(su^{-1})\, ds\, du.
\end{aligned}$$

In the first line we have used Fubini's theorem, which holds because

$$\int_\Omega \int_G |f(s)||\varphi(\gamma(su))|\, ds\, du \leq \|f\|_1 \|\varphi\|_\infty.$$

Thus, defining

$$\Psi(f)(r) = \int_\Omega \int_{\Omega r} f(su^{-1})\, ds\, du, \quad \forall f \in L_1(G),\ \forall r \in \Gamma,$$

the previous computations show that $\Psi(f) \in \ell_1(\Gamma)$ and that $\Phi = \Psi^*$, as in the proof of Proposition 2.7. We conclude that $\Phi : M_{p\text{-}cb}(\Gamma) \to M_{p\text{-}cb}(G)$ is weak*-weak* continuous. Finally, we need to check that Φ maps $A_p(\Gamma)$ to $A_p(G)$. Take $f \in \ell_p(\Gamma)$ and $g \in \ell_{p'}(\Gamma)$, and define

$$\tilde{f}(s) = f(\gamma(s)), \quad \tilde{g}(s) = g(\gamma(s)), \quad \forall s \in G.$$

Observe that

$$\int_G |\tilde{f}(s)|^p\, ds = \sum_{r\in\Gamma} \int_{\Omega r} |f(r)|^p\, ds = \|f\|_p^p.$$

This shows that $\tilde{f} \in L_p(G)$ and $\|\tilde{f}\|_p \leq \|f\|_p$. Analogously $\|\tilde{g}\|_{p'} \leq \|g\|_{p'}$. Moreover, for every $s \in G$,

$$\begin{aligned}
\tilde{g} * \check{\tilde{f}}(s) &= \tilde{f} * \check{\tilde{g}}(s^{-1}) \\
&= \int_G \tilde{f}(t)\tilde{g}(st)\, dt \\
&= \int_G f(\gamma(t))g(\gamma(st))\, dt \\
&= \sum_{r\in\Gamma} f(r) \int_{\Omega r} g(\gamma(st))\, dt \\
&= \sum_{r\in\Gamma} f(r) \int_{\Omega} g(\gamma(str))\, dt \\
&= \int_{\Omega} \sum_{r\in\Gamma} f(r) g(\gamma(st)r)\, dt \\
&= \int_{\Omega} f * \check{g}(\gamma(st)^{-1})\, dt \\
&= \int_{\Omega} g * \check{f}(\gamma(st))\, dt \\
&= \Phi(g * \check{f})(s).
\end{aligned}$$

Hence $\Phi(g * \check{f}) \in A_p(G)$ and $\|\Phi(g * \check{f})\|_{A_p(G)} \leq \|g\|_{p'}\|f\|_p$. By linearity we obtain

$$\left\| \Phi \left(\sum g_n * \check{f}_n \right) \right\|_{A_p(G)} \leq \sum \|g_n\|_{p'}\|f_n\|_p,$$

for all finite families (f_n) in $\ell_p(\Gamma)$ and (g_n) in $\ell_{p'}(\Gamma)$. Taking the infimum over all the decompositions we get $\|\Phi(a)\|_{A_p(G)} \leq \|a\|_{A_p(\Gamma)}$ for all $a \in \Lambda_p(\ell_{p'}(\Gamma) \otimes \ell_p(\Gamma)) \subset A_p(\Gamma)$, where Λ_p is the map defined in (1.6). By density the same holds for all $a \in A_p(\Gamma)$. Thus, if (φ_i) is a net in $A_p(\Gamma)$ such that $\varphi_i \to 1$ in $\sigma(M_{p\text{-}cb}(\Gamma), Q_{p\text{-}cb}(\Gamma))$, then $(\Phi(\varphi_i))$ is a net in $A_p(G)$ such that $\Phi(\varphi_i) \to 1$ in $\sigma(M_{p\text{-}cb}(G), Q_{p\text{-}cb}(G))$. □

2.4 The p-AP implies that the convoluters are pseudo-measures

Cowling [Cow98] showed that when G is weakly amenable, $PM_p(G) = CV_p(G)$ for every $1 < p < \infty$, suggesting that similar arguments would show that the same holds when G has the AP. This was explained in detail by Daws and Spronk [DS19]. With minor changes in their proof, we show that for $1 < p < \infty$ fixed, if G has the p-AP, then $PM_p(G) = CV_p(G)$. In terms of Proposition 2.4, this is (a priori) a stronger statement; however, we do not know if there exist groups satisfying the p-AP for some p, but failing to have the AP.

Theorem 2.12. *Let $1 < p < \infty$. If G has the p-AP, then $CV_p(G) = PM_p(G)$.*

Remark 2.13. *By Proposition 2.4, the previous theorem implies that, if G has the q-AP, then $CV_p(G) = PM_p(G)$ for all $1 < p, q < \infty$ such that $\left|\frac{1}{q} - \frac{1}{2}\right| \leq \left|\frac{1}{p} - \frac{1}{2}\right|$.*

Recall that $PM_p(G)$ is the dual of $A_p(G)$. Cowling showed [Cow98] that $CV_p(G)$ can also be viewed as a dual space. Let e be the identity element of G. For every compact neighbourhood K of e in G, let $L_p(K)$ be the subspace of $L_p(G)$ consisting of those functions supported on K, and let $\breve{A}_{p,K}(G)$ be the space of functions of the form

$$a = \sum_n g_n * \check{f}_n, \quad g_n \in L_{p'}(K),\ f_n \in L_p(K),$$

such that $\displaystyle\sum_n \|g_n\|_{p'}\|f_n\|_p < \infty$. Let $\breve{A}_p(G) = \displaystyle\bigcup_K \breve{A}_{p,K}(G)$ and

$$\|a\|_{\breve{A}_p} = \inf \sum_n \|g_n\|_{p'}\|f_n\|_p,$$

where the infimum ranges over all the decompositions $a = \sum g_n * \check{f}_n \in \breve{A}_{p,K}(G)$ and all the compact neighbourhoods K of e. Then $\breve{A}_p(G)^*$ may be identified with $CV_p(G)$ by $T \in CV_p(G) \mapsto \Phi_T \in \breve{A}_p(G)^*$, where

$$\langle \Phi_T, a\rangle = \sum \langle g_n, T(f_n)\rangle, \quad a = \sum g_n * \check{f}_n.$$

For every $\tau \in L_{p'}(G)\hat{\otimes}L_p(G)$, $T \in CV_p(G)$ and $\psi \in \breve{A}_p(G)$ such that $\psi \geq 0$ and $\int \psi = 1$, we can define $\mu \in M_{p\text{-}cb}(G)^*$ by

$$\langle \mu, \varphi\rangle = \langle T, (\psi * \varphi)\cdot\tau\rangle, \quad \forall \varphi \in M_{p\text{-}cb}(G),$$

using the duality $\mathcal{B}(L_p(G)) = (L_{p'}(G)\hat{\otimes}L_p(G))^*$. Here we view τ as a function on $G \times G$ and

$$\varphi\cdot\tau(s,t) = \varphi(st^{-1})\tau(s,t), \quad \forall\varphi \in M_{p\text{-}cb}(G),\ \forall\tau \in L_{p'}(G)\hat{\otimes}L_p(G).$$

Under these conditions, $\|\mu\| \leq \|\tau\|\|T\|$.

Proposition 2.14. *Let μ be as above. Then μ is $*$-weak continuous, that is, $\mu \in Q_{p\text{-}cb}(G)$.*

Proof. Since $C_c(G) \otimes C_c(G)$ is dense in $L_{p'}(G)\hat{\otimes}L_p(G)$, by continuity we may assume $\tau \in C_c(G) \otimes C_c(G)$. We will construct $g \in L_1(G)$ such that $\langle \mu, \varphi\rangle = \int \varphi g$ for all $\varphi \in M_{p\text{-}cb}(G)$. Recall the definition of Λ_p in (1.6) and observe that $a = \Lambda_p(\tau) \in \breve{A}_p(G)$. We have

$$(\psi * \varphi)(s)a(s) = (\psi * \chi_S\varphi)(s)a(s), \quad \forall s \in G,$$

where $S = \mathrm{supp}(\psi)^{-1}\mathrm{supp}(a)$ is a compact subset of G. Define $\psi_t(s) = \psi(st)$ and

$$g(t) = \chi_S(t)\Delta(t^{-1})\langle T, \psi_{t^{-1}}a\rangle, \quad t \in G.$$

Then $g \in L_1(G)$ and

$$\int_G \varphi(t)g(t)\,dt = \langle \mu, \varphi\rangle, \quad \forall\varphi \in M_{p\text{-}cb}(G).$$

See [DS19, Lemma 3.2] for details. □

Proof of Theorem 2.12. It is always true that $PM_p(G) \subseteq CV_p(G)$. Now let $T \in CV_p(G)$ and (φ_i) be a net in $A_{p,c}(G)$ such that $\varphi_i \to 1$ in $\sigma(M_{p\text{-}cb}(G), Q_{p\text{-}cb}(G))$. Fix $\psi \in \check{A}_p(G)$ as above and take $\tau \in L_{p'}(G)\hat{\otimes}L_p(G)$. By the previous proposition, there exists $\mu \in Q_{p\text{-}cb}(G)$ such that $\langle \varphi, \mu \rangle = \langle T, (\psi * \varphi) \cdot \tau \rangle$ for all $\varphi \in M_{p\text{-}cb}(G)$. Therefore

$$\begin{aligned} \lim_i \langle (\psi * \varphi_i) \cdot T, \tau \rangle &= \lim_i \langle T, (\psi * \varphi_i) \cdot \tau \rangle \\ &= \lim_i \langle \varphi_i, \mu \rangle \\ &= \langle 1, \mu \rangle \\ &= \langle T, (\psi * 1) \cdot \tau \rangle \\ &= \langle T, 1 \cdot \tau \rangle \\ &= \langle T, \tau \rangle. \end{aligned}$$

Here we have used the $M_{p\text{-}cb}(G)$-module structure of $CV_p(G)$ (see [DS19, Corollary 2.7]). This is valid for all $\tau \in L_{p'}(G)\hat{\otimes}L_p(G)$, so $(\psi * \varphi_i) \cdot T \to T$ in the weak* topology of $\mathcal{B}(L_p(G))$. Since ψ and φ_i have compact support, so does $\psi * \varphi_i$. Thus, by [DS19, Corollary 2.8], $(\psi * \varphi_i) \cdot T \in PM_p(G)$ for all i, and since $PM_p(G)$ is weak* closed, this implies that $T \in PM_p(G)$. □

Chapter 3

Simple Lie groups and the p-AP

The main goal of this chapter is to prove the following.

Theorem 3.1. *Let $1 < p < \infty$ and G be a connected simple Lie group with finite centre.*

a) G has the p-AP if and only if it has real rank 0 or 1.

b) If Γ is a lattice in G, then Γ has the p-AP if and only if G has real rank 0 or 1.

For this purpose, we concentrate on the groups $\mathrm{SL}(3,\mathbb{R})$ and $\mathrm{Sp}(2,\mathbb{R})$, and conclude using the stability properties proved in Section 2.3. For more details on Lie groups, we refer the reader to [Kna02] and [Hel78].

3.1 The KAK decomposition

If G is a connected semisimple Lie group, then its Lie algebra $\mathfrak{g}$ admits a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. The real rank of G is defined as the dimension of any maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. If G has finite centre, then it may be decomposed as $G = KAK$, where K is a maximal compact subgroup and $A = \exp\mathfrak{a}$. This decomposition is not unique; however, fixing a Weyl chamber $\mathfrak{a}^+$ and letting $A^+ = \exp\mathfrak{a}^+$, we still have $G = K\overline{A^+}K$, where $\overline{A^+}$ is the closure of A^+ in G. See [Hel78, §IX.1] for details. The advantage of this decomposition is that for every $g \in G$, if $g = kak'$ with $k, k' \in K$, $a \in \overline{A^+}$, then a is unique. This is the main tool that will allow us to work with averages of multipliers when G is $\mathrm{SL}(3,\mathbb{R})$ or $\mathrm{Sp}(2,\mathbb{R})$, since in those cases K and $\overline{A^+}$ are explicit and very well known.

3.1.1 The $K\overline{A^+}K$ decomposition of $\mathrm{SL}(3,\mathbb{R})$

The special linear group $G = \mathrm{SL}(3,\mathbb{R})$ is the Lie group of 3×3 matrices over $\mathbb{R}$ with determinant 1. The group $K = \mathrm{SO}(3,\mathbb{R}) = \{g \in G \ : \ g^t g = I\}$ is a maximal compact subgroup of G. If we put

$$D(r,s,t) = \begin{pmatrix} e^r & 0 & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^t \end{pmatrix}, \quad r,s,t \in \mathbb{R},$$

then $\overline{A^+}$ is given by

$$\overline{A^+} = \{D(r,s,t) \ : \ r \geq s \geq t, \ r+s+t=0\}.$$

3.1.2 The $K\overline{A^+}K$ decomposition of Sp$(2,\mathbb{R})$

The symplectic group $G = \mathrm{Sp}(2,\mathbb{R})$ is the Lie group of all 4×4 matrices over $\mathbb{R}$ that preserve the standard symplectic form $\omega(x,y) = \langle Jx, y\rangle$, where

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{3.1}$$

Namely, $G = \{g \in M_4(\mathbb{R}) \ : \ g^t J g = J\}$. A maximal compact subgroup of G is given by

$$K = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in M_4(\mathbb{R}) \ \middle| \ A + \mathrm{i}B \in \mathrm{U}(2) \right\}. \tag{3.2}$$

This group is isomorphic to U(2) via the following map

$$\begin{pmatrix} a+\mathrm{i}b & e+\mathrm{i}f \\ c+\mathrm{i}d & g+\mathrm{i}h \end{pmatrix} \mapsto \begin{pmatrix} a & e & -b & -f \\ c & g & -d & -h \\ b & f & a & e \\ d & h & c & g \end{pmatrix}. \tag{3.3}$$

Let

$$D(\beta,\gamma) = \begin{pmatrix} e^{\beta} & 0 & 0 & 0 \\ 0 & e^{\gamma} & 0 & 0 \\ 0 & 0 & e^{-\beta} & 0 \\ 0 & 0 & 0 & e^{-\gamma} \end{pmatrix}, \quad \beta,\gamma \in \mathbb{R}. \tag{3.4}$$

Then $\overline{A^+} = \{D(\beta,\gamma) \ : \ \beta \geq \gamma \geq 0\}$.

3.2 The special linear group SL$(3,\mathbb{R})$

This section is devoted to the proof of the following theorem.

Theorem 3.2. *The group $SL(3,\mathbb{R})$ does not have the p-AP for any $1 < p < \infty$.*

We shall fix $G = \mathrm{SL}(3,\mathbb{R})$, $K = \mathrm{SO}(3,\mathbb{R})$ and $p \in [2,\infty)$.

3.2.1 The operators T_δ and averages of multipliers

The idea of the proof of Theorem 3.2 is to find an element $\mu \in Q_{p\text{-}cb}(G)$ such that $\langle 1, \mu\rangle = 1$ and $\langle \varphi, \mu\rangle = 0$ for every $\varphi \in M_{p\text{-}cb}(G) \cap C_c(G)$. In order to do this, we will construct a Cauchy net of functions (f_i) in $L_1(G)$ such that

$$\int f_i = 1 \quad \text{and} \quad \int f_i \varphi \to 0,$$

for every $\varphi \in C_c(G)$. The main tool used for this construction is the family of operators (T_δ) considered in [Laf08, §2] and [LdlS11, §5], which can be viewed as operators on $L_p(K)$ when identifying the sphere S^2 with a quotient of $\mathrm{SO}(3,\mathbb{R})$. We detail this now. Let U be the subgroup of K given by

$$U = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{SO}(2,\mathbb{R}) \end{pmatrix}.$$

For $\delta \in [-1,1]$, we define $T_\delta \in \mathcal{B}(L_p(K))$ by

$$T_\delta f = \int_U \int_U \lambda_p(u k_\delta u') f \, du \, du', \quad \forall f \in L_p(G), \tag{3.5}$$

where

$$k_\delta = \begin{pmatrix} \delta & -\sqrt{1-\delta^2} & 0 \\ \sqrt{1-\delta^2} & \delta & 0 \\ 0 & 0 & 1 \end{pmatrix} \in K.$$

This operator is well defined because, for every $f \in L_p(G)$, the map

$$s \in G \mapsto \lambda_p(s) f \in L_p(G)$$

is continuous. Observe that $U \backslash K$ may be identified with the sphere S^2 by

$$Uk \mapsto k^{-1} e_1,$$

where $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. If we put

$${}^U L_2(K) = \{ f \in L_2(K) \ : \ \forall u \in U, \ \lambda_2(u) f = f \},$$

then this identification defines an isometry $\Psi : L_2(S^2) \to {}^U L_2(K)$ by

$$[\Psi f](k) = f(k^{-1} e_1), \quad \forall f \in C(S^2), \ \forall k \in K,$$

which extends to $L_2(S^2)$ by density. This isometry relates the operators T_δ with those defined in [Laf08]. Since $T_\delta(L_2(K)) \subseteq {}^U L_2(K)$, the operator $\Theta_\delta = \Psi^{-1} T_\delta \Psi$ is well defined on $L_2(S^2)$.

Proposition 3.3. *For all $f \in C(S^2)$, $x \in S^2$ and $\delta \in (-1,1)$, $[\Theta_\delta f](x)$ is the average of f on the circle $\{y \in S^2 : \langle x, y \rangle = \delta\}$. Moreover, $[\Theta_1 f](x) = f(x)$ and $[\Theta_{-1} f](x) = f(-x)$.*

Proof. First observe that all the vectors $y \in S^2$ that satisfy $\langle e_1, y \rangle = \delta$ are of the form

$$\begin{pmatrix} \delta \\ \sqrt{1-\delta^2} \cos\theta \\ \sqrt{1-\delta^2} \sin\theta \end{pmatrix},$$

with $\theta \in [0, 2\pi)$. So if we put $x = k^{-1}e_1$ with $k \in K$, then

$$\begin{aligned}\langle x, y\rangle = \delta \iff & \exists\theta \in [0, 2\pi), \quad y = k^{-1}\begin{pmatrix}\delta \\ \sqrt{1-\delta^2}\cos\theta \\ \sqrt{1-\delta^2}\sin\theta\end{pmatrix} \\ \iff & \exists\theta \in [0, 2\pi), \quad y = k^{-1}u(\theta)k_\delta e_1,\end{aligned}$$

where

$$u(\theta) = \begin{pmatrix}1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta\end{pmatrix}, \quad \theta \in [0, 2\pi).$$

Thus

$$\begin{aligned}[\Theta_\delta f](x) &= [T_\delta \Psi f](k) \\ &= \int_U \lambda_p(uk_\delta)[\Psi f](k)\, du \\ &= \int_U [\Psi f](k_\delta^{-1}u^{-1}k)\, du \\ &= \int_U f(k^{-1}uk_\delta e_1)\, du \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(k^{-1}u(\theta)k_\delta e_1)\, d\theta.\end{aligned}$$

So if $\delta \in (-1, 1)$,

$$[\Theta_\delta f](x) = \frac{1}{2\pi\sqrt{1-\delta^2}}\int_{\langle x,y\rangle=\delta} f(y)\, dy = ⨍_{\langle x,y\rangle=\delta} f(y)\, dy.$$

Finally, $[\Theta_1 f](x) = f(k^{-1}e_1) = f(x)$ and $[\Theta_{-1} f](x) = f(-k^{-1}e_1) = f(-x)$. □

The previous proposition implies that the operators Θ_δ are exactly those defined in [Laf08, §2]. The following estimate corresponds to [Laf08, Lemme 2.2.a] together with [HdL13, Lemma 3.11].

Lemma 3.4. *For all $\delta \in [-1, 1]$,*

$$\|\Theta_\delta - \Theta_0\|_{\mathcal{B}(L_2)} \leq 4|\delta|^{\frac{1}{2}}.$$

This inequality allows us to obtain the following estimate by interpolation.

Lemma 3.5. *For all $\delta \in [-1, 1]$,*

$$\|T_\delta - T_0\|_{L_p \to L_p} \leq 2^{1+\frac{2}{p}}|\delta|^{\frac{1}{p}}. \tag{3.6}$$

Proof. Observe that, for all $f \in L_2(K)$, there exists $\tilde{f} \in {}^U L_2(K)$ such that $T_\delta \tilde{f} = T_\delta f$ and $\|\tilde{f}\|_2 \leq \|f\|_2$, namely

$$\tilde{f} = \int_U \lambda_2(u) f \, du.$$

So, by Lemma 3.4, for all $f \in L_2(K)$,

$$\begin{aligned}
\|T_\delta f - T_0 f\|_2 &= \|T_\delta \tilde{f} - T_0 \tilde{f}\|_2 \\
&= \|\Psi(\Theta_\delta - \Theta_0)\Psi^{-1}\tilde{f}\|_2 \\
&\leq \|\Theta_\delta - \Theta_0\|_{\mathcal{B}(L_2)} \|\tilde{f}\|_2 \\
&\leq 4|\delta|^{\frac{1}{2}} \|f\|_2.
\end{aligned}$$

This proves the result when $p = 2$. If $p > 2$, choose θ and q such that $1 - \frac{2}{p} < \theta < 1$ and $\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{q}$. This implies that $q > p$. Observe that

$$\|T_\delta - T_0\|_{L_q \to L_q} \leq \|T_\delta\|_{L_q \to L_q} + \|T_0\|_{L_q \to L_q} \leq 2.$$

By interpolation we get

$$\|T_\delta - T_0\|_{L_p \to L_p} \leq (4|\delta|^{\frac{1}{2}})^{1-\theta} 2^\theta.$$

Taking $\theta \to 1 - \frac{2}{p}$, we obtain (3.6). $\square$

Remark 3.6. *In the previous proof, we don't take $q = \infty$ directly because in* (3.5)*, we defined T_δ only for $1 < p < \infty$, in which case the left regular representation is continuous for the strong operator topology. To avoid giving a precise definition of T_δ for $p = \infty$ and since this case is not relevant in this thesis, we make use of the limit process $q \to \infty$.*

For $\delta \in [-1, 1]$, define $\varepsilon(\delta) = 2^{1+\frac{2}{p}} |\delta|^{\frac{1}{p}}$. So the previous lemma states that

$$\|T_\delta - T_0\|_{\mathcal{B}(L_p(K))} \leq \varepsilon(\delta), \qquad \forall \delta \in [-1, 1].$$

Observe that, since the left regular representation commutes with the right regular representation, the same holds for the operators T_δ, namely,

$$T_\delta \rho_p(k) = \rho_p(k) T_\delta, \qquad \forall k \in K.$$

So $T_\delta \in CV_p(K)$. Furthermore, K is compact, so in particular it is amenable, and by [Her73, Theorem 5], $CV_p(K) = PM_p(K)$. Hence $T_\delta \in PM_p(K)$ for all $\delta \in [-1, 1]$.

Now we state the main inequality used in the proof of Theorem 3.2, which involves averages of multipliers. For a continuous function $\varphi : G \to \mathbb{C}$, we define

$$\tilde{\varphi}(t) = \int_K \int_K \varphi(ktk') \, dk \, dk', \qquad \forall t \in G. \tag{3.7}$$

This function is continuous and K-biinvariant. We will also consider the following family of elements of G,

$$D_a = \begin{pmatrix} e^a & 0 & 0 \\ 0 & e^{-\frac{a}{2}} & 0 \\ 0 & 0 & e^{-\frac{a}{2}} \end{pmatrix}, \qquad \forall a \in \mathbb{R}.$$

These matrices satisfy $uD_a = D_a u$ for every $u \in U$.

Lemma 3.7. *For every $\varphi \in M_{p\text{-}cb}(G)$, $a \in \mathbb{R}$ and $\delta \in [-1, 1]$,*

$$|\tilde{\varphi}(D_a k_\delta D_a) - \tilde{\varphi}(D_a k_0 D_a)| \leq \varepsilon(\delta)\|\varphi\|_{M_{p\text{-}cb}(G)}. \tag{3.8}$$

Proof. Consider the function $\phi : K \to \mathbb{C}$ given by

$$\phi(s) = \tilde{\varphi}(D_a s D_a) = \int_K \int_K \varphi(k D_a s D_a k')\, dk\, dk'.$$

Since $\varphi \in M_{p\text{-}cb}(G)$, for all $s, t \in K$,

$$\begin{aligned}
\phi(st^{-1}) &= \int_K \int_K \varphi(k D_a s (k'^{-1} D_a^{-1} t)^{-1})\, dk\, dk' \\
&= \int_K \int_K \langle \beta(k D_a s), \alpha(k'^{-1} D_a^{-1} t)\rangle\, dk\, dk' \\
&= \left\langle \int_K \beta(k D_a s)\, dk, \int_K \alpha(k^{-1} D_a^{-1} t)\, dk \right\rangle,
\end{aligned}$$

where α and β are as in (1.7). The integrals in the last expression are well defined because α and β are continuous and K is compact. So we have found functions $\tilde{\alpha} : K \to E$ and $\tilde{\beta} : K \to E^*$ given by

$$\tilde{\alpha}(t) = \int_K \alpha(k^{-1} D_a^{-1} t)\, dk, \qquad \tilde{\beta}(t) = \int_K \beta(k D_a t)\, dk, \qquad \forall t \in G,$$

such that

$$\phi(st^{-1}) = \langle \tilde{\beta}(s), \tilde{\alpha}(t)\rangle$$

and

$$\sup_{s\in K} \|\tilde{\beta}(s)\| \leq \sup_{s\in G} \|\beta(s)\|, \qquad \sup_{t\in K} \|\tilde{\alpha}(t)\| \leq \sup_{t\in G} \|\alpha(t)\|.$$

Finally, $\tilde{\alpha}$ and $\tilde{\beta}$ are continuous thanks to Lemma 2.3. We conclude that $\phi \in M_{p\text{-}cb}(K)$ and $\|\phi\|_{M_{p\text{-}cb}(K)} \leq \|\varphi\|_{M_{p\text{-}cb}(G)}$. Hence, by Proposition 2.2, $\phi \in A_p(K)$ and $\|\phi\|_{A_p(K)} = \|\phi\|_{M_{p\text{-}cb}(K)}$. Recall that $T_\delta \in PM_p(K) = A_p(K)^*$, thus, using Lemma 3.5,

$$|\langle T_\delta - T_0, \phi\rangle| \leq \|T_\delta - T_0\|_{\mathcal{B}(L_p(K))} \|\phi\|_{A_p(K)} \leq \varepsilon(\delta)\|\varphi\|_{M_{p\text{-}cb}(G)}.$$

On the other hand,

$$\begin{aligned}
\langle T_\delta, \phi\rangle &= \int_U \int_U \langle \lambda_p(u k_\delta u'), \phi\rangle\, du\, du' \\
&= \int_U \int_U \phi(u k_\delta u')\, du\, du' \\
&= \int_U \int_U \int_K \int_K \varphi(k D_a u k_\delta u' D_a k')\, dk\, dk'\, du\, du' \\
&= \int_U \int_U \int_K \int_K \varphi(k u D_a k_\delta D_a u' k')\, dk\, dk'\, du\, du' \\
&= \int_U \int_U \int_K \int_K \varphi(k D_a k_\delta D_a k')\, dk\, dk'\, du\, du' \\
&= \int_K \int_K \varphi(k D_a k_\delta D_a k')\, dk\, dk' \\
&= \tilde{\varphi}(D_a k_\delta D_a).
\end{aligned}$$

And so we obtain (3.8). $\square$

Remark 3.8. *We point out that the use of Proposition 2.2 is not necessary for proving Lemma 3.7. Indeed, by [Daw10, Lemma 8.2], there is a bounded linear map* $M_\phi : PM_p(K) \to PM_p(K)$ *such that* $\|M_\phi\| \leq \|\phi\|_{M_{p\text{-}cb}(K)}$ *and* $M_\phi \lambda_p(s) = \phi(s)\lambda_p(s)$*. By the same computations above, we obtain* $M_\phi T_\delta = \tilde{\varphi}(D_a k_\delta D_a) T_\delta$*. Then we apply* $M_\phi(T_\delta - T_0)$ *to the constant function 1 on* K *and conclude by taking the norm on* $L_p(K)$*.*

3.2.2 Proof of the theorem

This section follows the ideas of [Laf08, §2], which were later used in [LdlS11] and [HdL13]. The letters r, s, t will be now reserved for real numbers. Consider the set

$$\Lambda = \{(r,s,t) \in \mathbb{R}^3 \mid r \geq s \geq t,\ r+s+t = 0\}.$$

Recall that the $K\overline{A^+}K$ decomposition of $\mathrm{SL}(3,\mathbb{R})$ states that, for every $g \in G$, there exists a unique $(r,s,t) \in \Lambda$ such that $g \in KD(r,s,t)K$, where

$$D(r,s,t) = \begin{pmatrix} e^r & 0 & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^t \end{pmatrix}.$$

Therefore, we may define functions $\gamma_1, \gamma_2, \gamma_3 : G \to \mathbb{R}$ by

$$g \in KD(\gamma_1(g), \gamma_2(g), \gamma_3(g))K, \quad (\gamma_1(g), \gamma_2(g), \gamma_3(g)) \in \Lambda, \qquad \forall g \in G.$$

Observe that the eigenvalues of $g^t g$ are $e^{2\gamma_i(g)}$, $i = 1,2,3$. So these functions are continuous. Moreover, for every $\varphi \in M_{p\text{-}cb}(G)$,

$$\tilde{\varphi}(g) = \tilde{\varphi}(D(\gamma_1(g), \gamma_2(g), \gamma_3(g))),$$

where $\tilde{\varphi}$ is defined as in (3.7).

Lemma 3.9. *For all* $(r,s,t) \in \Lambda$ *and* $\varphi \in M_{p\text{-}cb}(G)$*,*

$$|\tilde{\varphi}(D(r,s,t)) - \tilde{\varphi}(D_{-t} k_0 D_{-t})| \leq \varepsilon(e^{r+2t}) \|\varphi\|_{M_{p\text{-}cb}(G)}. \tag{3.9}$$

Proof. Observe first that, since the elements of K are isometries of $\ell_2^3 = (\mathbb{R}^3, \|\cdot\|_2)$, for all $k, k' \in K$,

$$\|kD(r,s,t)^{-1}k'\|_{\mathcal{B}(\ell_2^3)} = \|D(r,s,t)^{-1}\|_{\mathcal{B}(\ell_2^3)} = e^{-t}.$$

Then

$$\begin{aligned} e^{-\gamma_3(D_a k_\delta D_a)} &= \|(D_a k_\delta D_a)^{-1}\|_{\mathcal{B}(\ell_2^3)} \\ &= \left\| \begin{pmatrix} \delta e^{2a} & -e^{\frac{a}{2}}\sqrt{1-\delta^2} & 0 \\ e^{\frac{a}{2}}\sqrt{1-\delta^2} & \delta e^{-a} & 0 \\ 0 & 0 & e^{-a} \end{pmatrix}^{-1} \right\|_{\mathcal{B}(\ell_2^3)} \\ &= \max\left\{ e^a, \left\| \begin{pmatrix} \delta e^{2a} & -e^{\frac{a}{2}}\sqrt{1-\delta^2} \\ e^{\frac{a}{2}}\sqrt{1-\delta^2} & \delta e^{-a} \end{pmatrix}^{-1} \right\|_{\mathcal{B}(\ell_2^2)} \right\}. \end{aligned}$$

But
$$\begin{pmatrix} \delta e^{2a} & -e^{\frac{a}{2}}\sqrt{1-\delta^2} \\ e^{\frac{a}{2}}\sqrt{1-\delta^2} & \delta e^{-a} \end{pmatrix} = \begin{pmatrix} e^a & 0 \\ 0 & e^{-\frac{a}{2}} \end{pmatrix} k \begin{pmatrix} e^a & 0 \\ 0 & e^{-\frac{a}{2}} \end{pmatrix}$$
with $k \in \mathrm{SO}(2,\mathbb{R})$. So, for $a > 0$,
$$\left\| \begin{pmatrix} \delta e^{2a} & -e^{\frac{a}{2}}\sqrt{1-\delta^2} \\ e^{\frac{a}{2}}\sqrt{1-\delta^2} & \delta e^{-a} \end{pmatrix}^{-1} \right\|_{\mathcal{B}(\ell_2^2)} \leq e^{\frac{a}{2}} e^{\frac{a}{2}} = e^a.$$
Therefore, if $a > 0$, then $\gamma_3(D_a k_\delta D_a) = -a$ for all $\delta \in [-1,1]$. On the other hand,
$$D_a k_1 D_a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{2a} & 0 & 0 \\ 0 & e^{-a} & 0 \\ 0 & 0 & e^{-a} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
and
$$D_a k_0 D_a = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\frac{a}{2}} & 0 & 0 \\ 0 & e^{\frac{a}{2}} & 0 \\ 0 & 0 & e^{-a} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$
Let $(r,s,t) \in \Lambda$. The previous equalities show that $\gamma_1(D_{-t}k_1D_{-t}) = -2t$ and $\gamma_1(D_{-t}k_0D_{-t}) = -\frac{t}{2}$. Moreover, observe that $-\frac{t}{2} \leq r \leq -2t$, so by continuity there exists $\delta \in [0,1]$ such that $\gamma_1(D_{-t}k_\delta D_{-t}) = r$. Furthermore, since $\gamma_3(D_{-t}k_\delta D_{-t}) = t$, we have $\gamma_2(D_{-t}k_\delta D_{-t}) = s$. Thus, using Lemma 3.7 and the equality $\tilde{\varphi}(D_{-t}k_\delta D_{-t}) = \tilde{\varphi}(D(r,s,t))$, we get
$$|\tilde{\varphi}(D(r,s,t)) - \tilde{\varphi}(D_{-t}k_0D_{-t})| \leq \varepsilon(\delta)\|\varphi\|_{M_{p\text{-}cb}(G)}.$$
Now observe that
$$\|D_{-t}k_\delta D_{-t}\|_{\mathcal{B}(\ell_2^3)} = \|D(r,s,t)\|_{\mathcal{B}(\ell_2^3)} = e^r$$
and
$$\|D_{-t}k_\delta D_{-t}e_1\|^2 = (\delta e^{-2t})^2 + e^{-t}(1-\delta^2) \geq (\delta e^{-2t})^2.$$
This implies $\delta \leq e^{r+2t}$ and so we get (3.9). □

Let us now define a family of measures in G by
$$\int_G f\, dm_g = \int_K \int_K f(kgk')\, dk\, dk', \quad \forall f \in C_b(G),\ \forall g \in G. \tag{3.10}$$
Since $M_{p\text{-}cb}(G) \hookrightarrow C_b(G)$ is contractive, m_g defines an element of $M_{p\text{-}cb}(G)^*$ of norm at most 1 for all $g \in G$. Then Lemma 3.9 implies the following.

Corollary 3.10. *Let $g, g' \in G$ such that $\gamma_3(g) = \gamma_3(g')$. Then*
$$\|m_g - m_{g'}\|_{M_{p\text{-}cb}(G)^*} \leq \varepsilon(e^{\gamma_1(g)+2\gamma_3(g)}) + \varepsilon(e^{\gamma_1(g')+2\gamma_3(g)}).$$

Proof. Observe that, for every $\varphi \in M_{p\text{-}cb}(G)$,

$$\langle m_g, \varphi \rangle = \int \varphi \, dm_g = \tilde{\varphi}(D(\gamma_1(g), \gamma_2(g), \gamma_3(g))).$$

Then Lemma 3.9 together with the triangle inequality implies

$$|\langle m_g - m_{g'}, \varphi \rangle| \leq \left(\varepsilon(e^{\gamma_1(g)+2\gamma_3(g)}) + \varepsilon(e^{\gamma_1(g')+2\gamma_3(g)})\right) \|\varphi\|_{M_{p\text{-}cb}(G)},$$

and the result follows. □

Moreover, if instead of fixing γ_3, we fix γ_1, we obtain a similar result.

Corollary 3.11. *Let $g, g' \in G$ such that $\gamma_1(g) = \gamma_1(g')$. Then*

$$\|m_g - m_{g'}\|_{M_{p\text{-}cb}(G)^*} \leq \varepsilon(e^{-2\gamma_1(g)-\gamma_3(g)}) + \varepsilon(e^{-2\gamma_1(g)-\gamma_3(g)}).$$

Proof. Consider the continuous group isomorphism $\theta : G \to G$ given by $\theta(g) = (g^t)^{-1}$. Then, for any $\varphi \in M_{p\text{-}cb}(G)$, we have that $\varphi \circ \theta \in M_{p\text{-}cb}(G)$ and $\|\varphi \circ \theta\|_{M_{p\text{-}cb}(G)} = \|\varphi\|_{M_{p\text{-}cb}(G)}$. Indeed,

$$\varphi \circ \theta(g_1 g_2^{-1}) = \varphi(\theta(g_1)\theta(g_2)^{-1}) = \langle \beta(\theta(g_1)), \alpha(\theta(g_2)) \rangle, \qquad \forall g_1, g_2 \in G,$$

where α and β are as in (1.7). Let $(r, s, t), (r', s', t') \in \Lambda$ such that $r = r'$. We may use Corollary 3.10 to get

$$\left|\widetilde{\varphi \circ \theta}(D(-t, -s, -r)) - \widetilde{\varphi \circ \theta}(D(-t', -s', -r'))\right| \\ \leq (\varepsilon(e^{-t-2r}) + \varepsilon(e^{-t'-2r}))\|\varphi\|_{M_{p\text{-}cb}(G)},$$

for every $\varphi \in M_{p\text{-}cb}(G)$. On the other hand, since θ restricted to K is the identity,

$$\widetilde{\varphi \circ \theta}(g) = \int_K \int_K \varphi(k\theta(g)k') \, dk \, dk' = \tilde{\varphi} \circ \theta(g), \quad \forall g \in G.$$

And

$$\begin{aligned} \theta(D(-t, -s, -r)) &= D(t, s, r) \\ &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} D(r, s, t) \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \end{aligned}$$

So

$$\widetilde{\varphi \circ \theta}(D(-t, -s, -r)) = \tilde{\varphi}(D(r, s, t)).$$

Therefore

$$|\tilde{\varphi}(D(r, s, t)) - \tilde{\varphi}(D(r', s', t'))| \leq (\varepsilon(e^{-t-2r}) + \varepsilon(e^{-t'-2r}))\|\varphi\|_{M_{p\text{-}cb}(G)},$$

and the result follows as in Corollary 3.10. □

We will use Corollaries 3.10 and 3.11 repeatedly on some particular paths joining two points of Λ in order to obtain the desired Cauchy net.

Lemma 3.12. *There exists a constant $C > 0$ such that, for all $g, g' \in G$ with $\gamma_3(g') \leq \gamma_3(g)$,*

$$\|m_g - m_{g'}\|_{M_{p\text{-}cb}(G)^*} \leq C e^{\frac{\gamma_3(g)}{p}}.$$

Proof. Put $(r, s, t) = (\gamma_1(g), \gamma_2(g), \gamma_3(g))$ and $(r', s', t') = (\gamma_1(g'), \gamma_2(g'), \gamma_3(g'))$. We shall consider first the case $t \leq -1$. Let $n \geq 0$ such that

$$t - 2(n+1) < t' \leq t - 2n.$$

Let $\varphi \in M_{p\text{-}cb}(G)$ with $\|\varphi\|_{M_{p\text{-}cb}(G)} = 1$ and define $\phi(r, s, t) = \tilde{\varphi}(D(r, s, t))$. Assume first that $s = s' = -1$. Then

$$\begin{aligned}
&\phi(1-t,-1,t) - \phi(1-t',-1,t') \\
&\quad = \sum_{i=0}^{n-1} \phi(1-t+2i,-1,t-2i) - \phi(1-t+2i,1,t-2(i+1)) \\
&\qquad + \sum_{i=0}^{n-1} \phi(1-t+2i,1,t-2(i+1)) - \phi(1-t+2(i+1),-1,t-2(i+1)) \\
&\qquad + \phi(1-t+2n,-1,t-2n) - \phi(1-t+2n,-1+t-2n-t',t') \\
&\qquad + \phi(1-t+2n,-1+t-2n-t',t') - \phi(1-t',-1,t').
\end{aligned}$$

Observe that $\varepsilon(e^{-a}) = \tilde{C} e^{-\frac{a}{p}}$ for all $a \geq 0$. Then, using Corollaries 3.10 and 3.11, we get

$$\begin{aligned}
&|\phi(1-t+2i,-1,t-2i) - \\
&\qquad \phi(1-t+2i,1,t-2(i+1))| \leq \tilde{C}(e^{\frac{t-2i-2}{p}} + e^{\frac{t-2i}{p}}) \\
&\qquad\qquad \leq 2\tilde{C} e^{\frac{t}{p}} e^{\frac{-2i}{p}},
\end{aligned}$$

$$\begin{aligned}
&|\phi(1-t+2i,1,t-2(i+1)) \\
&\qquad -\phi(1-t+2(i+1),-1,t-2(i+1))| \leq \tilde{C}(e^{\frac{t-2i-3}{p}} + e^{\frac{t-2i-1}{p}}) \\
&\qquad\qquad \leq 2\tilde{C} e^{\frac{t}{p}} e^{\frac{-2i}{p}},
\end{aligned}$$

and

$$\begin{aligned}
&|\phi(1-t+2n,-1,t-2n) \\
&\qquad -\phi(1-t+2n,-1+t-2n-t',t')| \leq \tilde{C}(e^{\frac{t-2n-2}{p}} + e^{\frac{2t-2-4n-t'}{p}}) \\
&\qquad\qquad \leq \tilde{C}(e^{\frac{t-2n-2}{p}} + e^{\frac{t-2n}{p}}) \\
&\qquad\qquad \leq 2\tilde{C} e^{\frac{t}{p}}.
\end{aligned}$$

Here we used the fact that $t-2(n+1)-t'<0$. We also get

$$\begin{aligned}|\phi(1-t+2n,-1+t-2n-t',t')\\ -\phi(1-t',-1,t')| &\leq \tilde{C}(e^{\frac{1-t+2n+t'}{p}}+e^{\frac{1+t'}{p}})\\ &\leq 2\tilde{C}e^{\frac{1+t'}{p}}\\ &\leq 2\tilde{C}e^{\frac{1}{p}}e^{\frac{t}{p}}.\end{aligned}$$

Putting everything together, we obtain

$$|\phi(1-t,-1,t)-\phi(1-t',-1,t')|\leq C_1e^{\frac{t}{p}},$$

with

$$C_1=2\tilde{C}\left(\frac{2}{1-e^{\frac{-2}{p}}}+1+e^{\frac{1}{p}}\right),$$

and this constant depends only on p. Now we deal with the general case. If $s>-1$, then

$$|\phi(r,s,t)-\phi(1-t,-1,t)|\leq\tilde{C}(e^{\frac{r+2t}{p}}+e^{\frac{1+t}{p}})\leq 2\tilde{C}e^{\frac{1}{p}}e^{\frac{t}{p}}.$$

If $s<-1$,

$$|\phi(r,s,t)-\phi(r,-1,1-r)|\leq\tilde{C}(e^{\frac{-2r-t}{p}}+e^{\frac{-r-1}{p}})\leq 2\tilde{C}e^{\frac{-2}{p}}e^{\frac{t}{p}}.$$

In both cases we found $(\tilde{r},-1,\tilde{t})\in\Lambda$ satisfying

$$|\phi(r,s,t)-\phi(\tilde{r},-1,\tilde{t})|\leq C_2e^{\frac{t}{p}},$$

with C_2 depending only on p, and such that $\tilde{t}\leq t$. Therefore

$$\begin{aligned}|\phi(r,s,t)-\phi(r',s',t')| &\leq |\phi(r,s,t)-\phi(\tilde{r},-1,\tilde{t})|+|\phi(\tilde{r},-1,\tilde{t})-\phi(\tilde{r}',-1,\tilde{t}')|\\ &\quad+|\phi(\tilde{r}',-1,\tilde{t}')-\phi(r',s',t')|\\ &\leq C_2e^{\frac{t}{p}}+C_1e^{\frac{1}{p}\max\{\tilde{t},\tilde{t}'\}}+C_2e^{\frac{t'}{p}}\\ &\leq C_3e^{\frac{t}{p}},\end{aligned}$$

with $C_3=(C_1+2C_2)$. This is valid when $t\leq -1$. If $t>-1$, then

$$\begin{aligned}|\phi(r,s,t)-\phi(r',s',t')| &\leq |\phi(r,s,t)|+|\phi(r',s',t')|\\ &\leq 2\|\varphi\|_{M_{p\text{-}cb}(G)}=2\\ &\leq C_4e^{\frac{t}{p}},\end{aligned}$$

with $C_4=2e^{\frac{1}{p}}$. We obtain the result taking $C=\max\{C_3,C_4\}$. □

Proof of Theorem 3.2. Let $p\in[2,\infty)$. Observe that, for $g\in G$, $\|g\|=e^{\gamma_1(g)}$ and $\gamma_3(g)\leq-\frac{1}{2}\gamma_1(g)$. So $\gamma_3(g)\to-\infty$ when $\|g\|\to\infty$. Then Lemma 3.12 implies the existence of $\mu \in M_{p\text{-}cb}(G)^*$ such that $m_g \to \mu$ as $\|g\| \to \infty$. Now define a new family of measures $\tilde{m}_g$ on G by

$$\begin{aligned} \int f\, d\tilde{m}_g &= \fint_{B_2} \int f(x)\, dm_{hg}(x)\, dh \\ &= \fint_{B_2} \int_K \int_K f(khgk')\, dk\, dk'\, dh, \quad \forall f \in C_c(G), \end{aligned} \tag{3.11}$$

where $B_2 = \{g \in G \ : \ \|g\| < 2\}$ and $\fint_{B_2} \cdots dh$ stands for the normalised integration over B_2. These again are probability measures for each $g \in G$. Moreover, observe that equation (3.11) says that $\tilde{m}_g = \nu_K * \chi_{B_2} * \delta_g * \nu_K$, where ν_K is the normalised Haar measure on K and χ_{B_2} is the normalised indicator function on B_2. Since $L_1(G)$ is a two-sided ideal in the convolution algebra of complex measures (see e.g. [Fol95, §2.5]), we have that $\tilde{m}_g \in L_1(G)$ for all $g \in G$. On the other hand, using Lemma 3.12, for all $\varphi \in M_{p\text{-}cb}(G)$ with $\|\varphi\|_{M_{p\text{-}cb}(G)} = 1$,

$$|\langle \mu - \tilde{m}_g, \varphi \rangle| \leq \fint_{B_2} |\langle \mu - m_{hg}, \varphi \rangle|\, dh \leq \fint_{B_2} C e^{\frac{\gamma_3(hg)}{p}}\, dh.$$

But $e^{\gamma_3(hg)} = \|(hg)^{-1}\|^{-1}$ and $\|g^{-1}\| \leq \|g^{-1}h^{-1}\|\|h\|$. Thus

$$e^{\gamma_3(hg)} \leq \|h\|\|g^{-1}\|^{-1} \leq 2e^{\gamma_3(g)}.$$

And so, $\tilde{m}_g$ converges to μ when $\|g\| \to \infty$. This implies that μ is in the adherence of $L_1(G)$ in $M_{p\text{-}cb}(G)^*$, that is, in $Q_{p\text{-}cb}(G)$. Now

$$\langle \mu, 1 \rangle = \lim_{g \to \infty} m_g(G) = 1.$$

And if $\varphi \in M_{p\text{-}cb}(G) \cap C_c(G)$, then for every g sufficiently large, $\varphi(g) = 0$. Since $\|kgk'\| = \|g\|$ for all $k, k' \in K$, we have also $\varphi(kgk') = 0$. Therefore

$$\langle \mu, \varphi \rangle = \lim_{g \to \infty} \int_K \int_K \varphi(kgk')\, dk\, dk' = 0.$$

Thus, if G had the p-AP, there would exist a net (φ_i) in $A_{p,c}(G)$ such that

$$0 = \langle \mu, \varphi_i \rangle \to \langle \mu, 1 \rangle = 1.$$

And this is a contradiction. Therefore, G does not have the p-AP for any $p \in [2, \infty)$. By Proposition 2.6, the same holds for $1 < p < 2$. □

3.3 The symplectic group Sp$(2, \mathbb{R})$

In this section we prove the following.

Theorem 3.13. *The group $Sp(2, \mathbb{R})$ does not have the p-AP for any $1 < p < \infty$.*

The strategy is the same as for $\mathrm{SL}(3,\mathbb{R})$, with the main difference being the choice of the paths in the Weyl chamber in order to establish a result analogous to Lemma 3.12. This is achieved by considering two different families of operators that will play the role of (T_δ), namely (3.12) and (3.13). These operators were defined in [dLdlS15]. From now on we fix $p \in [2,\infty)$, $G = \mathrm{Sp}(2,\mathbb{R})$ and K as in (3.2). Recall that K is isomorphic to $\mathrm{U}(2)$, and

$$\mathrm{SO}(2,\mathbb{R}) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \;\middle|\; \theta \in [0,2\pi) \right\}$$

may be viewed as a subgroup of $\mathrm{U}(2)$ by inclusion. Therefore, using the isomorphism (3.3), we may define a family of operators (T_θ) on $L_p(K)$ by

$$T_\theta f = \int_{\mathrm{SO}(2)} \int_{\mathrm{SO}(2)} \lambda(r d_\theta r') f \, dr \, dr', \quad \forall \theta \in [0,2\pi), \ \forall f \in L_p(K), \tag{3.12}$$

where

$$d_\theta = \begin{pmatrix} e^{\mathrm{i}\theta} & 0 \\ 0 & e^{-\mathrm{i}\theta} \end{pmatrix} \in \mathrm{SU}(2).$$

Here $\mathrm{SO}(2)$ is endowed with the normalised Haar measure.

Lemma 3.14. *There is a constant $C_1 > 0$ such that, for all $\theta \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$,*

$$\|T_\theta - T_{\frac{\pi}{4}}\|_{\mathcal{B}(L_p)} \leq C_1 \left|\theta - \tfrac{\pi}{4}\right|^{\frac{1}{p}}.$$

Proof. The inequality for $p = 2$ is given by [dLdlS15, Lemma 3.1]. The other cases follow by interpolation as in Lemma 3.5. □

We consider also

$$S_\theta = \frac{1}{2\pi} \int_0^{2\pi} \lambda(d_\varphi u_\theta d_{-\varphi}) \, d\varphi, \quad \forall \theta \in \mathbb{R}, \ \forall f \in L_p(K), \tag{3.13}$$

where

$$u_\theta = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{\mathrm{i}\theta} & -1 \\ 1 & e^{-\mathrm{i}\theta} \end{pmatrix}.$$

Lemma 3.15. *There is a constant $C_2 > 0$ such that, for all $\theta_1, \theta_2 \in \mathbb{R}$,*

$$\|S_{\theta_1} - S_{\theta_2}\|_{\mathcal{B}(L_p)} \leq C_2 \left|\theta_1 - \theta_2\right|^{\frac{1}{2p}}.$$

Proof. The case $p = 2$ is given by [dLdlS15, Lemma 3.2] and the others follow by interpolation as in Lemma 3.5. □

Again since $PM_p(K) = CV_p(K)$, the operators T_θ and S_θ belong to $PM_p(K)$. In order to obtain a result analogous to Lemma 3.7, we define the following elements of G:

$$D_a = \begin{pmatrix} e^a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{-a} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad D'_a = \begin{pmatrix} e^a & 0 & 0 & 0 \\ 0 & e^a & 0 & 0 \\ 0 & 0 & e^{-a} & 0 \\ 0 & 0 & 0 & e^{-a} \end{pmatrix}, \quad a \in \mathbb{R}.$$

Consider also

$$v = \frac{1}{\sqrt{2}} \begin{pmatrix} 1+\mathrm{i} & 0 \\ 0 & 1+\mathrm{i} \end{pmatrix} \in \mathrm{U}(2),$$

which again, by the isomorphism (3.3), defines an element of K. Then, defining the average $\tilde{\varphi}$ of a multiplier $\varphi \in M_{p\text{-}cb}(G)$ as in (3.7), we obtain the following two lemmas, which can be proved in the same way as Lemma 3.7, using the operators T_θ and S_θ instead of T_δ.

Lemma 3.16. *For every $\varphi \in M_{p\text{-}cb}(G)$, $a \in \mathbb{R}$ and $\theta \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$,*

$$|\tilde{\varphi}(D'_a d_\theta v D'_a) - \tilde{\varphi}(D'_{2a})| \leq C_1 \left|\theta - \tfrac{\pi}{4}\right|^{\frac{1}{p}} \|\varphi\|_{M_{p\text{-}cb}(G)}.$$

Lemma 3.17. *For every $\varphi \in M_{p\text{-}cb}(G)$ and $a, \theta_1, \theta_2 \in \mathbb{R}$,*

$$|\tilde{\varphi}(D_a u_{\theta_1} D_a) - \tilde{\varphi}(D_a u_{\theta_2} D_a)| \leq C_2 |\theta_1 - \theta_2|^{\frac{1}{2p}} \|\varphi\|_{M_{p\text{-}cb}(G)}.$$

The remaining of this section consists mostly of adaptations of some of the results in [HdL13, §3]. Recall that $\overline{A^+} = \{D(\beta, \gamma) \ : \ \beta \geq \gamma \geq 0\}$, where $D(\beta, \gamma)$ is defined as in (3.4).

Lemma 3.18. *[HdL13, Lemma 3.16] Let $\beta \geq \gamma \geq 0$. Then the equations*

$$\sinh^2(2s) + \sinh^2(s) = \sinh^2(\beta) + \sinh^2(\gamma)$$
$$\sinh(2t)\sinh(t) = \sinh(\beta)\sinh(\gamma)$$

have unique solutions $s = s(\beta, \gamma)$, $t = t(\beta, \gamma)$ in the interval $[0, \infty)$. Moreover,

$$s \geq \frac{\beta}{4}, \quad t \geq \frac{\gamma}{2}.$$

Lemma 3.19. *[HdL13, Lemma 3.19] Let $s \geq t \geq 0$. Then the system of equations*

$$\sinh^2(\beta) + \sinh^2(\gamma) = \sinh^2(2s) + \sinh^2(s)$$
$$\sinh(\beta)\sinh(\gamma) = \sinh(2t)\sinh(t)$$

has a unique solution $(\beta, \gamma) \in \mathbb{R}^2$ for which $\beta \geq \gamma \geq 0$. Moreover, if $1 \leq t \leq s \leq \frac{3t}{2}$, then

$$|\beta - 2s| \leq 1,$$
$$|\gamma + 2s - 3t| \leq 1.$$

Lemma 3.20. *There exists a constant $C_3 > 0$ such that whenever $\beta \geq \gamma \geq 0$ and $s = s(\beta, \gamma)$ is chosen as in Lemma 3.18, then*

$$|\tilde{\varphi}(D(\beta, \gamma)) - \tilde{\varphi}(D(2s, s))| \leq C_3 e^{-\frac{\beta - \gamma}{4p}} \|\varphi\|_{M_{p\text{-}cb}(G)},$$

for every $\varphi \in M_{p\text{-}cb}(G)$.

Proof. First assume that $\beta - \gamma \geq 8$. Then by (the proof of) [HdL13, Lemma 3.17], there exist $r_1, r_2 \in \left[0, \frac{1}{2}\right]$ and $w_1, w_2 \in \mathrm{SU}(2)$ given by

$$w_i = \begin{pmatrix} a_i + \mathrm{i}b_i & 0 \\ 0 & a_i - \mathrm{i}b_i \end{pmatrix}, \quad i = 1, 2,$$

where $a_i = \left(\dfrac{1 + r_i}{2}\right)^{\frac{1}{2}}$ and $b_i = \left(\dfrac{1 - r_i}{2}\right)^{\frac{1}{2}}$, such that

$$D'_a w_1 v D'_a \in KD(\beta, \gamma)K, \quad D'_a w_2 v D'_a \in KD(2s, s)K,$$

for some $a > 0$. Observe that $w_i = d_{\theta_i}$ with $\theta_i = \arctan\left(\frac{b_i}{a_i}\right)$, and

$$(\tan\theta_i)^2 = \frac{1 - r_i}{1 + r_i} \in \left[\tfrac{1}{3}, 1\right]$$

because $0 \leq r_i \leq \frac{1}{2}$. This implies that $\theta_i \in \left[\frac{\pi}{6}, \frac{\pi}{4}\right]$, so by Lemma 3.16,

$$|\tilde{\varphi}(D'_a w_i v D'_a) - \tilde{\varphi}(D'_{2a})| \leq C_1 \left|\theta_i - \tfrac{\pi}{4}\right|^{\frac{1}{p}} \|\varphi\|_{M_{p\text{-}cb}(G)},$$

for every $\varphi \in M_{p\text{-}cb}(G)$. Using the inequality $|\arctan(x) - \arctan(y)| \leq |x - y|$, we obtain

$$\left|\theta_i - \tfrac{\pi}{4}\right| \leq \left|\tfrac{b_i}{a_i} - 1\right| = 1 - \left(\frac{1 - r_i}{1 + r_i}\right)^{\frac{1}{2}} \leq 2r_i.$$

The last inequality can be justified as follows:

$$\begin{aligned} 1 - 2r_i \leq \left(\frac{1 - r_i}{1 + r_i}\right)^{\frac{1}{2}} &\iff 1 - 4r_i + 4r_i^2 \leq \frac{1 - r_i}{1 + r_i} \\ &\iff 1 - 4r_i + 4r_i^2 + r_i - 4r_i^2 + 4r_i^3 \leq 1 - r_i \\ &\iff 4r_i^3 \leq 2r_i \iff r_i^2 \leq \frac{1}{2}. \end{aligned}$$

Therefore, since $\tilde{\varphi}(D(\beta, \gamma)) = \tilde{\varphi}(D'_a w_1 v D'_a)$ and $\tilde{\varphi}(D(2s, s)) = \tilde{\varphi}(D'_a w_2 v D'_a)$,

$$\begin{aligned} |\tilde{\varphi}(D(\beta, \gamma)) - \tilde{\varphi}(D(2s, s))| &\leq |\tilde{\varphi}(D'_a w_1 v D'_a) - \tilde{\varphi}(D'_{2a})| \\ &\quad + |\tilde{\varphi}(D'_a w_2 v D'_a) - \tilde{\varphi}(D'_{2a})| \\ &\leq C_1 \left((2r_1)^{\frac{1}{p}} + (2r_2)^{\frac{1}{p}}\right) \|\varphi\|_{M_{p\text{-}cb}(G)}. \end{aligned}$$

Again by the proof of [HdL13, Lemma 3.17], $r_1, r_2 \leq 2e^{\frac{\gamma - \beta}{4}}$. So

$$C_1 \left((2r_1)^{\frac{1}{p}} + (2r_2)^{\frac{1}{p}}\right) \leq C e^{\frac{\gamma - \beta}{4p}},$$

where $C > 0$ depends only on p. Finally, if $\beta - \gamma \leq 8$, then $\frac{2}{p} + \frac{\gamma - \beta}{4p} \geq 0$ and

$$\begin{aligned} |\tilde{\varphi}(D(\beta, \gamma)) - \tilde{\varphi}(D(2s, s))| &\leq 2\|\tilde{\varphi}\|_\infty \\ &\leq 2\|\tilde{\varphi}\|_{M_{p\text{-}cb}(K)} \\ &\leq 2\|\varphi\|_{M_{p\text{-}cb}(G)} \\ &\leq C' e^{\frac{\gamma - \beta}{4p}} \|\varphi\|_{M_{p\text{-}cb}(G)}, \end{aligned}$$

with $C' = 2e^{\frac{2}{p}}$. Hence, the result follows with $C_3 = \max\{C, C'\}$. □

Lemma 3.21. *There exists a constant $C_4 > 0$ such that whenever $\beta \geq \gamma \geq 0$ and $t = t(\beta, \gamma)$ is chosen as in Lemma 3.18, then*

$$|\tilde{\varphi}(D(\beta,\gamma)) - \tilde{\varphi}(D(2t,t))| \leq C_4 e^{-\frac{\gamma}{4p}} \|\varphi\|_{M_{p\text{-}cb}(G)},$$

for every $\varphi \in M_{p\text{-}cb}(G)$.

Proof. Assume first that $\gamma \geq 2$. Then the proof of [HdL13, Lemma 3.18] gives the existence of $\theta_1, \theta_2 \in \mathbb{R}$ such that $|\theta_1 - \theta_2| \leq e^{-\frac{\gamma}{2}}$ and

$$D_a u_{\theta_1} D_a \in K D(\beta,\gamma) K, \quad D_a u_{\theta_2} D_a \in K D(2t,t) K,$$

for some $a > 0$. Then, using Lemma 3.17 we get

$$\begin{aligned}|\tilde{\varphi}(D(\beta,\gamma)) - \tilde{\varphi}(D(2t,t))| &= |\tilde{\varphi}(D_a u_{\theta_1} D_a) - \tilde{\varphi}(D_a u_{\theta_2} D_a)| \\ &\leq C_2 \, |\theta_1 - \theta_2|^{\frac{1}{2p}} \, \|\varphi\|_{M_{p\text{-}cb}(G)} \\ &\leq C e^{-\frac{\gamma}{4p}} \|\varphi\|_{M_{p\text{-}cb}(G)}.\end{aligned}$$

for every $\varphi \in M_{p\text{-}cb}(G)$, with $C > 0$ depending only on p. Finally, if $\gamma < 2$, then

$$|\tilde{\varphi}(D(\beta,\gamma)) - \tilde{\varphi}(D(2t,t))| \leq 2\|\tilde{\varphi}\|_\infty \leq 2\|\varphi\|_{M_{p\text{-}cb}(G)} \leq C' e^{-\frac{\gamma}{4p}} \|\varphi\|_{M_{p\text{-}cb}(G)},$$

with $C' = 2e^{\frac{1}{2p}}$. The result follows with $C_4 = \max\{C, C'\}$. ☐

Lemma 3.22. *There exists a constant $C_5 > 0$ such that whenever $s, t \geq 0$ satisfy $2 \leq t \leq s \leq \frac{6}{5}t$,*

$$|\tilde{\varphi}(D(2s,s)) - \tilde{\varphi}(D(2t,t))| \leq C_5 e^{-\frac{s}{8p}} \|\varphi\|_{M_{p\text{-}cb}(G)},$$

for every $\varphi \in M_{p\text{-}cb}(G)$.

Proof. Take $\beta \geq \gamma \geq 0$ given by Lemma 3.19. Then, by Propositions 3.20 and 3.21,

$$|\tilde{\varphi}(D(\beta,\gamma)) - \tilde{\varphi}(D(2s,s))| \leq C_3 e^{-\frac{\beta-\gamma}{4p}} \|\varphi\|_{M_{p\text{-}cb}(G)},$$

$$|\tilde{\varphi}(D(\beta,\gamma)) - \tilde{\varphi}(D(2t,t))| \leq C_4 e^{-\frac{\gamma}{4p}} \|\varphi\|_{M_{p\text{-}cb}(G)}.$$

As shown in [HdL13, Lemma 3.20], $\min\{\gamma, \beta - \gamma\} \geq \frac{s}{2} - 1$. Thus the result follows by the triangle inequality with $C_5 = e^{\frac{1}{4p}}(C_3 + C_4)$. ☐

Lemma 3.23. *There exists a constant $C_6 > 0$ such that whenever $s \geq t \geq 0$,*

$$|\tilde{\varphi}(D(2s,s)) - \tilde{\varphi}(D(2t,t))| \leq C_6 e^{-\frac{t}{8p}} \|\varphi\|_{M_{p\text{-}cb}(G)},$$

for every $\varphi \in M_{p\text{-}cb}(G)$.

Proof. Assume first that $t \geq 5$. Write $s = t + n + \delta$ where $n \geq 0$ is an integer and $\delta \in [0, 1)$. Then, by Lemma 3.22, for every $j \in \{0, 1, ..., n-1\}$,

$$|\tilde{\varphi}(D(2(t+j+1), t+j+1)) - \tilde{\varphi}(D(2(t+j), t+j))| \leq C_5 e^{-\frac{t+j+1}{8p}} \|\varphi\|_{M_{p\text{-}cb}(G)}.$$

And

$$|\tilde{\varphi}(D(2s, s)) - \tilde{\varphi}(D(2(t+n), t+n))| \leq C_5 e^{-\frac{s}{8p}} \|\varphi\|_{M_{p\text{-}cb}(G)}.$$

Thus

$$\begin{aligned} |\tilde{\varphi}(D(2s, s)) - \tilde{\varphi}(D(2t, t))| &\leq C_5 \left(\sum_{j=0}^{n} e^{-\frac{t+j}{8p}} \right) \|\varphi\|_{M_{p\text{-}cb}(G)} \\ &\leq C e^{-\frac{t}{8p}} \|\varphi\|_{M_{p\text{-}cb}(G)}, \end{aligned}$$

with $C > 0$ depending only on p. Finally, if $t < 5$, then

$$|\tilde{\varphi}(D(2s, s)) - \tilde{\varphi}(D(2t, t))| \leq 2\|\tilde{\varphi}\|_\infty \leq 2\|\varphi\|_{M_{p\text{-}cb}(G)}.$$

The result follows by taking $C_6 = \max\{C, 2e^{\frac{5}{8p}}\}$. □

Lemma 3.24. *There exists a constant $C_7 > 0$ such that for all $\beta_1 \geq \gamma_1 \geq 0$, $\beta_2 \geq \gamma_2 \geq 0$ with $\beta_1 \leq \beta_2$,*

$$|\tilde{\varphi}(D(\beta_1, \gamma_1)) - \tilde{\varphi}(D(\beta_2, \gamma_2))| \leq C_7 e^{-\frac{\beta_1}{32p}} \|\varphi\|_{M_{p\text{-}cb}(G)},$$

for every $\varphi \in M_{p\text{-}cb}(G)$.

Proof. If $\beta_i \geq 2\gamma_i$, then $\beta_i - \gamma_i \geq \frac{\beta_i}{2}$, so by Lemma 3.20,

$$|\tilde{\varphi}(D(\beta_i, \gamma_i)) - \tilde{\varphi}(D(2s_i, s_i))| \leq C_3 e^{-\frac{\beta_i}{8p}} \|\varphi\|_{M_{p\text{-}cb}(G)},$$

with $s_i \geq \frac{\beta_i}{4}$. If $\beta_i < 2\gamma_i$, by Lemma 3.21,

$$|\tilde{\varphi}(D(\beta_i, \gamma_i)) - \tilde{\varphi}(D(2t_i, t_i))| \leq C_4 e^{-\frac{\beta_i}{8p}} \|\varphi\|_{M_{p\text{-}cb}(G)},$$

with $t_i \geq \frac{\gamma_i}{2} \geq \frac{\beta_i}{4}$. In both cases there exists $r_i \geq \frac{\beta_i}{4}$ such that

$$|\tilde{\varphi}(D(\beta_i, \gamma_i)) - \tilde{\varphi}(D(2r_i, r_i))| \leq C e^{-\frac{\beta_i}{8p}} \|\varphi\|_{M_{p\text{-}cb}(G)},$$

with $C = \max\{C_3, C_4\}$. Moreover, by Lemma 3.23,

$$|\tilde{\varphi}(D(2r_1, r_1)) - \tilde{\varphi}(D(2r_2, r_2))| \leq C_6 e^{-\frac{r}{8p}} \|\varphi\|_{M_{p\text{-}cb}(G)},$$

with $r = \min\{r_1, r_2\}$. Observe that $r \geq \min\{\frac{\beta_1}{4}, \frac{\beta_2}{4}\} = \frac{\beta_1}{4}$. Putting everything together we get

$$|\tilde{\varphi}(D(\beta_1, \gamma_1)) - \tilde{\varphi}(D(\beta_2, \gamma_2))| \leq C_7 e^{-\frac{\beta_1}{32p}} \|\varphi\|_{M_{p\text{-}cb}(G)},$$

with $C_7 = 2C + C_6$. □

Now we are in position of proving Theorem 3.13 in the same way as Theorem 3.2. Define the functions $\beta, \gamma : G \to [0, \infty)$ by

$$g \in KD(\beta(g), \gamma(g))K, \quad \beta(g) \geq \gamma(g) \geq 0, \quad \forall g \in G,$$

and the family of measures m_g as in (3.10). Then Lemma 3.24 can be stated as follows.

Corollary 3.25. *There exists a constant $C > 0$ such that, for all $g, g' \in G$ with $\beta(g) \leq \beta(g')$,*

$$\|m_g - m_{g'}\|_{M_{p\text{-}cb}(G)^*} \leq Ce^{-\frac{\beta(g)}{32p}}.$$

Proof of Theorem 3.13. The proof is almost the same as that of Theorem 3.2, by using Corollary 3.25 instead of Lemma 3.12. Just observe that in this case, $\|g\| = e^{\beta(g)}$ and

$$e^{-\beta(hg)} = \|hg\|^{-1} \leq \|h^{-1}\|\|g\|^{-1} \leq \|h\|e^{-\beta(g)},$$

for every $g, h \in G$. The last inequality holds because $h^{-1} = J^{-1}h^tJ$, where J is the matrix defined in (3.1). □

3.4 Simple Lie groups with finite centre

Now we are ready to give the proof of Theorem 3.1, which relies on the following lemma.

Lemma 3.26. *Let $1 < p < \infty$ and G_1, G_2 be two locally isomorphic connected simple Lie groups with finite centre. Then G_1 has the p-AP if and only if G_2 has the p-AP.*

Proof. Since the groups are locally isomorphic, their Lie algebras are isomorphic, which in turn implies that their adjoint groups are isomorphic (see [Hel78, §II.5] for more details). If we denote the centre of G_i by $Z(G_i)$, then by [Hel78, Corollary II.5.2], $G_1/Z(G_1)$ and $G_2/Z(G_2)$ are isomorphic. Then, using Proposition 2.7, we see that G_i has the p-AP if and only if $G_i/Z(G_i)$ does. Since the quotients are isomorphic, the result follows. □

Proof of Theorem 3.1. First let us prove part (a). If G has real rank greater than 1, then by [Mar91, Proposition I.1.6.2], it has a subgroup H which is locally isomorphic to either $\mathrm{SL}(3, \mathbb{R})$ or $\mathrm{Sp}(2, \mathbb{R})$. Moreover, as was shown in [Dor96, §4], H is a closed subgroup of G. By Lemma 3.26, H cannot have the p-AP for any $1 < p < \infty$, or else it would contradict Theorem 3.2 or Theorem 3.13. Furthermore, by Proposition 2.9, the same holds for G. On the other hand, it was shown in [CH89] that if G has real rank 1, it is weakly amenable. And it follows from the KAK decomposition that if G has real rank 0, it is compact, which in turns implies that it is amenable. By [HK94, Theorem 1.12], these properties are stronger than the AP, so we conclude using Proposition 2.4. Part (b) is a consequence of (a) plus Propositions 2.9 and 2.11. □

Chapter 4

Radial Schur multipliers

This chapter concentrates on radial Schur multipliers on products of trees, products of hyperbolic graphs and finite dimensional CAT(0) cube complexes. The main goal is to provide the proofs of Theorems A, B and C below. The reason behind this new nomenclature is a practical one, as we will define sets $\mathcal{A}_N$ (resp. $\mathcal{B}_N$, $\mathcal{C}_N$) which correspond to the functions satisfying the hypotheses of Theorem A (resp. B, C) for each $N \geq 1$.

4.1 Main results

We will begin by stating in a more precise way the results described in the introduction. Recall that if X is a connected graph, a function $\phi : X \times X \to \mathbb{C}$ is said to be radial if there exists $\dot{\phi} : \mathbb{N} \to \mathbb{C}$ such that

$$\phi(x,y) = \dot{\phi}(d(x,y)), \quad \forall x, y \in X. \tag{4.1}$$

Conversely, we say that $\dot{\phi}$ defines a radial function $\phi : X \times X \to \mathbb{C}$ if (4.1) holds for ϕ. We also define the discrete derivatives

$$\begin{aligned} \mathfrak{d}_1\dot{\phi}(n) &= \dot{\phi}(n) - \dot{\phi}(n+1), \\ \mathfrak{d}_2\dot{\phi}(n) &= \dot{\phi}(n) - \dot{\phi}(n+2), \quad \forall n \in \mathbb{N}, \end{aligned}$$

and by induction, $\mathfrak{d}_j^{m+1}\dot{\phi}(n) = \mathfrak{d}_j(\mathfrak{d}_j^m\dot{\phi})(n)$ for $j = 1, 2$ and $m \geq 1$. These higher order derivatives admit the following expression,

$$\mathfrak{d}_j^m\dot{\phi}(n) = \sum_{k=0}^{m} \tbinom{N}{k}(-1)^k\dot{\phi}(n+jk). \tag{4.2}$$

Let $\mathcal{T}_d$ be the d-homogeneous tree. Recall that a Hankel matrix is an infinite matrix of the form $(a_{i+j})_{i,j\in\mathbb{N}}$, where (a_n) is a sequence of complex numbers. Haagerup, Steenstrup and Szwarc [HSS10] proved that a function $\dot{\phi} : \mathbb{N} \to \mathbb{C}$ defines a radial Schur multiplier on $\mathcal{T}_d$ ($3 \leq d \leq \infty$) if and only if the Hankel matrix

$$H = (\mathfrak{d}_2\dot{\phi}(i+j))_{i,j\in\mathbb{N}}$$

belongs to the trace class $S_1(\ell_2(\mathbb{N}))$ (see Appendix B for a definition). By Theorem C.1, this is equivalent to the fact that the analytic function $(1-z^2)\sum_{n\geq 0}\dot{\phi}(n)z^n$ belongs to the Besov space $B_1^1(\mathbb{T})$ (see Appendix C for more details). Moreover, they show that the associated Schur multiplier ϕ satisfies

$$\|\phi\|_{cb} = |c_+| + |c_-| + \begin{cases} \left(1-\frac{1}{d-1}\right)\left\|\left(1-\frac{1}{d-1}\tau\right)^{-1}H\right\|_{S_1}, & \text{if } 3\leq d<\infty, \\ \|H\|_{S_1}, & \text{if } d=\infty, \end{cases}$$

where

$$c_\pm = \frac{1}{2}\lim_{n\to\infty}\dot{\phi}(2n) \pm \frac{1}{2}\lim_{n\to\infty}\dot{\phi}(2n+1), \tag{4.3}$$

and $\tau : S_1(\ell_2(\mathbb{N})) \to S_1(\ell_2(\mathbb{N}))$ is defined by $\tau(A) = SAS^*$, where S is the forward shift operator on $\ell_2(\mathbb{N})$. In particular, for $3\leq d<\infty$,

$$\|\phi\|_{cb} \geq \frac{d-2}{d}\|H\|_{S_1} + |c_+| + |c_-|.$$

Since any tree $\mathcal{T}$ of minimum degree $d\geq 3$ admits isometric embeddings

$$\mathcal{T}_d \hookrightarrow \mathcal{T} \hookrightarrow \mathcal{T}_\infty,$$

and since, by Theorem 1.3, the restriction of a Schur multiplier to a subset is again a Schur multiplier, a corollary of their result is the following.

Theorem 4.1 (Haagerup–Steenstrup–Szwarc). *Let $\dot{\phi} : \mathbb{N}\to\mathbb{C}$ be a function. The following are equivalent.*

a) The function $\dot{\phi}$ defines a radial Schur multiplier on any tree of minimum degree $d\geq 3$.

b) The Hankel matrix

$$H = (\mathfrak{d}_2\dot{\phi}(i+j))_{i,j\in\mathbb{N}}$$

is an element of $S_1(\ell_2(\mathbb{N}))$.

c) The analytic function

$$z\mapsto (1-z^2)\sum_{n\geq 0}\dot{\phi}(n)z^n$$

belongs to the Besov space $B_1^1(\mathbb{T})$.

In that case, the following limits exist

$$\lim_{n\to\infty}\dot{\phi}(2n), \quad \lim_{n\to\infty}\dot{\phi}(2n+1),$$

and the corresponding Schur multiplier ϕ satisfies

$$\frac{d-2}{d}\|H\|_{S_1} + |c_+| + |c_-| \leq \|\phi\|_{cb} \leq \|H\|_{S_1} + |c_+| + |c_-|, \tag{4.4}$$

where c_+ and c_- are defined as in (4.3).

We extend this result to finite products of trees.

Theorem A. *Let $N \geq 1$ and let $\dot{\phi} : \mathbb{N} \to \mathbb{C}$ be a bounded function. The following are equivalent.*

a) The function $\dot{\phi}$ defines a radial Schur multiplier on any product of N infinite trees $T_1, ..., T_N$ of minimum degrees $d_1, ..., d_N \geq 3$.

b) The generalised Hankel matrix

$$H = \left(\binom{N+i-1}{N-1}^{\frac{1}{2}} \binom{N+j-1}{N-1}^{\frac{1}{2}} \mathfrak{d}_2^N \dot{\phi}(i+j) \right)_{i,j \in \mathbb{N}} \tag{4.5}$$

is an element of $S_1(\ell_2(\mathbb{N}))$.

c) The analytic function

$$z \mapsto (1-z^2)^N \sum_{n \geq 0} \dot{\phi}(n) z^n$$

belongs to the Besov space $B_1^N(\mathbb{T})$.

In that case, the following limits exist

$$\lim_{n\to\infty} \dot{\phi}(2n), \quad \lim_{n\to\infty} \dot{\phi}(2n+1),$$

and the corresponding Schur multiplier ϕ satisfies

$$\left[\prod_{i=1}^{N} \frac{d_i - 2}{d_i} \right] \|H\|_{S_1} + |c_+| + |c_-| \leq \|\phi\|_{cb} \leq \|H\|_{S_1} + |c_+| + |c_-|, \tag{4.6}$$

where c_+ and c_- are defined as in (4.3).

The proof of Theorem A uses the same ideas as [HSS10]; however, some new considerations must be made in order adapt them to products. Observe that we have added the hypothesis that $\dot{\phi}$ is bounded. In Theorem 4.1, this is a consequence of the fact that the Hankel matrix H is of trace class, but this is no longer true in Theorem A, as the function $\dot{\phi}(n) = n$ shows.

We also obtain a similar result for products of hyperbolic graphs. Using arguments inspired by [HSS10] and [Oza08], Mei and de la Salle [MdlS17] showed that a sufficient condition for a function $\dot{\phi} : \mathbb{N} \to \mathbb{C}$ to define a radial Schur multiplier on a hyperbolic graph of bounded degree is that the Hankel matrix

$$H = (\mathfrak{d}_1 \dot{\phi}(i+j))_{i,j \in \mathbb{N}}$$

belongs to $S_1(\ell_2(\mathbb{N}))$. Moreover, the following estimate holds:

$$\|\phi\|_{cb} \leq C \|H\|_{S_1} + |c|,$$

where $c = \lim_{n\to\infty} \dot{\phi}(n)$, and C is a constant depending on the graph, which is given by a construction by Ozawa [Oza08]. Furthermore, by the characterisation of radial Herz-Schur multipliers on some free products of groups proved by Wysoczański [Wys95], this condition turns out to be also necessary. This follows from the particular case of the hyperbolic group $(\mathbb{Z}/3\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z})$. More precisely, these results together yield the following.

Theorem 4.2 (Wysoczański, Mei–de la Salle)**.** *Let $\dot{\phi} : \mathbb{N} \to \mathbb{C}$ be a function. The following are equivalent.*

a) The function $\dot{\phi}$ defines a radial Schur multiplier on every hyperbolic graph with bounded degree.

b) The Hankel matrix

$$H = (\mathfrak{d}_1 \dot{\phi}(i+j))_{i,j\in\mathbb{N}}$$

is an element of $S_1(\ell_2(\mathbb{N}))$.

c) The analytic function

$$z \mapsto (1-z) \sum_{n\geq 0} \dot{\phi}(n) z^n$$

belongs to $B_1^1(\mathbb{T})$.

Moreover, in that case, $\dot{\phi}(n)$ converges to some $c \in \mathbb{C}$, and there exists $C > 0$ depending only on the graph, such that

$$\|\phi\|_{cb} \leq C\|H\|_{S_1} + |c|.$$

We also extend this characterisation to products.

Theorem B. *Let $N \geq 1$ and let $\dot{\phi} : \mathbb{N} \to \mathbb{C}$ be a bounded function. The following are equivalent.*

a) The function $\dot{\phi}$ defines a radial Schur multiplier on every product of N hyperbolic graphs with bounded degree $X_1, ..., X_N$.

b) The generalised Hankel matrix

$$H = \left(\binom{N+i-1}{N-1}^{\frac{1}{2}} \binom{N+j-1}{N-1}^{\frac{1}{2}} \mathfrak{d}_1^N \dot{\phi}(i+j) \right)_{i,j\in\mathbb{N}}$$

is an element of $S_1(\ell_2(\mathbb{N}))$.

c) The analytic function

$$z \mapsto (1-z)^N \sum_{n\geq 0} \dot{\phi}(n) z^n$$

belongs to $B_1^N(\mathbb{T})$.

Moreover, in that case, $\dot{\phi}(n)$ converges to some limit $c \in \mathbb{C}$, and there exists $C > 0$ depending only on the graphs $X_1, ..., X_N$, such that

$$\|\phi\|_{cb} \leq C\|H\|_{S_1} + |c|.$$

Observe that, once again, we must make the assumption that $\dot{\phi}$ is bounded. The argument that we use to show that the condition $H \in S_1$ is sufficient is a mix of the proof of Theorem A with the ideas of [MdlS17]. In order to prove that this condition is also necessary, we use some tools from [Wys95], but since we only deal with one particular hyperbolic graph, the proof gets reduced to studying a very specific product of homogeneous trees, and then applying some elements of the proof of Theorem A.

We also show that, as a consequence of Theorem B, groups acting properly on products of hyperbolic graphs with bounded degrees are weakly amenable. As far as we know, this result is new; however, we do not know if it provides new examples of weakly amenable groups.

Finally, we deal with multipliers on finite dimensional CAT(0) cube complexes. A cube complex X is a polyhedral complex in which each cell is isometric to the Euclidean cube $[0,1]^n$ for some $n \in \mathbb{N}$, and the gluing maps are isometries. The dimension of X is the maximum of all such n. The CAT(0) condition is defined in terms of the metric on X induced by the Euclidean metric on each cube. It also admits a combinatorial characterisation proved by Gromov [Gro87] by what is sometimes referred to as the *link condition*. However, thanks to a very nice result of Chepoi [Che00], we may define CAT(0) cube complexes as those whose 1-skeleton is a median graph. See Section 4.6 for a definition of median graphs. They generalise trees, in the sense that trees are exactly the 1-dimensional CAT(0) cube complexes. Furthermore, a product of N trees defines an N-dimensional CAT(0) cube complex. However, the class of finite dimensional CAT(0) cube complexes is far more general. Indeed, Chepoi and Hagen [CH13] gave an example of a uniformly locally finite CAT(0) cube complex of dimension 5 that cannot be embedded in a finite product of trees.

CAT(0) cube complexes have been widely studied and remain an object of great interest in geometric group theory. We refer the reader to [GH10, §2] and the references therein for a presentation close to the spirit of this thesis.

Observe that Theorem A provides a necessary condition for a function to define a radial Schur multiplier on every N-dimensional CAT(0) cube complex; however, we do not know whether this condition is also sufficient. The following result asserts that another (stronger) condition is sufficient.

Theorem C. *Let X be (the 0-skeleton of) a CAT(0) cube complex of dimension $N < \infty$. Let $\phi : X \times X \to \mathbb{C}$ be a radial function with $\phi = \dot{\phi} \circ d$. Then, the following conditions are equivalent.*

a) The generalised Hankel matrix

$$H = \left(\binom{N+i-1}{N-1}^{\frac{1}{2}} \binom{N+j-1}{N-1}^{\frac{1}{2}} \mathfrak{d}_2 \dot{\phi}(i+j) \right)_{i,j \in \mathbb{N}}$$

belongs to $S_1(\ell_2(\mathbb{N}))$.

b) The analytic function

$$z \mapsto (1-z^2)\sum_{n\geq 0}\dot{\phi}(n)z^n$$

belongs to $B_1^N(\mathbb{T})$.

In that case, the following limits exist:

$$\lim_{n\to\infty}\dot{\phi}(2n), \qquad \lim_{n\to\infty}\dot{\phi}(2n+1),$$

and ϕ *defines a radial Schur multiplier on* X *of norm at most*

$$M\|H\|_{S_1}+|c_+|+|c_-|, \tag{4.7}$$

where c_+ *and* c_- *are defined as in* (4.3)*, and* $M>0$ *is a constant depending only on the dimension* N*.*

Observe that that the matrix H is not the same as that of Theorem A. Indeed, it involves only the first derivative $\mathfrak{d}_2\dot{\phi}$, regardless of the dimension N. The reason for this is that the proof of Theorem C uses a construction of Mizuta [Miz08] that allows us to adapt the arguments of [HSS10] in this more general context. In fact, Mizuta used this construction to study a very particular family of radial Schur multipliers, in order to show that groups acting properly on finite dimensional CAT(0) cube complexes are weakly amenable. This was proved independently by Guentner and Higson [GH10] using uniformly bounded representations.

4.2 Multi-radial multipliers on products of trees

In this section we prove a more general form of Theorem A by studying what we will call multi-radial multipliers. These objects will also be useful in the proof of Theorem B. Let $X=X_1\times\cdots\times X_N$ be a product of N graphs. Observe that the combinatorial distance in this case is given by

$$d(x,y)=\sum_{i=1}^{N}d_i(x_i,y_i)\quad \forall x,y\in X,$$

where $x=(x_i)_{i=1}^N$, $y=(y_i)_{i=1}^N$, and d_i is the combinatorial distance on X_i. We say that $\phi: X\times X\to\mathbb{C}$ is a multi-radial function if there exists $\tilde{\phi}:\mathbb{N}^N\to\mathbb{C}$ such that

$$\phi(x,y)=\tilde{\phi}(d(x_1,y_1),...,d(x_N,y_N)),\quad \forall x,y\in X.$$

In order to precisely state the main result of this section, we need to fix some notation. For any integer $N\geq 1$, let $[N]$ denote the set $\{1,...,N\}$. For each $I\subset[N]$, we define a vector $\chi^I\in\{0,1\}^N$ by

$$\chi_i^I=\begin{cases}1 & \text{if } i\in I\\ 0 & \text{if } i\notin I.\end{cases} \tag{4.8}$$

We shall also write, for $n = (n_1, ..., n_N) \in \mathbb{N}^N$, $|n| = n_1 + \cdots + n_N$. Our goal is to prove the following characterisation of multi-radial multipliers in terms of operators on $\ell_2(\mathbb{N}^N)$.

Proposition 4.3. *Let $N \geq 1$ and let $\tilde{\phi} : \mathbb{N}^N \to \mathbb{C}$ be a function such that the limits*

$$l_0 = \lim_{\substack{|n|\to\infty \\ |n| \text{ even}}} \tilde{\phi}(n) \qquad\qquad l_1 = \lim_{\substack{|n|\to\infty \\ |n| \text{ odd}}} \tilde{\phi}(n) \tag{4.9}$$

exist. Then $\tilde{\phi}$ defines a multi-radial Schur multiplier on any product of N infinite trees of minimum degrees $d_1, ..., d_N \geq 3$ if and only if the operator $T = (T_{n,m})_{m,n\in\mathbb{N}^N}$ given by

$$T_{m,n} = \sum_{I\subset[N]} (-1)^{|I|}\tilde{\phi}(m + n + 2\chi^I), \quad \forall m, n \in \mathbb{N}^N \tag{4.10}$$

is an element of $\in S_1(\ell_2(\mathbb{N}^N))$. In that case

$$\left[\prod_{i=1}^{N} \frac{d_i - 2}{d_i}\right] \|T\|_{S_1} + |c_+| + |c_-| \leq \|\phi\|_{cb} \leq \|T\|_{S_1} + |c_+| + |c_-|,$$

where $c_\pm = l_0 \pm l_1$.

Remark 4.4. *In Proposition 4.3, we make the assumption that the limits l_0 and l_1 exist, whereas in Theorem A this is a consequence. The reason for this is that multi-radial functions can have completely different behaviours on each coordinate. Take for example $N = 2$ and define $\tilde{\phi}(n_1, n_2) = f(n_1) + g(n_2)$, where $f, g : \mathbb{N} \to \mathbb{C}$ are any bounded functions. Then*

$$\begin{aligned} T_{m,n} &= f(m_1 + n_1) + g(m_2 + n_2) - f(m_1 + n_1 + 2) - g(m_2 + n_2) \\ &\quad - f(m_1 + n_1) - g(m_2 + n_2 + 2) + f(m_1 + n_1 + 2) + g(m_2 + n_2 + 2) \\ &= 0, \end{aligned}$$

but for $\tilde{\phi}$ to define a Schur multiplier, f and g should at least satisfy the characterisation given by Theorem 4.1.

4.2.1 Proof of sufficiency

We begin with a general observation about bipartite graphs. A graph is said to be bipartite if it does not contain any odd-length cycles. This is the case of products of trees, and more generally, median graphs.

Lemma 4.5. *Let X be a connected bipartite graph. Then the function $\chi : X \times X \to \{-1, 1\}$ given by*

$$\chi(x, y) = (-1)^{d(x,y)}$$

is a Schur multiplier of norm 1.

Proof. Fix $x_0 \in X$ and define $P : X \to \mathbb{C}$ by $P(x) = (-1)^{d(x,x_0)}$. Hence, for all $x, y \in X$,

$$\langle P(x), P(y) \rangle = (-1)^{d(x,x_0)+d(y,x_0)}.$$

Observe that $d(x, x_0) + d(y, x_0) + d(x, y)$ is even because it is the length of a cycle. Thus

$$(-1)^{d(x,x_0)+d(y,x_0)} = (-1)^{d(x,y)} = \chi(x, y).$$

Therefore, by Theorem 1.3, χ is a Schur multiplier and $\|\chi\|_{cb} \leq 1$. The other inequality follows from the fact that the supremum norm is bounded above by the cb norm. $\square$

We shall fix $N \geq 1$, and consider, for each $i = 1, ..., N$, an infinite tree X_i endowed with the combinatorial distance d_i. We do not make any assumptions on the degrees. We follow the same strategy as in [HSS10]. For each $i \in \{1, ..., N\}$, fix an infinite geodesic $\omega_0^{(i)} : \mathbb{N} \to X_i$. This means that $\omega_0^{(i)}$ is injective and for each $n \in \mathbb{N}$, $\omega_0^{(i)}(n)$ is adjacent to $\omega_0^{(i)}(n+1)$. Observe that for each $x \in X_i$ there exists a unique infinite geodesic $\omega_x : \mathbb{N} \to X_i$ such that $\omega_x(0) = x$, and which eventually flows with $\omega_0^{(i)}$. The latter may be expressed as $|\omega_x \Delta \omega_0^{(i)}| < \infty$, viewing the geodesics as subsets of X_i. Moreover, for any $x, y \in X_i$, let

$$\begin{aligned} k_0 &= \min\{k \in \mathbb{N} : \omega_x(k) \in \omega_y\}, \\ m_0 &= \min\{m \in \mathbb{N} : \omega_y(m) \in \omega_x\}, \end{aligned} \tag{4.11}$$

and observe that $k_0 + m_0 = d(x, y)$. Then, for all $k, m \in \mathbb{N}$,

$$\omega_x(k) = \omega_y(l) \iff \exists\, j \in \mathbb{N},\ k = k_0 + j,\ m = m_0 + j.$$

With all these observations, we may now give the proof of the *if* part of Proposition 4.3.

Lemma 4.6. *Let $N \geq 1$ and let $\tilde{\phi} : \mathbb{N}^N \to \mathbb{C}$ be a function such that the limits*

$$l_0 = \lim_{\substack{|n| \to \infty \\ |n| \text{ even}}} \tilde{\phi}(n) \qquad\qquad l_1 = \lim_{\substack{|n| \to \infty \\ |n| \text{ odd}}} \tilde{\phi}(n)$$

exist, and such that the operator $T = (T_{n,m})_{m,n \in \mathbb{N}^N}$ given by

$$T_{n,m} = \sum_{I \subset [N]} (-1)^{|I|} \tilde{\phi}(m + n + 2\chi^I), \quad \forall m, n \in \mathbb{N}^N,$$

where χ^I is defined as in (4.8), is an element of $\in S_1(\ell_2(\mathbb{N}^N))$. Then $\tilde{\phi}$ defines a multi-radial Schur multiplier on $X = X_1 \times \cdots \times X_N$ of norm at most $\|T\|_{S_1} + |c_+| + |c_-|$.

Proof. We treat first the case $l_0 = l_1 = 0$. Let $S_2(\ell_2(\mathbb{N}^N))$ be the space of Hilbert–Schmidt operators on $\ell_2(\mathbb{N}^N)$ (see [Mur90, §2.4] for details). Using the polar decomposition, we can find $A, B \in S_2(\ell_2(\mathbb{N}^N))$ such that $T = A^*B$ and $\|T\|_{S_1} = \|A\|_{S_2}\|B\|_{S_2}$. Define now, for each $x = (x_i)_{i=1}^N \in X$,

$$P(x) = \sum_{k_1,...,k_N=0}^{\infty} \delta_{\omega_{x_1}(k_1)} \otimes \cdots \otimes \delta_{\omega_{x_N}(k_N)} \otimes Be_{(k_1,...,k_N)}$$

and

$$Q(x) = \sum_{m_1,...,m_N=0}^{\infty} \delta_{\omega_{x_1}(m_1)} \otimes \cdots \otimes \delta_{\omega_{x_N}(m_N)} \otimes Ae_{(m_1,...,m_N)},$$

where $\{e_n\}_{n\in\mathbb{N}^N}$ is the canonical orthonormal basis of $\ell_2(\mathbb{N}^N)$. Observe that

$$\|P(x)\|^2 = \sum_{n\in\mathbb{N}^N}^{\infty} \|Be_n\|^2 = \|B\|_{S_2}^2, \quad \forall x \in X.$$

Similarly, $\|Q(y)\|^2 = \|A\|_{S_2}^2$ for all $y \in X$. Now,

$$\langle P(x), Q(y)\rangle = \sum_{\substack{k_1,...,k_N=0\\ m_1,...,m_N=0}}^{\infty} \left(\prod_{i=1}^{N} \left\langle \delta_{\omega_{x_i}(k_i)}, \delta_{\omega_{y_i}(m_i)} \right\rangle\right) \left\langle A^*Be_{(k_1,...,k_N)}, e_{(m_1,...,m_N)} \right\rangle.$$

Recall that

$$\omega_{x_i}(k_i) = \omega_{y_i}(m_i) \iff \exists\, j_i \in \mathbb{N},\ k_i = k_0 + j_i,\ m_i = m_0 + j_i,$$

where k_0 and m_0 are defined as in (4.11), and satisfy $k_0 + m_0 = d_i(x_i, y_i)$. Moreover, in that case, since $T = A^*B$,

$$\langle A^*Be_{(k_1,...,k_N)}, e_{(m_1,...,m_N)}\rangle = \sum_{I\subset[N]} (-1)^{|I|}\tilde{\phi}\left(\vec{d}(x,y) + 2(j_1, ..., j_N) + 2\chi^I\right),$$

where

$$\vec{d}(x,y) = (d_1(x_1,y_1), ..., d_N(x_N,y_N)) \in \mathbb{N}^N.$$

Thus

$$\langle P(x), Q(y)\rangle = \sum_{j_1,...,j_N=0}^{\infty} \sum_{I\subset[N]} (-1)^{|I|}\tilde{\phi}\left(\vec{d}(x,y) + 2(j_1, ..., j_N) + 2\chi^I\right).$$

Now observe that this expression corresponds to N telescoping series. Indeed, fixing $j_1, ..., j_{N-1}$ and defining

$$a_j = \sum_{\substack{I\subset[N]\\ N\notin I}} (-1)^{|I|}\tilde{\phi}\left(\vec{d}(x,y) + 2(j_1, ..., j_{N-1}, j) + 2\chi^I\right),$$

we obtain

$$\sum_{j_N=0}^{\infty} \sum_{I\subset[N]} (-1)^{|I|}\tilde{\phi}\left(\vec{d}(x,y) + 2(j_1, ..., j_N) + 2\chi^I\right) = \sum_{j=0}^{\infty} a_j - a_{j+1}$$

and this equals a_0 because we have assumed $l_0 = l_1 = 0$. Hence

$$\langle P(x), Q(y)\rangle = \sum_{j_1,...,j_{N-1}=0}^{\infty} \sum_{\substack{I\subset[N]\\ N\notin I}} (-1)^{|I|}\tilde{\phi}\left(\vec{d}(x,y) + 2(j_1, ..., j_{N-1}, 0) + 2\chi^I\right).$$

Repeating this argument for the variables $j_1, ..., j_{N-1}$, we get

$$\langle P(x), Q(y)\rangle = \tilde{\phi}\left(d_1(x_1, y_1), ..., d_N(x_N, y_N)\right).$$

We conclude that this defines a multi-radial Schur multiplier ϕ on X, and

$$\|\phi\|_{cb} \leq \|A\|_{S_2}\|B\|_{S_2} = \|T\|_{S_1}.$$

Now we drop the assumption $l_0 = l_1 = 0$. Define

$$\tilde{\psi}(n) = \tilde{\phi}(n) - \left(c_+ + (-1)^{|n|}c_-\right), \quad \forall n \in \mathbb{N}^N,$$

where $c_\pm = \frac{1}{2}(l_0 \pm l_1)$. Observe that

$$\lim_{|n|\to\infty} \tilde{\psi}(n) = 0,$$

and

$$\sum_{I\subset[N]} (-1)^{|I|}\tilde{\psi}(n + 2\chi^I) = \sum_{I\subset[N]} (-1)^{|I|}\tilde{\phi}(n + 2\chi^I), \quad \forall n \in \mathbb{N}^N.$$

Hence, by the previous arguments, we know that $\tilde{\psi}$ defines a multi-radial Schur multiplier on X of norm at most $\|T\|_{S_1}$. Using Lemma 4.5, we conclude that ϕ is a Schur multiplier, and

$$\|\phi\|_{cb} \leq \|T\|_{S_1} + |c_+| + |c_-|.$$

□

4.2.2 Double commuting isometries

Now we will deal with the *only if* part of Proposition 4.3. Once again, we follow the strategy of [HSS10]. Their argument is based on the study of a certain isometry on the ℓ_2 space of a homogeneous tree. In our case, we need to consider N copies ($N \geq 1$) of that isometry and analyse them together, although they act independently, in some sense. For this purpose, we shall need some preliminaries on double commuting isometries.

Given an isometry V on a Hilbert space $\mathcal{H}$, the Wold–von Neumann theorem gives a decomposition of $\mathcal{H}$ as a direct sum of two subspaces such that, on one of them V acts as a unitary, and on the other it is a unilateral shift. We will need an extension of this result for N double commuting isometries. Given $V_1, ..., V_N$ isometries on a Hilbert space $\mathcal{H}$, we say that they double commute if, for all $i, j \in \{1, ..., N\}$ with $i \neq j$, V_i commutes with both V_j and V_j^*. Let us first state the Wold–von Neumann theorem. For a proof, see e.g. [Mur90, Theorem 3.5.17].

Theorem 4.7 (Wold–von Neumann)**.** *Let V be an isometry on a Hilbert space $\mathcal{H}$. Then $\mathcal{H}$ admits a decomposition*

$$\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_s \ ,$$

where

$$\mathcal{H}_u = \bigcap_{n \in \mathbb{N}} V^n \mathcal{H}, \qquad \mathcal{H}_s = \bigoplus_{n \in \mathbb{N}} V^n \left(Ker\ V^*\right).$$

Słociński [Sł80] generalised this decomposition to a certain type of commuting pairs. In particular, his result applies to double commuting isometric pairs (V_1, V_2). Inductively, we can obtain a similar decomposition for double commuting isometries $V_1, ..., V_N$. We present the proof here for the sake of completeness. But first we need to fix some notations. Recall that $[N]$ stands for the set $\{1, ..., N\}$. Given $I = \{i_1, ..., i_k\} \subset [N]$ with $i_1 < \cdots < i_k$, and $p = (p_1, ..., p_k) \in \mathbb{N}^I$, put

$$V_I^p = V_{i_1}^{p_1} \cdots V_{i_k}^{p_k}.$$

We also write $I^c = [N] \setminus I$.

Proposition 4.8. *Let $V_1, ..., V_N$ be double commuting isometries on a Hilbert space $\mathcal{H}$. Define, for each nonempty $I \subset [N]$,*

$$\mathcal{W}_I = \bigcap_{i \in I} Ker\ V_i^*.$$

Then $\mathcal{H}$ admits a decomposition

$$\mathcal{H} = \bigoplus_{I \subset [N]} \mathcal{H}_I,$$

where

$$\mathcal{H}_I = \bigoplus_{p \in \mathbb{N}^I} V_I^p \left(\bigcap_{q \in \mathbb{N}^{I^c}} V_{I^c}^q \mathcal{W}_I \right),$$

for all nonempty $I \subsetneq [N]$, and

$$\mathcal{H}_\varnothing = \bigcap_{q \in \mathbb{N}^N} V_{[N]}^q \mathcal{H}, \qquad \mathcal{H}_{[N]} = \bigoplus_{p \in \mathbb{N}^N} V_{[N]}^p \left(\mathcal{W}_{[N]}\right).$$

Proof. We proceed by induction on N. For $N = 1$, this is exactly the Wold–von Neumann decomposition. Suppose now that we have such a decomposition for some $N \geq 1$ and consider $V_1, ..., V_{N+1}$ double commuting isometries on a Hilbert space $\mathcal{H}$. Then, again by the Wold–von Neumann decomposition,

$$\mathcal{H} = \mathcal{H}^u \oplus \mathcal{H}^s,$$

where

$$\mathcal{H}^u = \bigcap_{n \geq 0} V_{N+1}^n \mathcal{H},$$

and

$$\mathcal{H}^s = \bigoplus_{n \geq 0} V_{N+1}^n \left(\text{Ker } V_{N+1}^*\right).$$

Since $V_1, ..., V_{N+1}$ are double commuting isometries, $\mathcal{H}^u$ and $\mathcal{H}^s$ are invariant subspaces for $V_1, ..., V_N$ and $V_1^*, ..., V_N^*$. Hence, by the induction hypothesis,

$$\mathcal{H}^u = \bigoplus_{I \subset [N]} \mathcal{H}_I^u, \qquad \mathcal{H}^s = \bigoplus_{I \subset [N]} \mathcal{H}_I^s.$$

For $I \neq \varnothing, [N]$,

$$\begin{aligned}
\mathcal{H}_I^u &= \bigoplus_{p \in \mathbb{N}^I} V_I^p \left(\bigcap_{q \in \mathbb{N}^{I^c}} V_{I^c}^q \left(\mathcal{H}^u \cap \bigcap_{i \in I} \text{Ker } V_i^* \right) \right) \\
&= \bigoplus_{p \in \mathbb{N}^I} V_I^p \left(\bigcap_{q \in \mathbb{N}^{I^c}} V_{I^c}^q \left(\bigcap_{n \geq 0} V_{N+1}^n \left(\bigcap_{i \in I} \text{Ker } V_i^* \right) \right) \right) \\
&= \bigoplus_{p \in \mathbb{N}^I} V_I^p \left(\bigcap_{q \in \mathbb{N}^J} V_J^q \left(\bigcap_{i \in [N+1] \setminus J} \text{Ker } V_i^* \right) \right),
\end{aligned}$$

where $J = I^c \cup \{N+1\}$. We have also

$$\begin{aligned}
\mathcal{H}_I^s &= \bigoplus_{p \in \mathbb{N}^I} V_I^p \left(\bigcap_{q \in \mathbb{N}^{I^c}} V_{I^c}^q \left(\mathcal{H}^s \cap \bigcap_{i \in I} \text{Ker } V_i^* \right) \right) \\
&= \bigoplus_{p \in \mathbb{N}^I} V_I^p \left(\bigcap_{q \in \mathbb{N}^{I^c}} V_{I^c}^q \left(\bigoplus_{n \geq 0} V_{N+1}^n \left(\text{Ker } V_{N+1}^*\right) \cap \bigcap_{i \in I} \text{Ker } V_i^* \right) \right). \qquad (4.12)
\end{aligned}$$

Now observe that, if $x \in V_{N+1}^n \left(\text{Ker } V_{N+1}^*\right) \cap \text{Ker } V_i^*$, then $x = V_{N+1}^n u$ with $u \in \text{Ker } V_{N+1}^*$. This yields

$$V_i^* u = V_i^* (V_{N+1}^*)^n x = (V_{N+1}^*)^n V_i^* x = 0,$$

which implies that $x \in V_{N+1}^n \left(\text{Ker } V_{N+1}^* \cap \text{Ker } V_i^*\right)$. Conversely, if

$$x \in V_{N+1}^n \left(\text{Ker } V_{N+1}^* \cap \text{Ker } V_i^*\right),$$

then the fact that V_{N+1} and V_i^* commute implies that $x \in \text{Ker } V_i^*$. Applying this argument to all $i \in I$, we see from (4.12) that

$$\begin{aligned}\mathcal{H}_I^s &= \bigoplus_{p\in\mathbb{N}^I} V_I^p \left(\bigcap_{q\in\mathbb{N}^{I^c}} V_{I^c}^q \left(\bigoplus_{n\geq 0} V_{N+1}^n \left(\text{Ker } V_{N+1}^* \cap \bigcap_{i\in I} \text{Ker } V_i^* \right)\right)\right) \\ &= \bigoplus_{p\in\mathbb{N}^K} V_K^p \left(\bigcap_{q\in\mathbb{N}^{I^c}} V_{I^c}^q \left(\bigcap_{i\in K} \text{Ker } V_i^* \right)\right),\end{aligned}$$

where $K = I \cup \{N+1\}$. This proves the result for $I \neq \varnothing, [N]$. The cases $I = \varnothing$ and $I = [N]$ are simpler and follow analogously. □

Corollary 4.9. *Let $V_1, ..., V_N$ be double commuting isometries on a Hilbert space $\mathcal{H}$. Then $\mathcal{H}$ admits a decomposition*

$$\mathcal{H} = \bigoplus_{I\subset[N]} \mathcal{H}_I, \tag{4.13}$$

where

$$\mathcal{H}_I \cong \ell_2\left(\mathbb{N}^I\right) \otimes Y_I,$$

with $Y_I = \bigcap_{q\in\mathbb{N}^{I^c}} V_{I^c}^q \mathcal{W}_I$ for $0 < |I| < N$, $Y_\varnothing = \bigcap_{q\in\mathbb{N}^N} V_{[N]}^q \mathcal{H}$, and $Y_{[N]} = \mathcal{W}_{[N]}$. Moreover, if $i \in I$, then V_i acts as a one coordinate shift on $\ell_2(\mathbb{N}^I)$, and if $i \notin I$, then V_i acts as a unitary on Y_I.

Proof. By construction, if $i \notin I$, then $V_i V_i^* x = x$ for all $x \in Y_I$. This proves that V_i acts as a unitary on Y_I. For the shift part, we shall only consider the case $I = [N]$ since it illustrates well all the other cases. We have

$$\mathcal{H}_{[N]} = \bigoplus_{p\in\mathbb{N}^N} V_{[N]}^p \left(\mathcal{W}_{[N]}\right).$$

Then we can define an isomorphism $\ell_2\left(\mathbb{N}^N\right) \otimes \left(\mathcal{W}_{[N]}\right) \cong \mathcal{H}_{[N]}$ by identifying

$$\delta_{n_1} \otimes \delta_{n_2} \otimes \cdots \otimes \delta_{n_N} \otimes w \;\longleftrightarrow\; V_1^{n_1} \cdots V_N^{n_N} w,$$

for all $(n_1, ..., n_N) \in \mathbb{N}^N$ and $w \in \mathcal{W}_{[N]}$. Then the action of V_1 on $\mathcal{H}_{[N]}$ is given by

$$\begin{aligned}V_1(\delta_{n_1} \otimes \delta_{n_2} \otimes \cdots \otimes \delta_{n_N} \otimes w) &\longleftrightarrow V_1^{n_1+1} V_2^{n_2} \cdots V_N^{n_N} w \\ &\longleftrightarrow \delta_{n_1+1} \otimes \delta_{n_2} \otimes \cdots \otimes \delta_{n_N} \otimes w,\end{aligned}$$

which corresponds to the forward shift operator on the first coordinate. For $V_2, ..., V_N$, the argument is analogous. □

4.2.3 Proof of necessity

From now on, we shall fix $X = X_1 \times \cdots \times X_N$, where X_i is a $(q_i + 1)$-regular tree ($2 \leq q_i < \infty$, $i = 1, ..., N$). Our aim is to prove the following result.

Lemma 4.10. *Let $\phi : X \times X \to \mathbb{C}$ be a multi-radial Schur multiplier with*

$$\phi(x, y) = \tilde{\phi}(d(x_1, y_1), ..., d(x_N, y_N)), \qquad \forall\, x, y \in X,$$

and such that the limits

$$l_0 = \lim_{\substack{|n|\to\infty \\ |n| \text{ even}}} \tilde{\phi}(n) \qquad\qquad l_1 = \lim_{\substack{|n|\to\infty \\ |n| \text{ odd}}} \tilde{\phi}(n)$$

exist. Then the operator $T = (T_{n,m})_{m,n\in\mathbb{N}^N}$ given by

$$T_{n,m} = \sum_{I\subset[N]} (-1)^{|I|}\tilde{\phi}(m + n + 2\chi^I), \quad \forall m, n \in \mathbb{N}^N,$$

where χ^I is defined as in (4.8), is an element of $\in S_1(\ell_2(\mathbb{N}^N))$ of norm at most

$$\left[\prod_{i=1}^{N} \frac{q_i + 1}{q_i - 1}\right] (\|\phi\|_{cb} - |c_+| - |c_-|)\,,$$

where $c_\pm = l_0 \pm l_1$.

Recall that on each X_i we may fix an infinite geodesic $\omega_0^{(i)} : \mathbb{N} \to X_i$, and for each $x \in X_i$ there exists a unique infinite geodesic $\omega_x : \mathbb{N} \to X_i$ such that $\omega_x(0) = x$, and $|\omega_x(\mathbb{N})\Delta\omega_0^{(i)}(\mathbb{N})| < \infty$. We define $U_i : \ell_2(X_i) \to \ell_2(X_i)$ by

$$U_i\delta_x = \frac{1}{\sqrt{q_i}} \sum_{\substack{y\in X_i \\ \omega_y(1)=x}} \delta_y, \quad \forall x \in X_i.$$

Since $\langle U_i\delta_x, U_i\delta_y\rangle = 0$ if $x \neq y$, this extends to an isometry on $\ell_2(X_i)$ whose adjoint is given by

$$U_i^*\delta_y = \frac{1}{\sqrt{q_i}}\delta_{\omega_y(1)}.$$

Consider now the C*-algebra generated by U_i,

$$C^*(U_i) = \overline{\text{span}}\{U_i^m(U_i^*)^n \;:\; m, n \in \mathbb{N}\} \subset \ell_2(X_i).$$

Let

$$\mathcal{A} = C^*(U_1) \otimes_{\min} \cdots \otimes_{\min} C^*(U_N)$$

denote their minimal tensor product. Since we have an explicit faithful representation of $C^*(U_i)$ on $\ell_2(X_i)$, $\mathcal{A}$ is naturally a subalgebra of $\mathcal{B}(\ell_2(X_1) \otimes \cdots \otimes \ell_2(X_N)) \cong$

$\mathcal{B}(\ell_2(X))$ (see [KR97, §11.3] for details). This implies that $U_1, ..., U_N$ extend to double commuting isometries on $\ell_2(X) \cong \ell_2(X_1) \otimes \cdots \otimes \ell_2(X_N)$ in the natural way (with a slight abuse of notation):

$$U_1 \delta_{(x_1,...,x_N)} = (U_1 \delta_{x_1}) \otimes \delta_{x_2} \otimes \cdots \otimes \delta_N,$$

and so forth. Define now, for $i \in \{1, ..., N\}$, $m, n \in \mathbb{N}$,

$$U_{i,m,n} = \left(1 - \frac{1}{q_i}\right)^{-1} \left(U_i^m (U_i^*)^n - \frac{1}{q_i} U_i^* U_i^m (U_i^*)^n U_i\right) \tag{4.14}$$

$$= \begin{cases} \left(1 - \frac{1}{q_i}\right)^{-1} \left(U_i^m (U_i^*)^n - \frac{1}{q_i} U_i^{m-1} (U_i^*)^{n-1}\right) & \text{if } m, n \geq 1 \\ U_i^m (U_i^*)^n & \text{if } \min\{m, n\} = 0, \end{cases} \tag{4.15}$$

and observe that we also have

$$C^*(U_i) = \overline{\operatorname{span}}\{U_{i,m,n} \ : \ m, n \in \mathbb{N}\}. \tag{4.16}$$

Since $U_1, ..., U_N$ are double commuting operators, there is no ambiguity in defining

$$U(m, n) = \prod_{i=1}^{N} U_{i,m_i,n_i} \in \mathcal{B}(\ell_2(X)), \tag{4.17}$$

for $m = (m_1, ..., m_N)$, $n = (n_1, ..., n_N) \in \mathbb{N}^N$.

Lemma 4.11. *Let $m, n \in \mathbb{N}^N$ and $x, y \in X$. If*

$$\langle U(m, n)\delta_y, \delta_x \rangle \neq 0,$$

then

$$d(x_i, y_i) = m_i + n_i, \qquad \forall\, i \in \{1, ..., N\}.$$

In particular, this implies that $d(x, y) = |m| + |n|$.

Proof. Observe that $\langle U(m, n)\delta_y, \delta_x \rangle \neq 0$ if and only if

$$\langle U_{i,m_i,n_i} \delta_{y_i}, \delta_{x_i} \rangle \neq 0, \qquad \forall\, i \in \{1, ..., N\},$$

where $x = (x_1, ..., x_N)$ and $y = (y_1, ..., y_N)$. So, by [HSS10, Lemma 2.6], we have $d(x_i, y_i) = m_i + n_i$. ☐

Lemma 4.12. *Let $\phi : X \times X \to \mathbb{C}$ be a multi-radial Schur multiplier. Then $\mathcal{A}$ is invariant under M_ϕ. Moreover, for all $m, n \in \mathbb{N}^N$*

$$M_\phi(U(m, n)) = \tilde{\phi}(m + n) U(m, n). \tag{4.18}$$

Proof. Using the fact that ϕ is multi-radial, together with Lemma 4.11, we get

$$\begin{aligned}\langle M_\phi(U(m,n))\delta_y, \delta_x\rangle &= \tilde{\phi}(d(x_1,y_1),...,d(x_N,y_N))\langle U(m,n)\delta_y,\delta_x\rangle \\ &= \tilde{\phi}(m+n)\langle U(m,n)\delta_y,\delta_x\rangle.\end{aligned}$$

This proves (4.18). Moreover, by (4.16), this also shows that $\mathcal{A}$ is invariant under M_ϕ. ∎

Lemma 4.13. *Let $\phi : X \times X \to \mathbb{C}$ be a multi-radial Schur multiplier. There exists a bounded linear functional $f_\phi : \mathcal{A} \to \mathbb{C}$ satisfying*

$$f_\phi(U(m,n)) = \tilde{\phi}(m+n), \qquad \forall\, m,n \in \mathbb{N}, \tag{4.19}$$

and $\|f_\phi\| \leq \|\phi\|_{cb}$.

Proof. By Coburn's theorem (cf. [Mur90, Theorem 3.5.18]), together with [Mur90, Theorem 3.5.11], there exists a $*$-homomorphism $\rho_i : C^*(U_i) \to C(\mathbb{T})$ such that $\rho_i(U_i)(z) = z$ for all $z \in \mathbb{T}$. Let $\gamma_0 : C(\mathbb{T}) \to \mathbb{C}$ be the pure state given by

$$\gamma_0(f) = f(1), \qquad \forall\, f \in C(\mathbb{T}).$$

Then $\gamma_i = \gamma_0 \circ \rho_i$ is a state on $C^*(U_i)$ and

$$\gamma_i(U_i^m (U_i^*)^n) = 1, \qquad \forall\, m,n \in \mathbb{N}.$$

Let $\gamma : \mathcal{A} \to \mathbb{C}$ be the product state $\gamma_1 \otimes \cdots \otimes \gamma_N$, which is uniquely defined by

$$\gamma(W_1 \otimes \cdots \otimes W_2) = \gamma_1(W_1)\cdots\gamma_N(W_N).$$

See [KR97, §11.3] for details. Then

$$\gamma(U_1^{m_1}(U_1^*)^{n_1} \otimes \cdots \otimes U_N^{m_N}(U_N^*)^{n_N}) = 1, \qquad \forall\, m_1,n_1,...,m_N,n_N \in \mathbb{N}.$$

Finally, define $f_\phi : \mathcal{A} \to \mathbb{C}$ by

$$f_\phi(W) = \gamma(M_\phi(W)), \qquad \forall\, W \in \mathcal{A}.$$

This map satisfies

$$\|f_\phi\| \leq \|M_\phi\| = \|\phi\|_{cb},$$

and by Lemma 4.12,

$$f_\phi(U(m,n)) = \tilde{\phi}(m+n)\gamma(U(m,n)) = \tilde{\phi}(m+n).$$

∎

Now we need to introduce some notation. Observe that we may write $\ell_2(\mathbb{N}^N) \cong \ell_2(\mathbb{N}_1) \otimes \cdots \otimes \ell_2(\mathbb{N}_N)$, where $\mathbb{N}_i$ is a copy of the natural numbers. If S_i denotes the forward shift operator on $\ell_2(\mathbb{N}_i)$, then we can repeat the previous arguments for S_i instead of U_i. In particular, $S_1, ..., S_N$ extend to double commuting isometries on $\ell_2(\mathbb{N}^N)$, and we can define S_{i,m_i,n_i} and

$$S(m,n) = \prod_{i=1}^{N} S_{i,m_i,n_i} \in \mathcal{B}(\ell_2(\mathbb{N}^N)), \tag{4.20}$$

as in (4.14) and (4.17). We call S_i the forward shift operator on the i-th coordinate. Now, for every non-empty $I \subset [N]$ and for every $m \in \mathbb{N}^N$, let m_I be the projection of m on $\mathbb{N}^I$, and define the operator

$$S_I^{m_I} = \prod_{i \in I} S_i^{m_i} \in \mathcal{B}(\ell_2(\mathbb{N}^I)).$$

Similarly,

$$(S_I^*)^{m_I} = \prod_{i \in I} (S_i^*)^{m_i}.$$

We can define $U_I^{m_I}, (U_I^*)^{m_I} \in \mathcal{B}\left(\ell_2\left(\prod_{i \in I} T_i\right)\right)$ analogously.

Lemma 4.14. *Let $f : \mathcal{A} \to \mathbb{C}$ be a bounded linear functional. Then there exists a family of bounded linear forms $\{f^I\}_{I \subset [N]}$ on $\mathcal{A}$ such that*

$$f = \sum_{I \subset [N]} f^I \quad \text{and} \quad \|f\| = \sum_{I \subset [N]} \left\|f^I\right\|. \tag{4.21}$$

Moreover, let $m, n \in \mathbb{N}^N$ and $I, J \subset [N]$.

(i) If $J \cap I = \varnothing$, then

$$f^I\left(U\left(m + k\chi^J, n + k\chi^J\right)\right) = f^I\left(U\left(m,n\right)\right), \quad \forall k \in \mathbb{N}.$$

(ii) If $J \cap I \neq \varnothing$, then

$$f^I\left(U\left(m + k\chi^J, n + k\chi^J\right)\right) \xrightarrow[k\to\infty]{} 0.$$

Proof. Let $(\pi, \mathcal{H})$ be the universal representation of $\mathcal{A}$. Then there exist $\xi, \eta \in \mathcal{H}$ such that

$$f(A) = \langle \pi(A)\xi, \eta \rangle, \qquad \forall\, A \in \mathcal{A},$$

and $\|f\| = \|\xi\|\|\eta\|$. On the other hand, since $U_1, ..., U_N$ are double commuting isometries, we can use Corollary 4.9 to get a decomposition $\mathcal{H} = \bigoplus_{I \subset [N]} \mathcal{H}_I$ with

$$\mathcal{H}_I \cong \ell_2\left(\mathbb{N}^I\right) \otimes Y_I.$$

And so

$$f(A) = \sum_{I \subset [N]} \langle \pi_I(A)\xi^I, \eta^I \rangle,$$

where π_I is the restriction of π to $\mathcal{H}_I$, and ξ^I (resp. η^I) is the projection of ξ (resp. η) on $\mathcal{H}_I$. Define, for each $I \subset [N]$,

$$f^I(A) = \langle \pi_I(A)\xi^I, \eta^I \rangle, \quad \forall A \in \mathcal{A}.$$

Hence,

$$\begin{aligned} \|f\| &\leq \sum_{I \subset [N]} \|f^I\| \\ &\leq \sum_{I \subset [N]} \|\xi^I\| \, \|\eta^I\| \\ &\leq \left(\sum_{I \subset [N]} \|\xi^I\|^2 \right)^{\frac{1}{2}} \left(\sum_{I \subset [N]} \|\eta^I\|^2 \right)^{\frac{1}{2}} \\ &= \|\xi\| \|\eta\| \\ &= \|f\|. \end{aligned}$$

and so we obtain (4.21). Now recall that $\pi_I(U_i)$ is a unitary for $i \in I^c$. Hence, for all $m, n \in \mathbb{N}^N$ and $I \subset [N]$, $\pi_I(U(m,n))$ equals

$$\left(\prod_{i \in I^c} \pi_I(U_i)^{m_i - n_i} \right) \left(\prod_{i \in I} \left(1 - \tfrac{1}{q_i}\right)^{-1} \pi_I \left(U_i^{m_i} (U_i^*)^{n_i} - \tfrac{1}{q_i} U_i^* U_i^{m_i} (U_i^*)^{n_i} U_i \right) \right).$$

This shows that, if $J \cap I = \varnothing$, then

$$\pi_I(U(m + k\chi^J, n + k\chi^J)) = \pi_I(U(m,n)), \quad \forall k \in \mathbb{N},$$

which proves (i). Now suppose that $J \cap I \neq \varnothing$ and observe that, for all $k \geq 1$, $\pi_I(U(m+k\chi^J, n+k\chi^J))$ can be written as the product of the three following expressions

$$\begin{aligned} &\left(\prod_{i \in I^c} \pi_I(U_i)^{m_i - n_i} \right) \\ &\left(\prod_{i \in I \cap J^c} \left(1 - \tfrac{1}{q_i}\right)^{-1} \pi_I \left(U_i^{m_i} (U_i^*)^{n_i} - \tfrac{1}{q_i} U_i^* U_i^{m_i} (U_i^*)^{n_i} U_i \right) \right) \\ &\left(\prod_{i \in I \cap J} \left(1 - \tfrac{1}{q_i}\right)^{-1} \pi_I \left(U_i^{m_i+k} (U_i^*)^{n_i+k} - \tfrac{1}{q_i} U_i^{m_i+k-1} (U_i^*)^{n_i+k-1} \right) \right). \end{aligned}$$

Recall that $\pi_I(U_i)$ is a shift if $i \in I$, which implies that $\pi_I\left(U_i^{m_i+k}(U_i^*)^{n_i+k}\right)$ converges to 0 in the weak operator topology (even more, this is true for the strong operator topology), hence the same holds for the last term above. Therefore $\pi_I(U(m+k\chi^J, n+k\chi^J))$ converges to 0 in WOT, which yields

$$f^I(U(m+k\chi^J, n+k\chi^J)) = \langle \pi_I(U(m+k\chi^J, n+k\chi^J))\xi^I, \eta^I \rangle \xrightarrow[k\to\infty]{} 0.$$

This proves (ii). □

Lemma 4.15. *Let $\phi : X \times X \to \mathbb{C}$ be a multi-radial Schur multiplier with*

$$\phi(x,y) = \tilde{\phi}(d(x_1,y_1), ..., d(x_N,y_N)), \qquad \forall\ x,y \in X,$$

and assume that the following limits exist:

$$l_0 = \lim_{\substack{|n|\to\infty \\ |n|\ even}} \tilde{\phi}(n), \qquad\qquad l_1 = \lim_{\substack{|n|\to\infty \\ |n|\ odd}} \tilde{\phi}(n).$$

Then, there exists a trace-class operator $T \in S_1(\ell_2(\mathbb{N}^N))$ such that for all $m,n \in \mathbb{N}^N$,

$$\tilde{\phi}(m+n) = c_+ + (-1)^{|m|+|n|}c_- + Tr(S(m,n)T), \tag{4.22}$$

where $S(m,n)$ is defined as in (4.20), and $c_\pm = \frac{1}{2}(l_0 \pm l_1)$. Moreover,

$$\|T\|_{S_1} + |c_+| + |c_-| \le \|\phi\|_{cb}.$$

Proof. By Lemma 4.13, There exists a bounded linear function $f_\phi : \mathcal{A} \to \mathbb{C}$ such that

$$\tilde{\phi}(m+n) = f_\phi(U(m,n)), \quad \forall m,n \in \mathbb{N}^N, \tag{4.23}$$

and $\|f_\phi\| \le \|\phi\|_{cb}$. Consider the decomposition

$$f_\phi = \sum_{I\subset[N]} f_\phi^I$$

given by Lemma 4.14. Take $J = [N]$ and observe that (4.23) together with (i) and (ii) in Lemma 4.14 imply that

$$\lim_{k\to\infty} \tilde{\phi}(m+n+2k\chi^J) = \lim_{k\to\infty} \sum_{I\subset[N]} f_\phi^I(U(m+k\chi^J, n+k\chi^J)) = f_\phi^\varnothing(U(m,n)),$$

for all $m,n \in \mathbb{N}^N$. So, by the hypothesis,

$$f_\phi^\varnothing(U(m,n)) = c_+ + (-1)^{|m|+|n|}c_-.$$

Now we shall prove by induction on $|I| \in \{1,...,N-1\}$ that $f_\phi^I = 0$. Consider first $I = \{i_0\}$ and $J = I^c$. Observe that I is the only nonempty subset of $[N]$ satisfying $I \cap J = \varnothing$. Then by the same arguments as before,

$$c_+ + (-1)^{|m|+|n|}c_- = \lim_{k\to\infty} \tilde{\phi}(m+n+2k\chi^J) = f_\phi^\varnothing(U(m,n)) + f_\phi^I(U(m,n)),$$

which proves that $f_\phi^I(U(m,n)) = 0$. Since the vector space spanned by $\{U(m,n)\,|\,m,n \in \mathbb{N}\}$ is dense in $\mathcal{A}$, this shows that $f_\phi^I = 0$ for all I with $|I| = 1$. Now suppose that this holds for $|I| \in \{1,..,l-1\}$ with $l < N$. Take $\tilde{I} \subset [N]$ with $|\tilde{I}| = l$ and $J = I^c$. Then, again by the same arguments,

$$\begin{aligned} c_+ + (-1)^{|m|+|n|}c_- &= \lim_{k\to\infty} \tilde{\phi}(m+n+2k\chi^J) \\ &= \lim_{k\to\infty} \sum_{\substack{I\subset[N] \\ |I|\geq l}} f_\phi^I(U(m+k\chi^J, n+k\chi^J)) \\ &= f_\phi^{\varnothing}(U(m,n)) + f_\phi^{\tilde{I}}(U(m,n)), \end{aligned}$$

which yields $f_\phi^{\tilde{I}}(U(m,n)) = 0$. We conclude that

$$\tilde{\phi}(m+n) = c_+ + (-1)^{|m|+|n|}c_- + f_\phi^{[N]}(U(m,n)), \quad \forall m,n \in \mathbb{N}^N.$$

Now recall the notations of Corollary 4.9, and that f_ϕ^I was defined as $\langle \pi_I(\cdot)\xi^I, \eta^I\rangle$ in the proof of Lemma 4.14. Write

$$\xi^{[N]} = \sum_{\lambda\in\Lambda} f_\lambda \otimes e_\lambda, \qquad \eta^{[N]} = \sum_{\lambda\in\Lambda} g_\lambda \otimes e_\lambda,$$

where $(e_\lambda)_{\lambda\in\Lambda}$ is an orthonormal basis of $Y_{[N]}$, and f_λ, g_λ are elements of $\ell_2(\mathbb{N}^N)$ such that

$$\|\xi^{[N]}\|^2 = \sum_{\lambda\in\Lambda} \|f_\lambda\|^2, \qquad \|\eta^{[N]}\|^2 = \sum_{\lambda\in\Lambda} \|g_\lambda\|^2.$$

We have

$$\langle \pi_{[N]}(U^m(U^*)^n)\xi^{[N]}, \eta^{[N]}\rangle = \sum_{\lambda\in\Lambda} \left\langle S_{[N]}^m \left(S_{[N]}^*\right)^n f_\lambda, g_\lambda\right\rangle = \mathrm{Tr}\left(S_{[N]}^m \left(S_{[N]}^*\right)^n T\right),$$

where

$$T = \sum_{\lambda\in\Lambda} f_\lambda \odot g_\lambda,$$

and $f_\lambda \odot g_\lambda \in S_1(\ell_2(\mathbb{N}^N))$ is the rank 1 operator defined by

$$(f_\lambda \odot g_\lambda)h = \langle h, g_\lambda\rangle f_\lambda, \qquad \forall\, h \in \ell_2(\mathbb{N}^N). \tag{4.24}$$

Hence,

$$f_\phi^{[N]}(U(m,n)) = \mathrm{Tr}\left(S(m,n)T\right).$$

This proves (4.22). Moreover,

$$\|T\|_{S_1} \leq \sum_{\lambda\in\Lambda} \|f_\lambda\|\|g_\lambda\| \leq \left(\sum_{\lambda\in\Lambda} \|f_\lambda\|^2\right)^{\frac{1}{2}} \left(\sum_{\lambda\in\Lambda} \|g_\lambda\|^2\right)^{\frac{1}{2}} = \|\xi^{[N]}\|\|\eta^{[N]}\|.$$

On the other hand, since $V = \pi_\varnothing(U_1)$ is a unitary, the C*-algebra $C^*(V)$ is isomorphic to $C(\sigma(V))$, where $\sigma(V) \subset \mathbb{T}$ is the spectrum of V. Hence, by the Riesz representation theorem, there exists a complex measure μ on $\mathbb{T}$ with $\mathrm{supp}(\mu) \subset \sigma(V)$ such that

$$\left\langle V^k \xi^\varnothing, \eta^\varnothing \right\rangle = \int_{\mathbb{T}} z^k \, d\mu(z), \quad \forall k \in \mathbb{Z},$$

and $\|\mu\| \leq \|\xi^\varnothing\|\|\eta^\varnothing\|$. Furthermore, let ν be the complex measure on $\mathbb{T}$ given by $\nu = c_+\delta_1 + c_-\delta_{-1}$. Then

$$\int_{\mathbb{T}} z^k \, d\nu(z) = c_+ + (-1)^k c_- = f_\phi^\varnothing(U_1^k) = \int_{\mathbb{T}} z^k \, d\mu(z), \qquad \forall k \geq 0,$$
$$\int_{\mathbb{T}} z^k \, d\nu(z) = c_+ + (-1)^k c_- = f_\phi^\varnothing((U_1^*)^{-k}) = \int_{\mathbb{T}} z^k \, d\mu(z), \qquad \forall k < 0,$$

which implies that $\mu = \nu$ and therefore $\|\mu\| = |c_+| + |c_-|$. We conclude that

$$\|T\|_{S_1} + |c_+| + |c_-| \leq \|\xi^{[N]}\|\|\eta^{[N]}\| + \|\xi^\varnothing\|\|\eta^\varnothing\| = \|f_\phi\| \leq \|\phi\|_{cb}.$$

□

Lemma 4.16. *Consider, for each $i = 1, ..., N$, the operator $\tau_i : \mathcal{B}(\ell_2(\mathbb{N}^N)) \to \mathcal{B}(\ell_2(\mathbb{N}^N))$ defined by*

$$\tau_i(T) = S_i T S_i^*.$$

Then for every $T \in S_1(\ell_2(\mathbb{N}^N))$, the operator

$$T' = \left[\prod_{i=1}^{N} \left(1 - \frac{1}{q_i}\right)^{-1} \left(I - \frac{\tau_i}{q_i}\right) \right] T$$

is again an element of $S_1(\ell_2(\mathbb{N}^N))$, and

$$\|T'\|_{S_1} \leq \left[\prod_{i=1}^{N} \frac{q_i + 1}{q_i - 1} \right] \|T\|_{S_1}.$$

Moreover, it satisfies

$$Tr\left(S(m,n)T\right) = Tr\left(S^m (S^*)^n T'\right), \quad \forall m, n \in \mathbb{N}^N. \tag{4.25}$$

Proof. First observe that τ_i is an injective $*$-homomorphism, hence it is an isometry on $\mathcal{B}(\ell_2(\mathbb{N}^N))$. Furthermore, it is also an isometry on $S_1(\ell_2(\mathbb{N}^N))$. Indeed, if $U|T|$ is the polar decomposition of $T \in S_1(\ell_2(\mathbb{N}^N))$, then $\tau_i(U)\tau_i(|T|)$ is the polar decomposition of $\tau_i(T)$. Therefore

$$\|\tau_i(T)\|_{S_1} = \mathrm{Tr}(\tau_i(|T|)) = \mathrm{Tr}(|T|) = \|T\|_{S_1}.$$

Thus

$$\begin{aligned}\|T'\|_{S_1} &= \left[\prod_{i=1}^{N}\left(1-\frac{1}{q_i}\right)^{-1}\right]\left\|\left[\prod_{i=1}^{N}\left(I-\frac{\tau_i}{q_i}\right)\right]T\right\|_{S_1}\\ &\leq \left[\prod_{i=1}^{N}\left(1-\frac{1}{q_i}\right)^{-1}\right]\left[\prod_{i=1}^{N}\left(1+\frac{1}{q_i}\right)\right]\|T\|_{S_1}\\ &= \left[\prod_{i=1}^{N}\frac{q_i+1}{q_i-1}\right]\|T\|_{S_1}. \end{aligned} \tag{4.26}$$

Finally, in order to obtain (4.25), we shall prove by induction on $k \in \{1, ..., N\}$ that for all $T \in S_1(\ell_2(\mathbb{N}^N))$,

$$\begin{aligned}&\operatorname{Tr}\left(\left[\prod_{i=1}^{k}\left(S_i^{m_i}(S_i^*)^{n_i}-\frac{1}{q_i}S_i^*S_i^{m_i}(S_i^*)^{n_i}S_i\right)\right]T\right)\\ &\qquad= \operatorname{Tr}\left(\left[\prod_{i=1}^{k}S_i^{m_i}(S_i^*)^{n_i}\right]\left[\prod_{i=1}^{k}\left(I-\frac{\tau_i}{q_i}\right)\right]T\right). \end{aligned} \tag{4.27}$$

Recall the identity $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$. Then, for $k = 1$, we have

$$\begin{aligned}&\operatorname{Tr}\left(\left(S_1^{m_1}(S_1^*)^{n_1}-\frac{1}{q_1}S_1^*S_1^{m_1}(S_1^*)^{n_1}S_1\right)T\right)\\ &\qquad= \operatorname{Tr}\left(S_1^{m_1}(S_1^*)^{n_1}T\right)-\frac{1}{q_1}\operatorname{Tr}\left(S_1^{m_1}(S_1^*)^{n_1}S_1TS_1^*\right)\\ &\qquad= \operatorname{Tr}\left(S_1^{m_1}(S_1^*)^{n_1}T-\frac{1}{q_1}S_1^{m_1}(S_1^*)^{n_1}\tau_1(T)\right)\\ &\qquad= \operatorname{Tr}\left(S_1^{m_1}(S_1^*)^{n_1}\left(I-\frac{\tau_1}{q_1}\right)T\right). \end{aligned} \tag{4.28}$$

Now suppose that (4.27) is true for some $k \in \{1, ..., N-1\}$, and define

$$\tilde{T} = \left(S_{k+1}^{m_{k+1}}(S_{k+1}^*)^{n_{k+1}}-\frac{1}{q_{k+1}}S_{k+1}^*S_{k+1}^{m_{k+1}}(S_{k+1}^*)^{n_{k+1}}S_{k+1}\right)T.$$

Then

$$\begin{aligned}&\operatorname{Tr}\left(\left[\prod_{i=1}^{k+1}\left(S_i^{m_i}(S_i^*)^{n_i}-\frac{1}{q_i}S_i^*S_i^{m_i}(S_i^*)^{n_i}S_i\right)\right]T\right)\\ &\qquad= \operatorname{Tr}\left(\left[\prod_{i=1}^{k}\left(S_i^{m_i}(S_i^*)^{n_i}-\frac{1}{q_i}S_i^*S_i^{m_i}(S_i^*)^{n_i}S_i\right)\right]\tilde{T}\right)\\ &\qquad= \operatorname{Tr}\left(\left[\prod_{i=1}^{k}S_i^{m_i}(S_i^*)^{n_i}\right]\left[\prod_{i=1}^{k}\left(I-\frac{\tau_i}{q_i}\right)\right]\tilde{T}\right). \end{aligned}$$

Repeating the computation (4.28) for S_{k+1} instead of S_1, this equals

$$\operatorname{Tr}\left(\left[\prod_{i=1}^{k+1} S_i^{m_i}(S_i^*)^{n_i}\right]\left[\prod_{i=1}^{k+1}\left(I-\frac{\tau_i}{q_i}\right)\right]T\right).$$

We have proven (4.27). Setting $k=N$ and multiplying by $\prod_{i=1}^N\left(1-\frac{1}{q_i}\right)^{-1}$, we obtain (4.25). $\square$

Now we are ready to prove Lemma 4.10.

Proof of Lemma 4.10. By Lemmas 4.15 and 4.16, there exists $T' \in S_1(\ell_2(\mathbb{N}^N))$ such that

$$\tilde{\phi}(m+n) = c_+ + (-1)^{|m|+|n|}c_- + \operatorname{Tr}\left(S^m(S^*)^nT'\right), \quad \forall m,n \in \mathbb{N}^N,$$

and

$$\|T'\|_{S_1} \leq \left[\prod_{i=1}^N \frac{q_i+1}{q_i-1}\right]\left(\|\phi\|_{cb} - |c_+| - |c_-|\right).$$

Hence, for all $m,n \in \mathbb{N}^N$ and $I \subset [N]$,

$$\tilde{\phi}(m+n+2\chi^I) = c_+ + (-1)^{|m|+|n|}c_- + \operatorname{Tr}\left(S^{m+\chi^I}(S^*)^{n+\chi^I}T'\right).$$

Now observe that for each $k \in \{0,...,N\}$ there are $\binom{N}{k}$ subsets $I \subset [N]$ of cardinality k. Thus

$$\sum_{I\subset[N]}(-1)^{|I|}\left(c_+ + (-1)^{|m|+|n|}c_-\right) = \left(c_+ + (-1)^{|m|+|n|}c_-\right)\sum_{k=0}^N \binom{N}{k}(-1)^k = 0.$$

On the other hand,

$$\begin{aligned}\operatorname{Tr}\left(S^{m+\chi^I}(S^*)^{n+\chi^I}T'\right) &= \sum_{q\in\mathbb{N}^N}\langle T'\delta_q, S^{n+\chi^I}(S^*)^{m+\chi^I}\delta_q\rangle \\ &= \sum_{q\geq m+\chi^I}\langle T'\delta_q, \delta_{q-m+n}\rangle \\ &= \sum_{q\geq\chi^I} T'_{n+q,m+q},\end{aligned}$$

where $q \geq \chi^I$ stands for the inequality in each coordinate. Thus,

$$\begin{aligned}\sum_{I\subset[N]}(-1)^{|I|}\tilde{\phi}(m+n+2\chi^I) &= \sum_{I\subset[N]}\sum_{q\geq\chi^I}(-1)^{|I|}T'_{n+q,m+q} \\ &= \sum_{q\in\mathbb{N}^N}T'_{n+q,m+q}\sum_{\substack{I\subset[N]\\ q\geq\chi^I}}(-1)^{|I|}.\end{aligned}$$

Finally, observe that

$$\sum_{\substack{I\subset[N]\\ q\geq\chi^I}} (-1)^{|I|} = \sum_{k=0}^{r} \tbinom{r}{j}(-1)^k,$$

where r is the cardinality of the set $\{i \in [N] \ : \ q_i > 0\}$. Hence, this sum equals 0 for all $q \in \mathbb{N}^N$, with the exception of $q = 0$, for which the sum is 1. We conclude that

$$\sum_{I\subset[N]} (-1)^{|I|}\tilde{\phi}(m+n+2\chi^I) = T'_{n,m}.$$

□

To end this section, we give the proof of Proposition 4.3 using Lemmas 4.6 and 4.10.

Proof of Proposition 4.3. Fix $N \geq 1$ and $X_1, ..., X_N$ infinite trees of minimum degrees $d_1, ..., d_N \geq 3$. Put $X = X_1 \times \cdots \times X_N$ and let $\tilde{\phi} : \mathbb{N}^N \to \mathbb{C}$ be a function such that the limits l_0 and l_1 (4.9) exist. Suppose first that

$$T = \left(\sum_{I\subset[N]} (-1)^{|I|}\tilde{\phi}(m+n+2\chi^I) \right)_{m,n\in\mathbb{N}^N}$$

belongs to $S_1(\ell_2(\mathbb{N}^N))$. Then, by Lemma 4.6, $\tilde{\phi}$ defines a multi-radial Schur multiplier ϕ on X, and

$$\|\phi\|_{cb} \leq \|T\|_{S_1} + |c_+| + |c_-|.$$

Conversely, assume that $\tilde{\phi}$ defines a Schur multiplier on X. Let $\mathcal{T}_d$ be the d-regular tree. Then there is an isometric embedding

$$\mathcal{T}_{d_1} \times \cdots \times \mathcal{T}_{d_N} \hookrightarrow X.$$

Hence, by restriction, $\tilde{\phi}$ defines a multi-radial Schur multiplier on $\mathcal{T}_{d_1} \times \cdots \times \mathcal{T}_{d_N}$ of norm at most $\|\phi\|_{cb}$. Thus, by Lemma 4.10, T is an element of $S_1(\ell_2(\mathbb{N}))$ of norm at most

$$\left[\prod_{i=1}^{N} \frac{d_i}{d_i-2}\right] \left(\|\phi\|_{cb} - |c_+| - |c_-|\right),$$

which completes the proof. □

4.3 Radial multipliers on finite products of trees

In this section, we return to radial multipliers and show how Proposition 4.3 implies Theorem A. First, we prove a general fact about trace-class operators that will provide a relation between the generalised Hankel matrices (4.5) and (4.10).

Lemma 4.17. *Let $N \geq 1$ and let $(a_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers. Define the following matrices*

$$H = \left(\binom{N+i-1}{N-1}^{\frac{1}{2}} \binom{N+j-1}{N-1}^{\frac{1}{2}} a_{i+j} \right)_{i,j\in\mathbb{N}}, \qquad T = \left(a_{|m|+|n|} \right)_{m,n\in\mathbb{N}^N}.$$

Then H belongs to $S_1(\ell_2(\mathbb{N}))$ if and only if T belongs to $S_1(\ell_2(\mathbb{N}^N))$. Moreover, in that case,

$$\|H\|_{S_1(\ell_2(\mathbb{N}))} = \|T\|_{S_1(\ell_2(\mathbb{N}^N))}.$$

Proof. Consider the following closed subspace of $\ell_2(\mathbb{N}^N)$,

$$E = \{ f \in \ell_2(\mathbb{N}^N) \ : \ \exists\, \dot{f} : \mathbb{N} \to \mathbb{C},\ f(m) = \dot{f}(|m|) \},$$

and define $V : \ell_2(\mathbb{N}) \to E$ by

$$V\delta_i = \binom{N+i-1}{N-1}^{-\frac{1}{2}} \sum_{|m|=i} \delta_m.$$

Since $\binom{N+i-1}{N-1}$ is exactly the cardinal of the set $\{m \in \mathbb{N}^N \ : \ |m| = i\}$, and since $\langle V\delta_i, V\delta_j \rangle = 0$ for $i \neq j$, V extends to a unitary from $\ell_2(\mathbb{N})$ to E. Suppose first that $T \in S_1(\ell_2(\mathbb{N}^N))$, and observe that both the ranges of T and T^* are contained in E. Hence, if $T = U|T|$ is the polar decomposition of T, then the same holds for the ranges of U and $|T|$. Thus we can write

$$V^*TV = V^*UVV^*|T|V,$$

and this is the polar decomposition of V^*TV. Thus

$$\|V^*TV\|_{S_1} = \mathrm{Tr}(V^*|T|V) = \mathrm{Tr}(|T|) = \|T\|_{S_1}.$$

Finally,

$$\begin{aligned}
\langle V^*TV\delta_j, \delta_i \rangle &= \langle TV\delta_j, V\delta_i \rangle \\
&= \binom{N+i-1}{N-1}^{-\frac{1}{2}} \binom{N+j-1}{N-1}^{-\frac{1}{2}} \sum_{|p|=i} \sum_{|q|=j} \langle T\delta_q, \delta_p \rangle \\
&= \binom{N+i-1}{N-1}^{-\frac{1}{2}} \binom{N+j-1}{N-1}^{-\frac{1}{2}} \sum_{|p|=i} \sum_{|q|=j} a_{|p|+|q|} \\
&= \binom{N+i-1}{N-1}^{\frac{1}{2}} \binom{N+j-1}{N-1}^{\frac{1}{2}} a_{i+j}.
\end{aligned}$$

Hence $V^*TV = H$, which proves one direction of the equivalence and the equality of the norms. Since V is a unitary, the argument for the other direction is analogous. □

The previous result links the characterisations given by Theorem A and Proposition 4.3. Since the existence of the limits l_0 and l_1 is one of the hypotheses of Proposition 4.3, we need a way to prove that they exist under the assumptions of Theorem A. This will be given by Lemma 4.19 below, which relies on the following elementary fact, whose proof we include for the reader's convenience.

Lemma 4.18. *Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence of complex numbers such that the sequence of differences $(a_n - a_{n+1})$ converges to $a \in \mathbb{C}$. Then $a = 0$.*

Proof. Suppose that $a \neq 0$. Then, there exists $n_0 \in \mathbb{N}$ such that

$$|a_n - a_{n+1} - a| \leq \frac{|a|}{2}, \quad \forall n \geq n_0.$$

Thus, for all $k \geq 1$,

$$|a_{n_0} - a_{n_0+k} - ka| \leq \sum_{n=0}^{k-1} |a_{n_0+n} - a_{n_0+n+1} - a| \leq k\frac{|a|}{2},$$

which implies that

$$|a_{n_0} - a_{n_0+k}| \geq k\frac{|a|}{2}, \quad \forall k \geq 1.$$

This contradicts the boundedness of (a_n). Therefore, $a = 0$. □

Recall the definition of the discrete derivative for a sequence of complex numbers $(a_n)_{n\in\mathbb{N}}$.

$$\begin{aligned}(\mathfrak{d}_1^0 a)_n &= a_n, \\ (\mathfrak{d}_1^{m+1} a)_n &= (\mathfrak{d}_1^m a)_n - (\mathfrak{d}_1^m a)_{n+1}, \quad \forall m, n \in \mathbb{N}.\end{aligned}$$

Lemma 4.19. *Let $m \geq 1$ and let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence of complex numbers such that*

$$\sum_{n\geq 0} \tbinom{m+n-1}{m-1} |(\mathfrak{d}_1^m a)_n| < \infty.$$

Then (a_n) converges.

Proof. We proceed by induction on m. For $m = 1$, we have

$$\sum_{n\geq 0} |a_n - a_{n+1}| < \infty.$$

By the triangle inequality, this implies that $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Hence, it converges. Now suppose that the result is true for some $m \geq 1$, and take $(a_n)_{n\in\mathbb{N}}$ such that

$$\sum_{n\geq 0} \tbinom{m+n}{m} \left|(\mathfrak{d}_1^{m+1} a)_n\right| < \infty. \tag{4.29}$$

In particular, the series

$$\sum_{n\geq 0} |(\mathfrak{d}_1^m a)_n - (\mathfrak{d}_1^m a)_{n+1}|$$

converges. Therefore, $((\mathfrak{d}_1^m a)_n)$ converges. Moreover, since $(\mathfrak{d}_1^m a)_n = (\mathfrak{d}_1^{m-1} a)_n - (\mathfrak{d}_1^{m-1} a)_{n+1}$, and since (a_n) is bounded, Lemma 4.18 implies that $((\mathfrak{d}_1^m a)_n)$ must converge to 0. Thus

$$(\mathfrak{d}_1^m a)_n = \sum_{j\geq 0} (\mathfrak{d}_1^m a)_{n+j} - (\mathfrak{d}_1^m a)_{n+j+1}, \quad \forall n \in \mathbb{N}.$$

Hence,

$$|(\mathfrak{d}_1^m a)_n| \leq \sum_{j\geq 0} \left|(\mathfrak{d}_1^{m+1} a)_{n+j}\right|,$$

which in turn implies

$$\begin{aligned}
\sum_{n\geq 0} \tbinom{m+n-1}{m-1} |(\mathfrak{d}_1^m a)_n| &\leq \sum_{n\geq 0} \tbinom{m+n-1}{m-1} \sum_{j\geq 0} \left|(\mathfrak{d}_1^{m+1} a)_{n+j}\right| \\
&\leq \sum_{n\geq 0} \sum_{j\geq 0} \tbinom{m+n+j-1}{m-1} \left|(\mathfrak{d}_1^{m+1} a)_{n+j}\right| \\
&= \sum_{k\geq 0} (k+1) \tbinom{m+k-1}{m-1} \left|(\mathfrak{d}_1^{m+1} a)_k\right| \\
&= \sum_{k\geq 0} \frac{m(k+1)}{m+k} \tbinom{m+k}{m} \left|(\mathfrak{d}_1^{m+1} a)_k\right|.
\end{aligned}$$

Since $\frac{m(k+1)}{m+k} \leq m$, by (4.29), this is finite. Therefore, by the induction hypothesis, (a_n) converges. $\square$

Proof of Theorem A. Fix $N \geq 1$ and $X_1, ..., X_N$ infinite trees of minimum degrees $d_1, ..., d_N \geq 3$. Put $X = X_1 \times \cdots \times X_N$ and let $\dot{\phi} : \mathbb{N} \to \mathbb{C}$ be a bounded function. Assume first that the matrix

$$H = \left(\tbinom{N+i-1}{N-1}^{\frac{1}{2}} \tbinom{N+j-1}{N-1}^{\frac{1}{2}} \mathfrak{d}_2^N \dot{\phi}(i+j) \right)_{i,j\in\mathbb{N}}$$

belongs to $S_1(\ell_2(\mathbb{N}))$. Then its diagonal defines an element of $\ell_1(\mathbb{N})$, so, by Lemma 4.19 applied to $a_n = \dot{\phi}(2n)$, the following limit exists

$$\lim_{n\to\infty} \dot{\phi}(2n).$$

Moreover, if S denotes the forward shift operator on $\ell_2(\mathbb{N})$, then HS is also of trace class, and

$$(HS)_{i,j} = \tbinom{N+i-1}{N-1}^{\frac{1}{2}} \tbinom{N+j}{N-1}^{\frac{1}{2}} \mathfrak{d}_2^N \dot{\phi}(i+j+1).$$

Since $\binom{N+j-1}{N-1} \leq \binom{N+j}{N-1}$, we can apply again Lemma 4.19 to the sequence $a_n = \dot{\phi}(2n+1)$ to get the existence of

$$\lim_{n\to\infty} \dot{\phi}(2n+1).$$

Now define $\tilde{\phi} : \mathbb{N}^N \to \mathbb{C}$ by $\tilde{\phi}(n) = \dot{\phi}(|n|)$. Then

$$\begin{aligned}\sum_{I\subset[N]} (-1)^{|I|}\tilde{\phi}(m+n+2\chi^I) &= \sum_{I\subset[N]} (-1)^{|I|}\dot{\phi}\left(|m|+|n|+2|I|\right) \\ &= \sum_{k=0}^{N} \tbinom{N}{k}(-1)^k\dot{\phi}\left(|m|+|n|+2k\right) \\ &= \mathfrak{d}_2^N\dot{\phi}\left(|m|+|n|\right).\end{aligned}$$

So by Lemma 4.17, the operator

$$T = \left(\sum_{I\subset[N]} (-1)^{|I|}\tilde{\phi}(m+n+2\chi^I)\right)_{m,n\in\mathbb{N}^N}$$

belongs to $S_1(\ell_2(\mathbb{N}^N))$. Hence, the (multi-)radial function $\phi(x,y) = \dot{\phi}(d(x,y))$ satisfies the hypotheses of Proposition 4.3, and therefore it is a Schur multiplier satisfying

$$\left[\prod_{i=1}^{N} \frac{d_i-2}{d_i}\right] \|T\|_{S_1} + |c_+| + |c_-| \leq \|\phi\|_{cb} \leq \|T\|_{S_1} + |c_+| + |c_-|, \tag{4.30}$$

where

$$c_\pm = \frac{1}{2}\lim_{n\to\infty}\dot{\phi}(2n) \pm \frac{1}{2}\lim_{n\to\infty}\dot{\phi}(2n+1).$$

Conversely, assume that $\dot{\phi}$ defines a radial Schur multiplier on X. By restriction, it defines a radial Schur multiplier on the tree X_1, so by Theorem 4.1, the limits $\lim_{n\to\infty}\dot{\phi}(2n)$ and $\lim_{n\to\infty}\dot{\phi}(2n+1)$ exist. Therefore, by Proposition 4.3 and the same computations as before, the operator T is an element of $S_1(\ell_2(\mathbb{N}^N))$ satisfying the estimates (4.30). This proves the equivalence (a) $\iff$ (b). The fact that (c) is also equivalent is given by Lemma 4.37. $\square$

4.4 Sufficient condition for a product of hyperbolic graphs

Now we turn to products of hyperbolic graphs and prove that (b) $\implies$ (a) in Theorem B. We also show that groups acting properly on such graphs are weakly amenable.

4.4.1 Sufficient condition

We shall fix now $N \geq 1$ and a family $X_1, ..., X_N$ of hyperbolic graphs with bounded degree. Define $X = X_1 \times \cdots \times X_N$. The following result proves part of Theorem B.

Proposition 4.20. *Let $\dot{\phi} : \mathbb{N} \to \mathbb{C}$ be a bounded function such that the generalised Hankel matrix*

$$H = \left(\binom{N+i-1}{N-1}^{\frac{1}{2}}\binom{N+j-1}{N-1}^{\frac{1}{2}}\mathfrak{d}_1^N\dot{\phi}(i+j)\right)_{i,j\in\mathbb{N}}$$

belongs to $S_1(\ell_2(\mathbb{N}))$. Then $\dot{\phi}$ defines a radial Schur multiplier ϕ on X. Moreover, $\dot{\phi}(n)$ converges to some $c \in \mathbb{C}$, and there exists $C > 0$ depending only on X, such that

$$\|\phi\|_{cb} \leq C\|H\|_{S_1} + |c|.$$

As in [MdlS17], the main tool that we will use is this remarkable construction by Ozawa.

Theorem 4.21. *[Oza08, Proposition 10] Let X be a hyperbolic graph with bounded degree. Then there is a Hilbert space $\mathcal{H}$, a constant $C_0 > 0$ and functions $\eta_k^+, \eta_k^- : X \to \mathcal{H}$ such that*

a) $\langle \eta_k^\pm(x), \eta_l^\pm(x)\rangle = 0$ for all $x \in X$ and $k, l \in \mathbb{N}$ such that $|k - l| \geq 2$.

b) $\|\eta_k^\pm(x)\|^2 \leq C_0$ for all $x \in X$ and all $k \in \mathbb{N}$.

c) For all $n \in \mathbb{N}$ and all $x, y \in X$,

$$\sum_{k=0}^{n} \langle \eta_k^+(x), \eta_{n-k}^-(y)\rangle = \begin{cases} 1 & \text{if } d(x,y) \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Like we did in Section 4.2, we shall obtain first a more general result involving multi-radial multipliers.

Lemma 4.22. *Let $N \geq 1$ and let $\tilde{\phi} : \mathbb{N}^N \to \mathbb{C}$ be a function such that the limit*

$$c = \lim_{|n|\to\infty} \tilde{\phi}(n)$$

exists, and such that the operator $T = (T_{n,m})_{m,n\in\mathbb{N}^N}$ given by

$$T_{n,m} = \sum_{I\subset[N]} (-1)^{|I|}\tilde{\phi}(m + n + \chi^I), \quad \forall m, n \in \mathbb{N}^N,$$

where χ^I is defined as in (4.8), is an element of $\in S_1(\ell_2(\mathbb{N}^N))$. Then $\tilde{\phi}$ defines a multi-radial Schur multiplier ϕ on X, and there exists $C > 0$ depending only on X, such that

$$\|\phi\|_{cb} \leq C\|T\|_{S_1} + |c|.$$

Proof. The proof follows the same lines as that of Lemma 4.6. Assume first that $c = 0$. Take $A, B \in S_2(\ell_2(\mathbb{N}^N))$ such that $T = A^*B$ and $\|T\|_{S_1} = \|A\|_{S_2}\|B\|_{S_2}$. Consider, for each $i = 1, ..., N$, the functions $\eta_k^\pm$ given by Theorem 4.21. Observe that these functions are not the same for different hyperbolic graphs; however, we shall make no distinction in the notation since they are defined in different spaces and will not interact with each other. Furthermore, we can let C_0 be the maximum of the N constants given by the theorem. Define now, for each $x = (x_i)_{i=1}^N \in X$,

$$P(x) = \sum_{k_1,...,k_N=0}^{\infty} \eta_{k_1}^+(x_1) \otimes \cdots \otimes \eta_{k_N}^+(x_N) \otimes Be_{(k_1,...,k_N)}$$

and

$$Q(x) = \sum_{l_1,...,l_N=0}^{\infty} \eta_{l_1}^-(x_1) \otimes \cdots \otimes \eta_{l_N}^-(x_N) \otimes Ae_{(l_1,...,l_N)},$$

where $\{e_n\}_{n\in\mathbb{N}^N}$ is the canonical orthonormal basis of $\ell_2(\mathbb{N}^N)$. Observe that, for all $x \in X$,

$$P(x) = \sum_{j_1,...,j_N=0}^{1} \sum_{m_1,...,m_N=0}^{\infty} \eta_{2m_1+j_1}^+(x_1) \otimes \cdots \otimes \eta_{2m_N+j_N}^+(x_N) \otimes Be_{(2m_1+j_1,...,2m_N+j_N)}.$$

Then, using parts (a) and (b) of Theorem 4.21,

$$\begin{aligned}
\|P(x)\|^2 &\leq 2^N \sum_{j_1,...,j_N=0}^{1} \left\| \sum_{m_1,...,m_N=0}^{\infty} \eta_{2m_1+j_1}^+(x_1) \otimes \cdots \otimes \eta_{2m_N+j_N}^+(x_N) \otimes Be_{(2m_1+j_1,...,2m_N+j_N)} \right\|^2 \\
&= 2^N \sum_{j_1,...,j_N=0}^{1} \sum_{m_1,...,m_N=0}^{\infty} \left\|\eta_{2m_1+j_1}^+(x_1)\right\|^2 \cdots \left\|\eta_{2m_N+j_N}^+(x_N)\right\|^2 \left\|Be_{(2m_1+j_1,...,2m_N+j_N)}\right\|^2 \\
&\leq (2C_0)^N \sum_{n\in\mathbb{N}^N}^{\infty} \|Be_n\|^2 \\
&= (2C_0)^N \|B\|_{S_2}^2.
\end{aligned}$$

Similarly, we obtain $\|Q(y)\|^2 \leq (2C_0)^N \|A\|_{S_2}^2$ for all $y \in X$. Now, we may write $\langle P(x), Q(y)\rangle$ as

$$\sum_{n_1=0}^{\infty} \sum_{m_1=0}^{n_1} \cdots \sum_{n_N=0}^{\infty} \sum_{m_N=0}^{n_N} \langle \eta_{m_1}^+(x_1), \eta_{n_1-m_1}^-(y_1)\rangle \cdots \langle \eta_{m_N}^+(x_N), \eta_{n_N-m_N}^-(y_N)\rangle \langle Be_{(m_1,...,m_N)}, Ae_{(n_1-m_1,...,n_N-m_N)}\rangle.$$

So, using Theorem 4.21(c) and the fact that $T = A^*B$,

$$\langle P(x), Q(y)\rangle = \sum_{\substack{n_1=d(x_1,y_1)\\ \vdots \\ n_N=d(x_N,y_N)}}^{\infty} \sum_{I\subset[N]} (-1)^{|I|}\tilde{\phi}((n_1,...,n_N)+\chi^I)$$

By the same inductive argument as in the proof of Lemma 4.6, together with the fact that $c = 0$, one shows that this equals $\tilde{\phi}(d(x_1, y_1), ..., d(x_N, y_N))$. We conclude that $\tilde{\phi}$ defines a multi-radial Schur multiplier ϕ such that

$$\|\phi\|_{cb} \leq (2C_0)^N \|A\|_{S_2}\|B\|_{S_2} \leq (2C_0)^N \|T\|_{S_1}.$$

In the general case, we use the previous argument for $\phi - c$ and conclude that

$$\|\phi\|_{cb} \leq (2C_0)^N \|T\|_{S_1} + |c|.$$

□

Proof of Proposition 4.20. Since H is of trace class, its diagonal belongs to $\ell_1(\mathbb{N})$, which implies, by Lemma 4.19, that $c = \lim_n \dot{\phi}(n)$ exists. By Lemma 4.17, the multi-radial function $\tilde{\phi}(n) = \dot{\phi}(|n|)$ satisfies the hypotheses of Lemma 4.22, which yields the conclusion. □

4.4.2 Bounded sequences of multipliers

To end this section, we show how Proposition 4.20 allows us to prove that groups acting properly on finite products of hyperbolic graphs of bounded degree are weakly amenable. The idea of the proof was essentially devised by Haagerup [Haa79] for the free group $\mathbb{F}_2$, and it was later exploited in [Oza08] and [Miz08] for hyperbolic groups and CAT(0) cubical groups respectively.

First, we give a proof of the formula (4.2).

Lemma 4.23. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers, then for all $n \in \mathbb{N}$ and $m \geq 1$,*

$$(\mathfrak{d}_1^m a)_n = \sum_{j=0}^{m} \binom{m}{j}(-1)^j a_{n+j}.$$

Proof. We proceed by induction on m. For $m = 1$ it is just the definition of $\mathfrak{d}_1 a$. Now suppose that the formula holds for some $m \geq 1$. Then

$$\begin{aligned}
(\mathfrak{d}_1^{m+1}a)_n &= (\mathfrak{d}_1^m(\mathfrak{d}_1 a))_n \\
&= \sum_{j=0}^{m} \tbinom{m}{j}(-1)^j \left(a_{n+j} - a_{n+j+1}\right) \\
&= \sum_{j=0}^{m} \tbinom{m}{j}(-1)^j a_{n+j} - \sum_{j=1}^{m+1} \tbinom{m}{j-1}(-1)^{j-1} a_{n+j} \\
&= a_n + \sum_{j=1}^{m}(-1)^j \left(\tbinom{m}{j} + \tbinom{m}{j-1}\right) a_{n+j} - (-1)^m a_{n+m+1} \\
&= a_n + \sum_{j=1}^{m}(-1)^j \tbinom{m+1}{j} a_{n+j} + (-1)^{m+1} a_{n+m+1} \\
&= \sum_{j=0}^{m+1}(-1)^j \tbinom{m+1}{j} a_{n+j}.
\end{aligned}$$

□

As before, we fix $N \geq 1$ and X a product of N hyperbolic graphs with bounded degrees.

Lemma 4.24. *For all $r \in (0,1)$, the function $\phi_r(x,y) = r^{d(x,y)}$ is a Schur multiplier on X. Moreover, $\|\phi_r\|_{cb} \leq C$, where C is the constant given by Proposition 4.20.*

Proof. First observe that

$$\sum_{k=0}^{N} \tbinom{N}{k}(-1)^k r^{i+j+k} = (1-r)^N r^{i+j},$$

for all $i, j \in \mathbb{N}$. Define $f \in \ell_2(\mathbb{N})$ by $f(i) = \binom{N-1+i}{N-1}^{\frac{1}{2}} r^i$, and let H be the positive rank-1 operator given by $H = (1-r)^N f \odot f$, which is defined as in (4.24). Then we have

$$H_{i,j} = \tbinom{N-1+i}{N-1}^{\frac{1}{2}} \tbinom{N-1+j}{N-1}^{\frac{1}{2}} (1-r)^N r^{i+j}, \quad \forall i, j \in \mathbb{N},$$

and

$$\|H\|_{S_1} = \mathrm{Tr}(H) = (1-r)^N \sum_{j \geq 0} \tbinom{N-1+j}{N-1} r^{2j}.$$

Observe that $\sum_{j\geq 0} \binom{N-1+j}{N-1} z^{2j}$ is the power series around 0 of the analytic function $(1-z^2)^{-N}$. Hence

$$\|H\|_{S_1} = \frac{(1-r)^N}{(1-r^2)^N} = \frac{1}{(1+r)^N} < 1.$$

The result follows from Proposition 4.20. □

Lemma 4.25. *For every $n \in \mathbb{N}$, define $\varphi_n : X \times X \to \mathbb{C}$ by*

$$\varphi_n(x,y) = \begin{cases} 1 & \text{if } d(x,y) = n \\ 0 & \text{otherwise.} \end{cases}$$

There exists a constant $C > 0$ such that, for every $n \in \mathbb{N}$,

$$\|\varphi_n\|_{cb} \leq C(n+1)^N. \tag{4.31}$$

Proof. Write $\varphi_n(x,y) = \dot{\varphi}_n(d(x,y))$. Then

$$\sum_{k=0}^{N} \binom{N}{k} (-1)^k \dot{\varphi}_n(i+j+k) = \begin{cases} \binom{N}{n-i-j}(-1)^{n-i-j} & \text{if } n-N \leq i+j \leq n \\ 0 & \text{otherwise.} \end{cases} \tag{4.32}$$

Define

$$H_n = \left((1+i+j)^{N-1} \mathfrak{d}_1^N \dot{\varphi}_n(i+j)\right)_{i,j\in\mathbb{N}},$$

and let $D^{(l)}$ ($l \in \mathbb{N}$) be the Hankel matrix given by

$$D^{(l)}_{i,j} = \begin{cases} 1 & \text{if } i+j = l \\ 0 & \text{otherwise.} \end{cases}$$

Then, by (4.32), H_n belongs to the linear span of $\left(D^{(l)}\right)_{l\in\mathbb{N}}$. In particular, for $n \geq N$,

$$H_n = \sum_{l=n-N}^{n} (l+1)^{N-1} \binom{N}{n-l} (-1)^{n-l} D^{(l)}.$$

Observe that $\left(D^{(l)}\right)^* D^{(l)}$ is a diagonal matrix whose first $l+1$ diagonal entries are 1 and the rest are 0. Hence

$$\|D^{(l)}\|_{S_1} = \operatorname{Tr}\left(\left(\left(D^{(l)}\right)^* D^{(l)}\right)^{\frac{1}{2}}\right) = \operatorname{Tr}\left(\left(D^{(l)}\right)^* D^{(l)}\right) = l+1.$$

Therefore,

$$\begin{aligned} \|H\|_{S_1} &\leq \sum_{k=0}^{N} \binom{N}{k} (n-k+1)^N \\ &\leq (n+1)^N \sum_{k=0}^{N} \binom{N}{k} \\ &= 2^N (1+n)^N. \end{aligned}$$

This, together with Lemma C.3 (with $\alpha + \beta = N-1$) and Proposition 4.20, proves the result for $n \geq N$. Taking C big enough, we obtain the estimate (4.31) for all n. □

By Proposition D.2, these two results together yield the following.

Corollary 4.26. *Let Γ be a countable discrete group acting properly by isometries on a finite product of hyperbolic graphs with bounded degrees. Then Γ is weakly amenable.*

Remark 4.27. *Using Theorem A instead of Proposition 4.20, the same arguments show that if a group acts properly on a finite product of trees, then it is weakly amenable. Here we do not make any assumptions on the degrees.*

4.5 Necessary condition for products of the Cayley graph of $(\mathbb{Z}/3\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z})$

Now we shall give the proof of (a) $\implies$ (b) in Theorem B by studying a very particular hyperbolic graph. The main tool here is the tree of Serre. For any group which is a free product of other groups, Serre [Ser80] constructed a certain tree with some very nice properties. We describe it now, following the presentation of [Wys95, §4].

Let $(G_i)_{i\in I}$ be a family of groups and let $G = *_{i\in I} G_i$ be their free product. We define the tree $\Gamma(G)$ in the following way:

(i) The set of vertices X consists of two disjoints subsets X_0 and X_1, where

$$X_0 = G,$$
$$X_1 = \{gG_i \,:\, g \in G,\ i \in I\}.$$

So the vertices are all the elements of G and all the cosets with respect to the subgroups G_i, $i \in I$.

(ii) The edges are all the pairs $\{g, gG_i\}$, with $g \in G$ and $i \in I$.

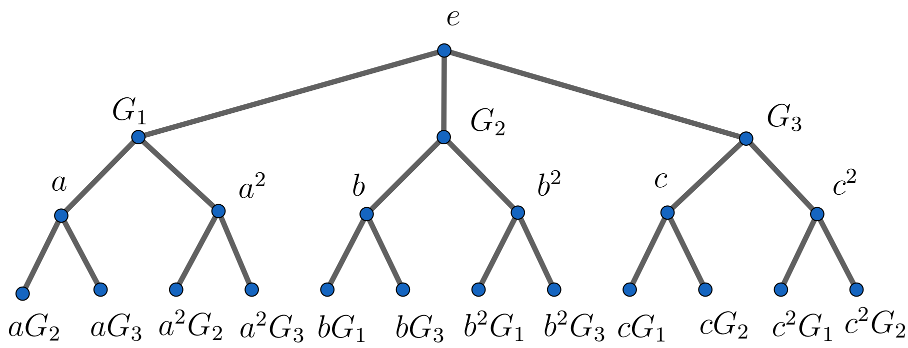
This implies that the elements of X_0 have all degree $|I|$, and an element $gG_i \in X_1$ has degree $|G_i|$. From now on, we shall fix $I = \{1,2,3\}$, and $G_i = \mathbb{Z}/3\mathbb{Z}$ for all i. Hence,

$$G = (\mathbb{Z}/3\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z}).$$

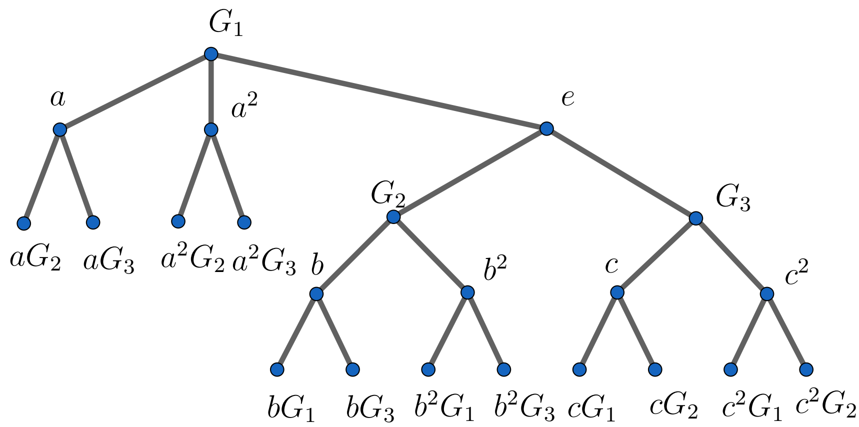
Observe that, in this case, $\Gamma(G)$ is the 3-homogeneous tree $\mathcal{T}_3$. Let X be the Cayley graph of G with generating set $G_1 \cup G_2 \cup G_3$. Then X is a hyperbolic graph. Let d and $d_{\Gamma(G)}$ denote the combinatorial distances of X and $\Gamma(G)$ respectively. Then, the previous construction gives a map $\Psi : X \hookrightarrow \Gamma(G)$ satisfying

$$d_{\Gamma(G)}(\Psi(x), \Psi(y)) = 2d(x,y), \quad \forall x, y \in X. \tag{4.33}$$

Furthermore, let $f : \Gamma(G) \to \Gamma(G)$ be the automorphism described as follows. We view the empty word e as the root of $\Gamma(G)$:



Then f moves G_1 to the root, dragging every vertex in order to define an isometry:



A more precise description is the following: The tree $\mathcal{T}_3$ may be viewed as the Cayley graph of $\mathbb{Z} * (\mathbb{Z}/2\mathbb{Z})$ with generators $-1, 1 \in \mathbb{Z}$ and $1 \in (\mathbb{Z}/2\mathbb{Z})$. Under this identification, f corresponds to the action by multiplication on the left by $1 \in \mathbb{Z}$. Moreover, observe that, for every vertex, f changes its distance to the root by 1. Since f is bijective, this implies that $\Gamma(G)$ is the disjoint union of $\Psi(X)$ and $f(\Psi(X))$.

Now we will consider products of $\Gamma(G)$, in order to relate them to the products of X. The goal is to be able to apply Proposition 4.3 to $\Gamma(G)^N$. From now on, if there is no ambiguity, the letter d will denote the distance in whichever space we are considering.

Proposition 4.28. *Let $N \geq 1$ and let $\phi : X^N \times X^N \to \mathbb{C}$ be a radial Schur multiplier with $\phi = \dot{\phi} \circ d$. Then $\dot{\phi}(n)$ converges to some limit $c \in \mathbb{C}$, and the generalised Hankel matrix*

$$H = \left(\binom{N+i-1}{N-1}^{\frac{1}{2}} \binom{N+j-1}{N-1}^{\frac{1}{2}} \mathfrak{d}_1^N \dot{\phi}(i+j) \right)_{i,j \in \mathbb{N}}$$

is an element of $S_1(\ell_2(\mathbb{N}))$ *of norm at most*

$$3^N \left(\|\phi\|_{cb} + |c|\right).$$

Proof. By restriction, ϕ defines a radial Schur multiplier on X. Hence, by [Wys95, Theorem 6.1], $c = \lim_{k\to\infty} \dot{\phi}(k)$ exists. We shall treat first the case when $c = 0$. By Theorem 4.1, there is a Hilbert space $\mathcal{H}$ and bounded functions $P, Q : X^N \to \mathcal{H}$ such that

$$\phi(x, y) = \langle P(x), Q(y)\rangle, \quad \forall x, y \in X^N.$$

Recall that $\Gamma(G)$ is the disjoint union of $A = \Psi(X)$ and $B = f(\Psi(X))$. Define functions $J : \Gamma(G)^N \to \{0,1\}^N$ and $\psi : \Gamma(G) \to X$ by

$$J(u)_i = \begin{cases} 0, & \text{if } u_i \in A \\ 1, & \text{if } u_i \in B, \end{cases} \quad \forall i \in \{1, ..., N\}.$$

$$\psi(\omega) = \begin{cases} \Psi^{-1}(\omega), & \text{if } \omega \in A \\ \Psi^{-1} \circ f^{-1}(\omega), & \text{if } \omega \in B. \end{cases}$$

Observe that $\Gamma(G)$ is a bipartite graph, and the equality $J(u) = J(v)$ is equivalent to the fact that $d(u_i, v_i)$ is even for all $i \in \{1, ..., N\}$. Consider now the Hilbert space $\tilde{\mathcal{H}} = \mathcal{H} \otimes \ell_2\left(\{0,1\}^N\right)$ and define functions $\tilde{P}, \tilde{Q} : \Gamma(G)^N \to \tilde{\mathcal{H}}$ by

$$\begin{aligned} \tilde{P}(u) &= P(\psi(u_1), ..., \psi(u_N)) \otimes \delta_{J(u)}, \\ \tilde{Q}(u) &= Q(\psi(u_1), ..., \psi(u_N)) \otimes \delta_{J(u)}. \end{aligned}$$

Observe that $\|\tilde{P}\|_\infty = \|P\|_\infty$ and $\|\tilde{Q}\|_\infty = \|Q\|_\infty$. Moreover, if $u, v \in \Gamma(G)^N$ satisfy $J(u) = J(v)$, then

$$\begin{aligned} \langle \tilde{P}(u), \tilde{Q}(v)\rangle &= \langle P(\psi(u_1), ..., \psi(u_N)), Q(\psi(v_1), ..., \psi(v_N))\rangle \\ &= \dot{\phi}(d(\psi(u_1), \psi(v_1)) + \cdots + d(\psi(u_N), \psi(v_N))) \\ &= \dot{\phi}\left(\tfrac{1}{2}d(u_1, v_1) + \cdots + \tfrac{1}{2}d(u_N, v_N)\right) \\ &= \dot{\phi}\left(\tfrac{1}{2}d(u, v)\right). \end{aligned}$$

On the other hand, if $J(u) \neq J(v)$, then $\langle \tilde{P}(u), \tilde{Q}(v)\rangle = 0$. We conclude that the function $\varphi : \Gamma(G)^N \times \Gamma(G)^N \to \mathbb{C}$ given by

$$\varphi(u, v) = \langle \tilde{P}(u), \tilde{Q}(v)\rangle, \quad \forall u, v \in \Gamma(G)^N, \tag{4.34}$$

is a Schur multiplier on the product of N copies of the 3-homogeneous tree, such that $\|\varphi\|_{cb} \leq \|\phi\|_{cb}$ and

$$\lim_{|n|\to\infty} \varphi(n) = 0.$$

Moreover, φ is a multi-radial function. That is, there exists $\tilde{\varphi} : \mathbb{N}^N \to \mathbb{C}$ such that

$$\varphi(u,v) = \tilde{\varphi}(d(u_1,v_1), ..., d(u_N,v_N)),$$

for all $u, v \in \Gamma(G)^N$. Namely,

$$\tilde{\varphi}(n) = \begin{cases} \dot{\phi}\left(\frac{1}{2}|n|\right), & \text{if } n \in (2\mathbb{N})^N \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, by Proposition 4.3, the operator

$$T = \left(\sum_{I \subset [N]} (-1)^{|I|} \tilde{\varphi}(m + n + 2\chi^I) \right)_{m,n \in \mathbb{N}^N}$$

is an element of $\in S_1(\ell_2(\mathbb{N}^N))$ of norm at most $3^N \|\phi\|_{cb}$. Recall that $\chi^I \in \{0,1\}^N$ is defined as

$$\chi_i^I = \begin{cases} 1 & \text{if } i \in I \\ 0 & \text{if } i \notin I. \end{cases}$$

Define now, for each $n \in \mathbb{N}^N$,

$$V\delta_n = \delta_{2n}.$$

Then V extends to an isometry on $\ell_2(\mathbb{N}^N)$, which implies that the operator $\tilde{T} = V^*TV$ is an element of $\in S_1(\ell_2(\mathbb{N}^N))$ of norm at most $3^N \|\phi\|_{cb}$. Furthermore, for all $m, n \in \mathbb{N}^N$,

$$\begin{aligned} \tilde{T}_{m,n} &= T_{2m,2n} \\ &= \sum_{I \subset [N]} (-1)^{|I|} \tilde{\varphi}\left(2(m + n + \chi^I)\right) \\ &= \sum_{I \subset [N]} (-1)^{|I|} \dot{\phi}\left(|m| + |n| + |I|\right) \\ &= \sum_{k=0}^{N} \binom{N}{k} (-1)^k \dot{\phi}\left(|m| + |n| + k\right) \\ &= \mathfrak{d}_1^N \dot{\phi}\left(|m| + |n|\right). \end{aligned}$$

Hence, by Lemma 4.17, the generalised Hankel matrix

$$H = \left(\binom{N+i-1}{N-1}^{\frac{1}{2}} \binom{N+j-1}{N-1}^{\frac{1}{2}} \mathfrak{d}_1^N \dot{\phi}(i+j) \right)_{i,j \in \mathbb{N}}$$

is an element of $S_1(\ell_2(\mathbb{N}))$ of norm at most $3^N \|\phi\|_{cb}$. This proves the result in the case $c = 0$. If $c \neq 0$, we repeat the previous argument for $\phi - c$, and since the derivative of a constant function is 0, we obtain the same conclusion with $\|H\|_{S_1} \leq 3^N \left(\|\phi\|_{cb} + |c|\right)$. ☐

This completes the proof of the equivalence (a) $\iff$ (b) in Theorem B, since we found a particular case for which the condition $H \in S_1(\ell_2(\mathbb{N}))$ is also necessary. The fact that (c) is also equivalent will be given by Lemma 4.37.

4.6 Sufficient condition for finite dimensional CAT(0) cube complexes

In this section we prove Theorem C. We begin by defining median graphs. Let X be (the set of vertices of) a connected graph. For all $x, y \in X$, we define

$$I(x,y) = \{u \in X \,:\, d(x,y) = d(x,u) + d(u,y)\},$$

where $d : X \times X \to \mathbb{N}$ is the combinatorial distance. Observe that $I(x,y)$ is the union of all the vertices lying in a geodesic joining x to y. We call it the interval between x and y. Now define

$$I(x,y,z) = I(x,y) \cap I(y,z) \cap I(z,x),$$

for all $x, y, z \in X$. We say that X is a median graph if

$$|I(x,y,z)| = 1, \quad \forall x,y,z \in X.$$

In that case, we call median of x, y, z, the unique element $\mu(x,y,z) \in I(x,y,z)$. It is not hard to check that trees, and more generally, products of trees are median. The following theorem of Chepoi relates median graphs to CAT(0) cube complexes. For details on CAT(0) cube complexes, see [GH10, §2].

Theorem 4.29. *[Che00, Theorem 6.1] Median graphs are exactly the 1-skeletons of CAT(0) cube complexes.*

This powerful result allows us to analyse these objects from two different points of view. We shall focus mainly on the median graph structure; however, the notion hyperplane will be useful. A hyperplane in a CAT(0) cube complex is an equivalence class of edges under the equivalence relation generated by

$$\{x,y\} \sim \{u,v\} \quad \text{if} \quad \{x,y,v,u\} \text{ is a square.}$$

If H is a hyperplane and $\{x,y\} \in H$, we say that H separates x from y, and that $\{x,y\}$ crosses H. More generally, we say that a path crosses H if one of its edges does.

Theorem 4.30. *[Sag95, Theorem 4.13] Let x, y be two vertices in a CAT(0) cube complex X, and let γ be a geodesic joining x and y. Then γ crosses every hyperplane separating x from y, and it does so only once. Moreover, γ does not cross any other hyperplane.*

Mizuta [Miz08] proved that groups acting properly on finite dimensional CAT(0) cube complexes are weakly amenable by making use of their median structure. We quickly describe his construction, as it will be our main tool to prove Theorem C. Let X be a median graph of dimension $N < \infty$ (as a cube complex) and $\mu : X^3 \to X$ its median function. Like we did in the case of trees, we fix an infinite geodesic $\omega_o : \mathbb{N} \to X$. Observe that we can always assume that such a geodesic exists since the fact of adding an infinite ray starting from a vertex of the complex preserves both the fact of being median and the dimension.

Lemma 4.31. *[Miz08, Lemma 2] Let $x_1, x_2 \in X$. There exists a unique point $m(x_1, x_2) \in X$ such that, for all but finitely many $z \in \omega_o$, $\mu(x_1, x_2, z) = m(x_1, x_2)$.*

For $x \in X$ and $k \in \mathbb{N}$, we put

$$A(x, k) = \{y \in X \ : \ \exists\, \omega_x \text{ infinite geodesic}, \omega_x(0) = x, \omega_x(k) = y, |\omega_x \Delta \omega_0| < \infty\}.$$

Observe that, for a tree, $A(x, k) = \{\omega_x(k)\}$, where ω_x is the unique geodesic that satisfies $\omega_x(0) = x$ and $|\omega_x \Delta \omega_0| < \infty$. The sets $A(x, k)$ can be endowed with a polytopal structure as follows. We define 0-polytopes as the set of vertices of X. For $l \in \{1, ..., N-1\}$, we say that $P \subset X$ is an l-polytope if there exists an $(l+1)$-cube C, a vertex $w \in C$ and $j \in \{1, ..., l\}$ such that P is the set of points at distance j from w, lying in a geodesic between w and $d_C(w)$, where $d_C(w)$ is the point diagonal to w with respect to C. For $l \in \{0, ..., N-1\}$, we define $\mathcal{A}(x, k)^{(l)}$ as the set of l-polytopes contained in $A(x, k)$. Hence, the set of all polytopes in $A(x, k)$ is

$$\mathcal{A}(x, k) = \bigcup_{l=0}^{N-1} \mathcal{A}(x, k)^{(l)}.$$

These sets of polytopes will play the role of the delta functions δ_{ω_x} in the proof of Lemma 4.6. More precisely, for $k \in \mathbb{N}$ and $l \in \{0, ..., N-1\}$, define maps $f_k^{(l)} : X \to \bigoplus_{j=0}^{N-1} \ell_2(\mathcal{X}^{(j)})$, where $\mathcal{X}^{(j)}$ is the set of all j-polytopes in X, by

$$f_k^{(l)}(x) = \sum_{P \in \mathcal{A}(x,k)^{(l)}} \delta_P \in \ell_2(\mathcal{X}^{(l)}) \subseteq \bigoplus_{j=0}^{N-1} \ell_2(\mathcal{X}^{(j)}).$$

It follows from the definition that $\|f_k^{(l)}(x)\|^2 = |\mathcal{A}(x, k)^{(l)}|$. Now put

$$P_k(x) = \sum_{l=0}^{N-1} f_k^{(l)}(x) \quad \text{and} \quad Q_k(x) = \sum_{l=0}^{N-1} (-1)^l f_k^{(l)}(x). \tag{4.35}$$

Observe that, if $k \neq j$, then $\langle P_k(x), P_j(x) \rangle = 0$ and $\langle Q_k(x), Q_j(x) \rangle = 0$. The following result is implicit in the proof of [Miz08, Theorem 2].

Proposition 4.32. *Let $x_1, x_2 \in X$ and $k_1, k_2 \in \mathbb{N}$. Then*

$$\langle P_{k_1}(x_1), Q_{k_2}(x_2) \rangle = \begin{cases} 1 & \text{if } \exists j \geq 0,\ k_i = l_i + j\ (i = 1, 2) \\ 0 & \text{if not} \end{cases},$$

where $l_i = d(x_i, m(x_1, x_2))$ $(i = 1, 2)$ and $m(x_1, x_2)$ is the point given by Lemma 4.31.

Moreover, from the definition we have

$$\|P_k(x)\|^2 = \sum_{l=0}^{N-1} \|f_k^{(l)}(x)\|^2 = \sum_{l=0}^{N-1} |\mathcal{A}(x, k)^{(l)}| = |\mathcal{A}(x, k)|.$$

And the same holds for $\|Q_k(x)\|^2$. The following lemma says that $|A(x, k)| = O(k^{N-1})$. We will prove that this is also true for $|\mathcal{A}(x, k)|$, and therefore, for $\|P_k(x)\|^2$ and $\|Q_k(x)\|^2$.

Lemma 4.33. *[Miz08, Lemma 5] For all $x \in X$ and $k \in \mathbb{N}$,*

$$|A(x,k)| \leq \binom{N-1+k}{N-1}.$$

Observe that, if we assume that the degrees of the points in X are uniformly bounded, we can conclude that there exists a constant $M > 0$ such that every vertex belongs to at most M polytopes. In what follows, we show that this is always true when we restrict ourselves to the polytopes in $\mathcal{A}(x,k)$. For $x \in X$, $k \in \mathbb{N}$, $y \in A(x,k)$ and $i = 0, ..., \min\{N-1, k\}$ define

$$B_i(x,y) = \{w \in A(x, k-i) \ : \ y \in A(w,i)\}.$$

Lemma 4.34. *For all $x \in X$, $k \in \mathbb{N}$, $y \in A(x,k)$ and $i = 0, ..., \min\{N-1, k\}$, we have*

$$|B_i(x,y)| \leq N^i.$$

Proof. We proceed by induction on i. Observe first that, for every x, k and y as above, $B_0(x,y) = \{y\}$, hence the result holds with equality for $i = 0$. If $k \geq 1$, for every $z \in B_1(x,y)$, let H_z be the hyperplane separating z from y. Observe that, by Theorem 4.30, H_z also separates y from x, since z lies in a geodesic joining x and y. Then [GH10, Proposition 2.8] implies that there is a cube C where all the hyperplanes H_z intersect. Hence $B_1(x,y) \cup \{y\}$ is included in C, which is of dimension at most N. Therefore $|B_1(x,y)| \leq N$. Finally assume that the result holds for some $i < N-1$, and take $x \in X$, $k \geq i+1$, $y \in A(x,k)$. Then

$$B_{i+1}(x,y) = \bigcup_{z \in B_1(x,y)} B_i(x,z).$$

Thus

$$|B_{i+1}(x,y)| \leq |B_1(x,y)| \left(\sup_{z \in B_1(x,y)} |B_i(x,z)| \right) \leq N N^i = N^{i+1}.$$

□

Lemma 4.35. *There is a constant $M > 0$, depending only on the dimension N, such that for every $x \in X$, $k \in \mathbb{N}$ and $y \in A(x,k)$,*

$$|\{P \in \mathcal{A}(x,k) \ : \ y \in P\}| \leq M.$$

Proof. Fix x, k and y as above and put $\mathcal{Q} = \{P \in \mathcal{A}(x,k) \ : \ y \in P\}$. If $P \in \mathcal{Q}$, then P is an l-polytope ($0 \leq l \leq N-1$) and by definition there exist an $(l+1)$-cube C and $z \in C$ such that P is the subset of C consisting of elements at a fixed distance from z. Let $\tilde{z} \in C$ be the point diagonal to z with respect to C. Without loss of generality, we may assume that $d(x,z) \leq d(x,\tilde{z})$. By [Miz08, Lemma 4], $P \subseteq A(z,i)$

and $z \in A(x, k-i)$, where $i = d(z, P) \in \{0, ..., \min\{k, N-1\}\}$. This means that $z \in B_i(x, y)$. So, if we put

$$A = \bigcup_{i=0}^{\min\{k,N-1\}} \bigcup_{z \in B_i(x,y)} A(z, i),$$

then $P \subseteq A$. Using Lemmas 4.33 and 4.34, we get

$$\begin{aligned} |A| &\leq \sum_{i=0}^{\min\{k,N-1\}} \sum_{z \in B_i(x,y)} |A(z,i)| \\ &\leq \sum_{i=0}^{\min\{k,N-1\}} |B_i(x,y)| \binom{N-1+i}{N-1} \\ &\leq \sum_{i=0}^{N-1} N^i \binom{N-1+i}{N-1}, \end{aligned}$$

and this depends only on N. Since $\mathcal{Q} \subseteq \mathcal{P}(A)$, we have $|\mathcal{Q}| \leq 2^{|A|}$, and the result follows. □

Lemma 4.36. *Let $N \geq 1$ and let M be the constant in Lemma 4.35. Then, for all $x \in X$ and $k \in \mathbb{N}$,*

$$|\mathcal{A}(x, k)| \leq M \binom{N-1+k}{N-1}.$$

Hence, the same holds for $\|P_k(x)\|^2$ and $\|Q_k(x)\|^2$.

Proof. Let $x \in X$ and $k \in \mathbb{N}$. Observe that

$$\mathcal{A}(x, k) = \bigcup_{y \in A(x,k)} \{P \in \mathcal{A}(x, k) \ : \ y \in P\}.$$

Thus, by Lemmas 4.35 and 4.33,

$$|\mathcal{A}(x, k)| \leq M|A(x, k)| \leq M \binom{N-1+k}{N-1}.$$

□

Proof of Theorem C. The equivalence between (a) and (b) will be given by Lemma 4.37. Hence, we will assume that (a) holds and prove that $\dot{\phi}$ defines a radial Schur multiplier satisfying (4.7). The proof follows the same idea as that of Lemma 4.6, by replacing the delta functions $\delta_{\omega_x(k)}$ by the vectors $P_k(x)$ and $Q_k(x)$ defined in (4.35). Let $D \in \mathcal{B}(\ell_2(\mathbb{N}))$ be the diagonal matrix defined by

$$D_{i,i} = \binom{N+i-1}{N-1}^{-\frac{1}{2}},$$

and put $\tilde{H} = DHD = (\mathfrak{d}_2\dot{\phi}(i+j))_{i,j \geq 0}$. Then $\tilde{H} \in S_1(\ell_2(\mathbb{N}))$, and by Theorem 4.1, we know that the limits $\lim_{n\to\infty} \dot{\phi}(2n), \lim_{n\to\infty} \dot{\phi}(2n+1)$ exist. Now take $A, B \in$

$S_2(\ell_2(\mathbb{N}))$ such that $H = A^*B$ and $\|H\|_{S_1} = \|A\|_{S_2}\|B\|_{S_2}$. We have $\tilde{H} = \tilde{A}^*\tilde{B}$, where $\tilde{A} = AD \in S_2(\ell_2(\mathbb{N}))$ and $\tilde{B} = BD \in S_2(\ell_2(\mathbb{N}))$. Define functions P and Q by

$$P(x) = \sum_{k\geq 0} P_k(x) \otimes \tilde{B}e_k, \quad Q(x) = \sum_{k\geq 0} Q_k(x) \otimes \tilde{A}e_k, \quad \forall x \in X,$$

where P_k and Q_k are as in (4.35), and $\{e_k\}_{k\in\mathbb{N}}$ is the canonical orthonormal basis of $\ell_2(\mathbb{N})$. By Proposition 4.32, we have

$$\langle P_{k_1}(x), Q_{k_2}(y)\rangle = \begin{cases} 1 & \text{if } \exists j \geq 0,\ k_i = l_i + j\ (i = 1, 2) \\ 0 & \text{if not} \end{cases},$$

where $l_1 = d(x, m(x, y))$ and $l_2 = d(y, m(x, y))$. This implies in particular that $l_1 + l_2 = d(x, y)$. We obtain

$$\begin{aligned}
\langle P(x), Q(y)\rangle &= \sum_{j\geq 0} \langle \tilde{A}^*\tilde{B}e_{l_1+j}, e_{l_2+j}\rangle \\
&= \sum_{j\geq 0} \tilde{H}_{l_1+j,l_2+j} \\
&= \sum_{j\geq 0} \mathfrak{d}_2\dot{\phi}(d(x, y) + 2j) \\
&= \sum_{j\geq 0} \dot{\phi}(d(x, y) + 2j) - \dot{\phi}(d(x, y) + 2(j + 1)) \\
&= \dot{\phi}(d(x, y)) - \lim_{n\to\infty} \dot{\phi}(d(x, y) + 2n) \\
&= \dot{\phi}(d(x, y)) - \left(c_+ + c_-(-1)^{d(x,y)}\right).
\end{aligned}$$

Moreover, by Lemma 4.36, we have

$$\begin{aligned}
\|P(x)\|^2 &= \sum_{k\geq 0} \|P_k(x)\|^2\|\tilde{B}e_k\|^2 \\
&= \sum_{k\geq 0} |\mathcal{A}(x, k)|\|BDe_k\|^2 \\
&\leq M \sum_{k\geq 0} \tbinom{N-1+k}{N-1}\tbinom{N-1+k}{N-1}^{-1}\|Be_k\|^2 \\
&= M\|B\|_{S_2}^2.
\end{aligned}$$

Similar computations show that $\|Q(y)\|^2 \leq M\|A\|_{S_2}^2$, and these bounds do not depend on x or y. This, together with Lemma 4.5, implies that ϕ is a Schur multiplier and

$$\|\phi\|_{cb} \leq M\|H\|_{S_1} + |c_+| + |c_-|.$$

□

4.7 Inclusions of sets of multipliers

In this section, we show how the conditions in Theorems A, B and C relate to each other. For this purpose, let us introduce some notation. For each $N \geq 1$ and $\dot{\phi} : \mathbb{N} \to \mathbb{C}$, define the following infinite matrices,

$$A(N,\dot{\phi}) = \Big((1+i+j)^{N-1}\mathfrak{d}_2^N\dot{\phi}(i+j)\Big)_{i,j\in\mathbb{N}}$$
$$B(N,\dot{\phi}) = \Big((1+i+j)^{N-1}\mathfrak{d}_1^N\dot{\phi}(i+j)\Big)_{i,j\in\mathbb{N}}$$
$$C(N,\dot{\phi}) = \Big((1+i+j)^{N-1}\mathfrak{d}_2\dot{\phi}(i+j)\Big)_{i,j\in\mathbb{N}},$$

and consider the following subspaces of $\ell_\infty(\mathbb{N})$,

$$\mathcal{A}_N = \Big\{\dot{\phi}\in\ell_\infty(\mathbb{N}) \,:\, A(N,\dot{\phi})\in S_1(\ell_2(\mathbb{N}))\Big\}$$
$$\mathcal{B}_N = \Big\{\dot{\phi}\in\ell_\infty(\mathbb{N}) \,:\, B(N,\dot{\phi})\in S_1(\ell_2(\mathbb{N}))\Big\}$$
$$\mathcal{C}_N = \Big\{\dot{\phi}\in\ell_\infty(\mathbb{N}) \,:\, C(N,\dot{\phi})\in S_1(\ell_2(\mathbb{N}))\Big\}.$$

In other words, thanks to Lemma C.3, a function $\dot{\phi} : \mathbb{N} \to \mathbb{C}$ belongs to $\mathcal{A}_N$ (resp. $\mathcal{B}_N$, $\mathcal{C}_N$) if and only if the generalised Hankel matrix defined in Theorem A (resp. B, C) is of trace class. Now we give the proof of the characterisations in terms of Besov spaces, which completes the proofs of Theorems A, B and C.

Lemma 4.37. *Let $N \geq 1$ and $\dot{\phi} : \mathbb{N} \to \mathbb{C}$ be a bounded function. Then $\dot{\phi}$ belongs to $\mathcal{A}_N$ (resp. $\mathcal{B}_N$, $\mathcal{C}_N$) if and only if the analytic function*

$$(z^2-1)^N\sum_{n\geq 0}\dot{\phi}(n)z^n \quad \left(\textit{resp. } (z-1)^N\sum_{n\geq 0}\dot{\phi}(n)z^n, \quad (z^2-1)\sum_{n\geq 0}\dot{\phi}(n)z^n\right)$$

belongs to $B_1^N(\mathbb{T})$.

Proof. We shall only prove the first case since the other ones are analogous. Extend $\dot{\phi}$ to $\mathbb{Z}$ by setting $\dot{\phi}(n) = 0$ for $n < 0$, and observe that

$$\begin{aligned}(z^2-1)^N\sum_{n\geq 0}\dot{\phi}(n)z^n &= \sum_{k=0}^{N}\sum_{n\geq 0}\tbinom{N}{k}(-1)^k\dot{\phi}(n)z^{n+2N-2k}\\ &= \sum_{k=0}^{N}\sum_{n\geq 0}\tbinom{N}{k}(-1)^k\dot{\phi}(n-2N+2k)z^{n}\\ &= \sum_{n\geq 0}\mathfrak{d}_2^N\dot{\phi}(n-2N)z^n.\end{aligned}$$

This, together with the fact that polynomials belong to $B_1^N(\mathbb{T})$, implies that $(z^2-1)^N\sum_{n\geq 0}\dot{\phi}(n)z^n$ belongs to $B_1^N(\mathbb{T})$ if and only if

$$\sum_{n\geq 2N}\mathfrak{d}_2^N\dot{\phi}(n-2N)z^n$$

does. By Proposition C.2, this is equivalent to

$$\sum_{n\geq 0} \mathfrak{d}_2^N \dot{\phi}(n) z^n \in B_1^N(\mathbb{T}),$$

which, by Theorem C.1, is equivalent to the fact that $\dot{\phi} \in \mathcal{A}_N$. □

The main goal in this section is to prove the following.

Proposition 4.38. *The sets $\mathcal{A}_N$, $\mathcal{B}_N$ and $\mathcal{C}_N$ satisfy the following relations.*

a) For all $N \geq 1$, $\mathcal{A}_{N+1} \subsetneq \mathcal{A}_N$, $\mathcal{B}_{N+1} \subsetneq \mathcal{B}_N$ and $\mathcal{C}_{N+1} \subsetneq \mathcal{C}_N$.

b) For all $N \geq 2$, $\mathcal{C}_N \subsetneq \mathcal{A}_N$ and $\bigcap_{m\geq 1} \mathcal{A}_m \not\subseteq \mathcal{C}_N$.

c) For all $N \geq 1$, $\mathcal{B}_N \subsetneq \mathcal{A}_N$ and $\bigcap_{m\geq 1} \mathcal{A}_m \not\subseteq \mathcal{B}_N$. Furthermore, $\bigcap_{m\geq 1} \mathcal{C}_m \not\subseteq \mathcal{B}_N$.

We will concentrate first in the strict inclusion $\mathcal{B}_{N+1} \subsetneq \mathcal{B}_N$ by studying the function

$$\dot{\phi}(n) = \frac{(-1)^n}{(n+1)^{\alpha+1}}, \tag{4.36}$$

for different values of $\alpha \geq 0$. The inclusion $\mathcal{A}_{N+1} \subsetneq \mathcal{A}_N$ follows by the same arguments, considering the function $n \mapsto \frac{\mathrm{i}^n}{(n+1)^{\alpha+1}}$ instead. The verifications will be left to the reader.

Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences of complex numbers. We will write $a_n \sim b_n$ if there exist $C_1, C_2 > 0$ such that

$$C_1|b_n| \leq |a_n| \leq C_2|b_n|, \quad \forall n \in \mathbb{N}.$$

Lemma 4.39. *Let $\alpha > 0$ and define*

$$a_n = \frac{(-1)^n}{(n+1)^\alpha}, \quad \forall n \in \mathbb{N}.$$

Then $(\mathfrak{d}_1^m a)_n \sim \frac{1}{(n+1)^\alpha}$ for all $m \in \mathbb{N}$.

Proof. By Lemma 4.23,

$$(\mathfrak{d}_1^m a)_n = \sum_{k=0}^{m} \binom{m}{k} (-1)^k \frac{(-1)^{n+k}}{(n+k+1)^\alpha} = (-1)^n \sum_{k=0}^{m} \binom{m}{k} \frac{1}{(n+k+1)^\alpha} \sim \frac{1}{(n+1)^\alpha}.$$

□

Observe that the sets $\mathcal{A}_N$, $\mathcal{B}_N$ and $\mathcal{C}_N$ are defined in terms of Hankel matrices, but in general it is not easy to determine if a sequence (a_n) defines a Hankel matrix in $S_1(\ell_2(\mathbb{N}))$. The following theorem of Bonsall [Bon86] provides a sufficient condition.

Theorem 4.40. *[Bon86, Theorem 3.1] Let (a_n) be a sequence of complex numbers converging to 0 and such that*

$$\sum_{n\geq 2} |a_{n-1} - a_n| n \log n < \infty.$$

Then the Hankel matrix $(a_{i+j})_{i,j\in\mathbb{N}}$ belongs to $S_1(\ell_2(\mathbb{N}))$.

Lemma 4.41. *Let $\alpha \geq 0$ and $m \in \mathbb{N}$. Define $\dot{\phi} : \mathbb{N} \to \mathbb{C}$ by*

$$\dot{\phi}(n) = \frac{(-1)^n}{(n+1)^{\alpha+1}}.$$

Then

$$\begin{aligned}
((1+i+j)^s \mathfrak{d}_1^m \dot{\phi}(i+j))_{i,j\in\mathbb{N}} &\notin S_1(\ell_2(\mathbb{N})), \quad \forall s \geq \alpha \\
((1+i+j)^s \mathfrak{d}_1^m \dot{\phi}(i+j))_{i,j\in\mathbb{N}} &\in S_1(\ell_2(\mathbb{N})), \quad \forall s < \alpha - 1.
\end{aligned}$$

Proof. Consider first $s \geq \alpha$ and observe that the series $\sum_{i\geq 0}(2i+1)^s \mathfrak{d}_1^m \dot{\phi}(2i)$ diverges because $\mathfrak{d}_1^m \dot{\phi}(n) \sim (n+1)^{-\alpha-1}$, by Lemma 4.39. This implies that

$$((1+i+j)^s \mathfrak{d}_1^m \dot{\phi}(i+j))_{i,j\in\mathbb{N}} \notin S_1(\ell_2(\mathbb{N})),$$

since otherwise its diagonal would be an element of ℓ_1. Now suppose that $s < \alpha - 1$. Then, again by Lemma 4.39,

$$\begin{aligned}
\left| n^s \mathfrak{d}_1^m \dot{\phi}(n-1) - (n+1)^s \mathfrak{d}_1^m \dot{\phi}(n) \right| &\leq n^s \left| \mathfrak{d}_1^m \dot{\phi}(n-1) \right| + (n+1)^s \left| \mathfrak{d}_1^m \dot{\phi}(n) \right| \\
&\sim n^s n^{-\alpha-1}.
\end{aligned}$$

Hence

$$\sum_{n\geq 2} \left| n^s \mathfrak{d}_1^m \dot{\phi}(n-1) - (n+1)^s \mathfrak{d}_1^m \dot{\phi}(n) \right| n \log n \leq C \sum_{n\geq 2} n^{s-\alpha} \log n < \infty.$$

By Theorem 4.40, $((1+i+j)^s \mathfrak{d}_1^m \dot{\phi}(i+j))_{i,j\in\mathbb{N}} \in S_1(\ell_2(\mathbb{N}))$. □

The following is a direct consequence of Lemma 4.41.

Corollary 4.42. *Let $N \geq 1$ and $\alpha \in (N, N+1]$. Then the function $\dot{\phi} : \mathbb{N} \to \mathbb{C}$ given by*

$$\dot{\phi}(n) = \frac{(-1)^n}{(n+1)^{\alpha+1}}, \quad \forall n \in \mathbb{N},$$

belongs to $\mathcal{B}_N \setminus \mathcal{B}_{N+2}$.

Observe that, by restriction, Theorem B implies that for all $N \geq 1$, $\mathcal{B}_{N+2} \subseteq \mathcal{B}_{N+1} \subseteq \mathcal{B}_N$. Corollary 4.42 says that one of these inclusions is strict. In order to show that both of them are, we will use the following identity.

Lemma 4.43. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers. For all $n, m \in \mathbb{N}$,*

$$(n+1)\sum_{k=0}^{m}\tbinom{m}{k}(-1)^k a_{n+k} = \sum_{k=0}^{m}\tbinom{m}{k}(-1)^k(n+k+1)a_{n+k} + m\sum_{k=0}^{m-1}\tbinom{m-1}{k}(-1)^k a_{n+k+1}.$$

Proof. Observe that the identity $m\binom{m-1}{k} = (k+1)\binom{m}{k+1}$ implies

$$m\sum_{k=0}^{m-1}\tbinom{m-1}{k}(-1)^k a_{n+k+1} = -\sum_{k=1}^{m} k\tbinom{m}{k}(-1)^k a_{n+k}.$$

Hence

$$\begin{aligned}\sum_{k=0}^{m}\tbinom{m}{k}(-1)^k(n+k+1)a_{n+k} + m\sum_{k=0}^{m-1}\tbinom{m-1}{k}(-1)^k a_{n+k+1} \\ &= \sum_{k=0}^{m}\tbinom{m}{k}(-1)^k(n+k+1-k)a_{n+k} \\ &= (n+1)\sum_{k=0}^{m}\tbinom{m}{k}(-1)^k a_{n+k}.\end{aligned}$$

□

Lemma 4.44. *For all $N \geq 1$, $\mathcal{B}_{N+1} \subsetneq \mathcal{B}_N$.*

Proof. Since, by Propositions 4.20 and 4.28, $\mathcal{B}_N$ corresponds to the set of radial Schur multipliers on the product of N copies of the Cayley graph of $(\mathbb{Z}/3\mathbb{Z})*(\mathbb{Z}/3\mathbb{Z})*(\mathbb{Z}/3\mathbb{Z})$, we have $\mathcal{B}_{N+1} \subseteq \mathcal{B}_N$. Suppose now that there exists $N \geq 1$ such that $\mathcal{B}_{N+1} = \mathcal{B}_N$, and take $\dot{\phi} \in \mathcal{B}_N$. Then the Hankel matrix

$$\Big((i+j+1)^N \mathfrak{d}_1^{N+1}\dot{\phi}(i+j)\Big)_{i,j\in\mathbb{N}}$$

belongs to $S_1(\ell_2(\mathbb{N}))$. On the other hand, applying Lemma 4.43 to the sequence $a_n = \mathfrak{d}_1\dot{\phi}(n)$, we obtain

$$\begin{aligned}(n+1)\sum_{k=0}^{N}\tbinom{N}{k}(-1)^k\mathfrak{d}_1\dot{\phi}(n+k) &= \sum_{k=0}^{N}\tbinom{N}{k}(-1)^k\dot{\psi}(n+k) \\ &\quad + N\sum_{k=0}^{N-1}\tbinom{N-1}{k}(-1)^k\mathfrak{d}_1\dot{\phi}(n+k+1),\end{aligned}$$

where $\dot{\psi}(n) = (n+1)\mathfrak{d}_1\dot{\phi}(n)$. By Lemma 4.23, this may be rewritten as

$$(i+j+1)\mathfrak{d}_1^{N+1}\dot{\phi}(i+j) = \mathfrak{d}_1^N\dot{\psi}(i+j) + N\mathfrak{d}_1^N\dot{\phi}(i+j+1), \tag{4.37}$$

Since $\dot{\phi} \in \mathcal{B}_N$, by Lemma 4.37 together with Proposition C.2, the matrix

$$\Big((i+j+1)^{N-1}\mathfrak{d}_1^N\dot{\phi}(i+j+1)\Big)_{i,j\in\mathbb{N}}$$

belongs to $S_1(\ell_2(\mathbb{N}))$. Hence, by (4.37), the same holds for

$$\Big((i+j+1)^{N-1}\mathfrak{d}_1^N\dot{\psi}(i+j)\Big)_{i,j\in\mathbb{N}}.$$

This says that $\dot{\psi}$ belongs to $\mathcal{B}_N$, which we have assumed is equal to $\mathcal{B}_{N+1}$. Thus

$$\Big((i+j+1)^{N}\mathfrak{d}_1^{N+1}\dot{\psi}(i+j)\Big)_{i,j\in\mathbb{N}} \in S_1(\ell_2(\mathbb{N})).$$

Again, by Lemmas 4.43 and 4.23,

$$\begin{aligned}(i+j+1)^{N+1}\mathfrak{d}_1^{N+2}\dot{\phi}(i+j) =&(i+j+1)^{N}\mathfrak{d}_1^{N+1}\dot{\psi}(i+j)\\ &+(N+1)(i+j+1)^{N}\mathfrak{d}_1^{N+1}\dot{\phi}(i+j+1),\end{aligned}$$

Hence,

$$\Big((i+j+1)^{N+1}\mathfrak{d}_1^{N+2}\dot{\phi}(i+j)\Big)_{i,j\in\mathbb{N}} \in S_1(\ell_2(\mathbb{N})),$$

which means that $\dot{\phi} \in \mathcal{B}_{N+2}$. Since $\dot{\phi}$ is arbitrary, this implies that $\mathcal{B}_N = \mathcal{B}_{N+2}$, which contradicts Corollary 4.42. We conclude that $\mathcal{B}_{N+1} \neq \mathcal{B}_N$. □

We have proved that $\mathcal{B}_{N+1} \subsetneq \mathcal{B}_N$. The same kind of argument shows that $\mathcal{A}_{N+1} \subsetneq \mathcal{A}_N$. In order to prove that $\mathcal{C}_{N+1} \subsetneq \mathcal{C}_N$, we shall use another result of Bonsall.

Theorem 4.45. *[Bon86, Corollary 3.3] Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that, for $m = 0, 1, 2$,*

$$(\mathfrak{d}_1^m a)_n \geq 0, \quad \forall n \in \mathbb{N}.$$

Then the Hankel matrix $(a_{i+j})_{i,j\in\mathbb{N}}$ belongs to $S_1(\ell_2(\mathbb{N}))$ if and only if

$$\sum_{n\geq 0} a_n < \infty.$$

Lemma 4.46. *Let $f : [0,\infty) \to \mathbb{C}$ be a smooth function. Define $\dot{\phi} : \mathbb{N} \to \mathbb{C}$ by $\dot{\phi}(n) = f(n)$. Then, for all $n \in \mathbb{N}$ and $m \geq 1$,*

$$\mathfrak{d}_2^m\dot{\phi}(n) = (-1)^m \int_0^2 \cdots \int_0^2 f^{(m)}(n+t_1+\cdots+t_m)\,dt_1\cdots dt_m. \tag{4.38}$$

Proof. We proceed by induction on m. For $m = 1$, we have

$$\mathfrak{d}_2\dot{\phi}(n) = f(n) - f(n+2) = -\int_0^2 f'(n+t)\,dt.$$

Now suppose that (4.38) holds for some $m \geq 1$. Then

$$\mathfrak{d}_2^{m+1}\dot{\phi}(n) = \mathfrak{d}_2^m\mathfrak{d}_2\dot{\phi}(n) = (-1)^m \int_0^2 \cdots \int_0^2 g^{(m)}(n+t_1+\cdots+t_m)\,dt_1\cdots dt_m,$$

with $g(t) = f(t) - f(t+2)$. Since

$$g^{(m)}(t) = f^{(m)}(t) - f^{(m)}(t+2) = -\int_0^2 f^{(m+1)}(t+s)\,ds,$$

we obtain

$$\mathfrak{d}_2^{m+1}\dot{\phi}(n) = (-1)^{m+1}\int_0^2 \cdots \int_0^2 f^{(m+1)}(n + t_1 + \cdots + t_{m+1})\,dt_1 \cdots dt_{m+1},$$

which concludes the proof. □

Lemma 4.47. *Let $N \geq 1$ and $\dot{\phi} : \mathbb{N} \to \mathbb{C}$ be given by*

$$\dot{\phi}(n) = \begin{cases} -\sum_{j=0}^{k-1}(2j+1)^{-N-\frac{1}{2}}, & n = 2k \\ -\sum_{j=1}^{k}(2j)^{-N-\frac{1}{2}}, & n = 2k+1. \end{cases}$$

Then $\dot{\phi} \in \mathcal{C}_N \setminus \mathcal{C}_{N+1}$. Moreover, if $N \geq 2$, $\dot{\phi} \in \bigcap_{m\geq 1} \mathcal{A}_m$.

Proof. Observe that $\dot{\phi}$ is bounded and

$$\dot{\phi}(n) - \dot{\phi}(n+2) = (n+1)^{-N-\frac{1}{2}}, \quad \forall n \in \mathbb{N}.$$

Therefore, for all $m \in \mathbb{N}$,

$$(n+1)^m \mathfrak{d}_2\dot{\phi}(n) = (n+1)^{m-N-\frac{1}{2}}, \quad \forall n \in \mathbb{N}.$$

If $m < N + \frac{1}{2}$, then $a_n = (n+1)^{m-N-\frac{1}{2}}$ satisfies the hypotheses of Theorem 4.45, and hence, $(a_{i+j})_{i,j\in\mathbb{N}} \in S_1(\ell_2(\mathbb{N}))$ if and only if $m - N - \frac{1}{2} < -1$. Therefore

$$\Big((i+j+1)^{N-1}\mathfrak{d}_2\dot{\phi}(i+j)\Big)_{i,j\in\mathbb{N}} \in S_1(\ell_2(\mathbb{N}))$$
$$\Big((i+j+1)^{N}\mathfrak{d}_2\dot{\phi}(i+j)\Big)_{i,j\in\mathbb{N}} \notin S_1(\ell_2(\mathbb{N})).$$

This means that $\dot{\phi} \in \mathcal{C}_N \setminus \mathcal{C}_{N+1}$. Now let $\Gamma(z) = \int_0^\infty t^{z-1}e^{-1}\,dt$ be the Gamma function. By Lemma 4.46 applied to the function $f(t) = (t+1)^{-N-\frac{1}{2}}$,

$$\mathfrak{d}_2^{m+1}\dot{\phi}(n) = \frac{\Gamma(N+\frac{1}{2}+m)}{\Gamma(N+\frac{1}{2})}\int_0^2 \cdots \int_0^2 (1+n+t_1+\cdots+t_m)^{-N-\frac{1}{2}-m}\,dt_1\cdots dt_m.$$

for all $m \geq 0$. This gives

$$\frac{2^m\Gamma(\alpha+m)}{\Gamma(\alpha)}(n+1+2m)^{-N-\frac{1}{2}-m} \leq \mathfrak{d}_2^{m+1}\dot{\phi}(n) \leq \frac{2^m\Gamma(\alpha+m)}{\Gamma(\alpha)}(n+1)^{-N-\frac{1}{2}-m},$$

which implies that $\mathfrak{d}_2^m\dot{\phi}(n) \sim (1+n)^{-N+\frac{1}{2}-m}$ for all $m \geq 1$. Hence

$$(1+n)^m \mathfrak{d}_2^m \dot{\phi}(n) \sim (1+n)^{-N+\frac{1}{2}}, \quad \forall m \geq 1.$$

Putting $a_n = (1+n)^{-N+\frac{1}{2}}$, we get

$$a_n - a_{n+1} \sim (1+n)^{-N-\frac{1}{2}}.$$

Thus

$$\sum_{n\geq 2} |a_{n-1} - a_n| n \log n \leq C \sum_{n\geq 2} n^{-N+\frac{1}{2}} \log n,$$

and this is finite whenever $N \geq 2$. By Theorem 4.40, $\dot{\phi} \in \mathcal{A}_m$, which concludes the proof. □

Now we deal with the strict inclusion $\mathcal{C}_N \subsetneq \mathcal{A}_N$. For this purpose, we shall consider the same function as in (4.36), but without the alternating factor $(-1)^n$. The reason for this is that, in this case, each derivation will increase the decay rate at infinity.

Lemma 4.48. *Let $N \geq 2$ and $\alpha \in (0, N-1)$. Define $\dot{\phi} : \mathbb{N} \to \mathbb{C}$ by*

$$\dot{\phi}(n) = \frac{1}{(n+1)^\alpha}, \quad \forall n \in \mathbb{N}.$$

Then $\dot{\phi} \notin \mathcal{C}_N$ and $\mathfrak{d}_2^m \dot{\phi} \in \mathcal{C}_N$ for all $m > N - \alpha$.

Proof. As in the proof of Lemma 4.47, we see that $\mathfrak{d}_2^m\dot{\phi}(n) \sim (1+n)^{-\alpha-m}$ for all $m \in \mathbb{N}$. Hence

$$\sum_{j\geq 0} (1+2j)^{N-1} \mathfrak{d}_2 \dot{\phi}(2j) = \infty.$$

This proves that $\dot{\phi} \notin \mathcal{C}_N$, since otherwise the trace of the matrix $C(N, \dot{\phi})$ would be finite. Now take $m > N - \alpha$ and observe that

$$\begin{aligned}
&\sum_{n\geq 2} \left| (n+1)^{N-1} \mathfrak{d}_2^{m+1} \dot{\phi}(n) - n^{N-1} \mathfrak{d}_2^{m+1} \dot{\phi}(n-1) \right| n \log n \\
&\leq \sum_{n\geq 3} \left| \mathfrak{d}_2^{m+1} \dot{\phi}(n-1) \right| n^N \log(n-1) + \sum_{n\geq 2} \left| \mathfrak{d}_2^{m+1} \dot{\phi}(n-1) \right| n^N \log n \\
&\leq 2 \sum_{n\geq 2} \left| \mathfrak{d}_2^{m+1} \dot{\phi}(n-1) \right| n^N \log n \\
&\leq C \sum_{n\geq 2} n^{N-\alpha-m-1} \log n,
\end{aligned}$$

and this is finite because $N - \alpha - m < 0$. By Theorem 4.40, the matrix

$$\left((1+i+j)^{N-1} \mathfrak{d}_2^{m+1} \dot{\phi}(i+j) \right)_{i,j\in\mathbb{N}}$$

belongs to $S_1(\ell_2(\mathbb{N}))$, which means that $\mathfrak{d}_2^m \dot{\phi} \in \mathcal{C}_N$. □

Corollary 4.49. *For all $N \geq 2$, $\mathcal{C}_N \subsetneq \mathcal{A}_N$.*

Proof. The inclusion $\mathcal{C}_N \subset \mathcal{A}_N$ is given by Theorems A and C, together with the fact that a product of N trees is an N-dimensional CAT(0) cube complex. Now observe that, by definition, for every $\dot{\phi} \in \ell_\infty(\mathbb{N})$, $\dot{\phi} \in \mathcal{A}_N$ if and only if $\mathfrak{d}_2^{N-1}\dot{\phi} \in \mathcal{C}_N$. If we assume that $\mathcal{A}_N = \mathcal{C}_N$, then by induction,

$$\dot{\phi} \in \mathcal{C}_N \iff \mathfrak{d}_2^{j(N-1)}\dot{\phi} \in \mathcal{C}_N,$$

for all $j \in \mathbb{N}$. Taking j big enough, this contradicts Lemma 4.48. Therefore $\mathcal{A}_N \neq \mathcal{C}_N$. □

Proof of Proposition 4.38. The inclusion $\mathcal{B}_{N+1} \subsetneq \mathcal{B}_N$ is given by Lemma 4.44, and the fact that $\mathcal{A}_{N+1} \subsetneq \mathcal{A}_N$ follows analogously. Lemma 4.47 shows that $\mathcal{C}_{N+1} \subsetneq \mathcal{C}_N$. This proves part (a). Part (b) corresponds to Corollary 4.49 together with Lemma 4.47. Finally, the inclusion $\mathcal{B}_N \subset \mathcal{A}_N$ is given by Theorems A and B, together with the fact that every tree is a hyperbolic graph. Moreover, by taking $\dot{\phi}(n) = (-1)^n$, we see that all the entries of the matrix $C(m, \dot{\phi})$ $(m \geq 1)$ are 0, and on the other hand, the trace of $B(N, \dot{\phi})$ does not converge. This proves that $\dot{\phi} \in \bigcap_{m\geq 1} \mathcal{C}_m \setminus \mathcal{B}_N$, for all $N \geq 1$. Hence $\bigcap_{m\geq 1} \mathcal{C}_m \nsubseteq \mathcal{B}_N$. Since $\mathcal{C}_m \subseteq \mathcal{A}_m$, this implies that $\bigcap_{m\geq 1} \mathcal{A}_m \nsubseteq \mathcal{B}_N$. □

Appendix A

Approximation properties of operator spaces

The aim of this appendix is to provide the definitions of the approximation properties mentioned in Section 1.4.3. For a much more detailed treatment, we refer the reader to [BO08], [Pis03] and [ALR10].

A.1 Tensor products

Let A and B be two C*-algebras. In general, the algebraic tensor product $A \odot B$ can be endowed with more than one C*-norm. We describe here two of them, which are of particular interest.

Suppose that we have faithful representations $\pi : A \to \mathcal{B}(\mathcal{H})$ and $\sigma : B \to \mathcal{B}(\mathcal{K})$. The minimal (or spatial) C*-norm on $A \odot B$ is defined as

$$\left\| \sum a_i \otimes b_i \right\|_{\min} = \left\| \sum \pi(a_i) \otimes \sigma(b_i) \right\|_{\mathcal{B}(\mathcal{H} \otimes \mathcal{K})}.$$

The minimal tensor product $A \otimes_{\min} B$ is defined as the completion of $A \odot B$ for this norm. This definition does not depend on the (faithful) representations π and σ. The minimal tensor product of operator spaces $E \otimes_{\min} F$ is defined analogously by means of Ruan's theorem (Theorem 1.2).

The above definition is given by the construction of a $*$-homomorphism from $A \odot B$ to the algebra of bounded operators on a Hilbert space, and then taking the norm closure. The maximal tensor product is defined by considering all such possible morphisms. Namely,

$$\|x\|_{\max} = \sup \left\{ \|\pi(x)\| \;\middle|\; \begin{array}{c} \pi : A \odot B \to \mathcal{B}(\mathcal{H}) \text{ is a } *\text{-homomorphism,} \\ \mathcal{H} \text{ a Hilbert space} \end{array} \right\}.$$

We denote by $A \otimes_{\max} B$ the completion of $A \odot B$ for this norm.

It was proven by Takesaki [Tak64] that every C*-norm $\| \cdot \|_\alpha$ on $A \odot B$ satisfies

$$\| \cdot \|_{\min} \leq \| \cdot \|_\alpha \leq \| \cdot \|_{\max},$$

and hence, there are natural surjective $*$-homomorphisms

$$A \otimes_{\max} B \to A \otimes_\alpha B \to A \otimes_{\min} B,$$

where $A \otimes_\alpha B$ stands for the completion of $A \odot B$ for the norm $\| \cdot \|_\alpha$.

If $M \subset \mathcal{B}(\mathcal{H})$ and $N \subset \mathcal{B}(\mathcal{K})$ are von Neumann algebras, the von Neumann algebraic tensor product $M \bar{\otimes} N$ is defined as the weak closure of $M \odot N$ in $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$.

A.2 C*-algebras and operator spaces

The starting point here is the notion of nuclearity. A C*-algebra A is said to be nuclear if, for every C*-algebra B, the algebraic tensor product $A \odot B$ admits a unique C*-norm. In other words, $A \otimes_{\min} B = A \otimes_{\max} B$. In [CE78] and [Kir77], it was proved that nuclearity may be viewed as an approximation property.

Definition A.1. *We say that a C*-algebra A has the completely positive approximation property (CPAP) if there exist a net of finite-rank completely positive maps $T_i : A \to A$ satisfying*

$$\lim_i \|T_i x - x\| = 0, \quad \forall x \in A.$$

Theorem A.2 (Choi-Effros [CE78], Kirchberg [Kir77])**.** *A C*-algebra is nuclear if and only if it has the CPAP.*

The fact that every completely positive map is completely bounded allows us to consider a weaker property by requiring the maps T_i to be only completely bounded. We define it in the more general context of operator spaces.

Definition A.3. *We say that an operator space E has the completely bounded approximation property (CBAP) if there exist a net of finite-rank maps $T_i : E \to E$ and a constant $C > 0$ such that*

$$\sup_i \|T_i\|_{cb} \leq C,$$

and

$$\lim_i \|T_i x - x\| = 0, \quad \forall x \in E.$$

The infimum of all such C over all possible nets is the CBAP constant of E. We denote it by $\Lambda_{cb}(E)$.

Finally, we describe an even weaker property. Let $\mathcal{K}(\ell_2)$ denote the space of compact operators on $\ell_2(\mathbb{N})$.

Definition A.4. *We say that an operator space E has the operator approximation property (OAP) if there exist a net of finite-rank maps $T_i : E \to E$ such that*

$$\lim_i \|(id \otimes T_i)x - x\| = 0, \quad \forall x \in \mathcal{K}(\ell_2) \otimes_{min} E.$$

A.3 von Neumann algebras

There exist three properties, defined for von Neumann algebras, which are in some sense analogous to the ones above. Let us first recall that the ultraweak topology on $\mathcal{B}(\mathcal{H})$ is the topology induced by the family of seminorms

$$T \mapsto \left| \sum_i \langle T\xi_i, \eta_i \rangle \right|,$$

for all sequences $(\xi_i)_i$ and $(\eta_i)_i$ in $\mathcal{H}$ such that $\sum \|\xi_i\|^2 < \infty$ and $\sum \|\eta_i\|^2 < \infty$. Every von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$ is closed for this topology.

Definition A.5. *A von Neumann algebra M is said to be semidiscrete if there exist a net of finite-rank completely positive ultraweakly continuous maps $T_i : M \to M$ such that $T_i(1_M) = 1_M$ and*

$$T_i x \to x, \quad \forall x \in M$$

in the ultraweak topology.

Definition A.6. *A von Neumann algebra M is said to have the weak* CBAP if there exist a net of finite-rank ultraweakly continuous maps $T_i : M \to M$ and a constant $C > 0$ such that*

$$\sup_i \|T_i\|_{cb} \leq C,$$

and

$$T_i x \to x, \quad \forall x \in M$$

in the ultraweak topology. The infimum of all such C over all possible nets is the weak CBAP constant of M. We denote it by $\Lambda_{cb*}(M)$.*

Definition A.7. *A von Neumann algebra M is said to have the weak* OAP if there exist a net of finite-rank ultraweakly continuous maps $T_i : M \to M$ such that*

$$(id \otimes T_i)x \to x, \quad \forall x \in \mathcal{B}(\ell_2)\bar{\otimes}M,$$

in the ultraweak topology of $\mathcal{B}(\ell_2)\bar{\otimes}M$.

A.4 p-Operator spaces

Similar properties can be defined in the context of p-operator spaces. For this purpose, instead of von Neumann algebras, we need to consider weak*-closed subspaces of $\mathcal{B}(L_p(X,\mu))$.

Definition A.8. *We say that a p-operator space E has the p-completely bounded approximation property (p-CBAP) if there exist a net of finite-rank maps $T_i : E \to E$ and a constant $C > 0$ such that*

$$\sup_i \|T_i\|_{p\text{-}cb} \leq C,$$

and

$$\lim_i \|T_i x - x\| = 0, \quad \forall x \in E.$$

The infimum of all such C over all possible nets is the p-CBAP constant of E. We denote it by $\Lambda_{p\text{-}cb}(E)$.

Definition A.9. *A weak*-closed subspace E of $\mathcal{B}(L_p)$ is said to have the weak* p-CBAP if there exist a net of finite-rank weak*-continuous maps $T_i : E \to E$ and a constant $C > 0$ such that*

$$\sup_i \|T_i\|_{p\text{-}cb} \leq C,$$

and

$$T_i x \to x, \quad \forall x \in E$$

in the weak topology. The infimum of all such C over all possible nets is the weak* p-CBAP constant of E. We denote it by $\Lambda_{p\text{-}cb*}(M)$.*

In what follows, we define p-OAP and weak* p-OAP. For this purpose, we need the notion of p-operator space injective tensor product. Let E and F be two p-operator spaces, and denote by $\mathcal{CB}_p(E, F)$ the vector space of p-completely bounded maps from E to F. This space has a natural p-operator space structure given by

$$M_n(\mathcal{CB}_p(E, F)) = \mathcal{CB}_p(E, M_n(F)).$$

In particular, the dual space $E^* = \mathcal{CB}_p(E, \mathbb{C})$ is a p-operator space. Now observe that there is an injective embedding $\theta : E \odot F \to \mathcal{CB}_p(E^*, F)$ given by

$$\theta(x \otimes y)f = f(x)y, \quad \forall x \in E, \ \forall y \in F, \ \forall f \in E^*.$$

We denote by $E \overset{\vee_p}{\otimes} F$ the completion of $E \odot F$ in $\mathcal{CB}_p(E^*, F)$, and we call it the p-operator space injective tensor product of E and F, which is endowed with the p-operator space structure inherited from $\mathcal{CB}_p(E^*, F)$.

Definition A.10. *A p-operator space E is said to have the p-operator approximation property (p-OAP) if there exist a net of finite-rank maps $T_i : E \to E$ such that*

$$\lim_i \|(id \otimes T_i)x - x\| = 0, \quad \forall x \in \mathcal{K}(\ell_p) \overset{\vee_p}{\otimes} E.$$

Now suppose that E and F are weak*-closed subspaces of $\mathcal{B}(L_p(\mu))$ and $\mathcal{B}(L_p(\nu))$ respectively. Then $E\bar{\otimes}F$ is defined as the weak* closure of $E \odot F$ in $\mathcal{B}(L_p(\mu \otimes \nu))$.

Definition A.11. *A weak*-closed subspace E of $\mathcal{B}(L_p)$ is said to have the weak* p-OAP if there exist a net of weak*-continuous finite-rank maps $T_i : E \to E$ such that*

$$(id \otimes T_i)x \to x, \quad \forall x \in \mathcal{B}(\ell_p)\bar{\otimes}E,$$

in the weak topology of $\mathcal{B}(\ell_p)\bar{\otimes}E$.*

Appendix B

Trace class operators

We give here a very quick introduction to trace class operators. For more details, we refer the reader to [Mur90, §2.4] and [Bla06, §I.8.5].

Let $\mathcal{H}$ a Hilbert space. Then every bounded operator $T \in \mathcal{B}(\mathcal{H})$ admits a polar decomposition

$$T = U|T|,$$

where U is a partial isometry satisfying $\operatorname{Ker} U = \operatorname{Ker} T$, and $|T|$ is the positive operator defined by functional calculus as follows,

$$|T| = (T^*T)^{\frac{1}{2}}.$$

For a proof of this fact, see e.g. [Mur90, Theorem 2.3.4].

Let $(e_i)_i$ be an orthonormal basis of $\mathcal{H}$. We say T is of trace class if

$$\operatorname{Tr}(|T|) = \sum_i \langle |T| e_i, e_i \rangle < \infty.$$

The space of trace class operators on $\mathcal{H}$ is denoted by $S_1(\mathcal{H})$, and it becomes a Banach space with the norm $\|T\|_{S_1} = \operatorname{Tr}(|T|)$. The value of the trace does not depend on the choice of the orthonormal basis (e_i).

We say that T is a Hilbert–Schmidt operator if

$$\operatorname{Tr}(T^*T) = \sum_i \|Te_i\|^2 < \infty.$$

The space of Hilbert–Schmidt operators on $\mathcal{H}$ is denoted by $S_2(\mathcal{H})$, and it becomes a Banach space with the norm $\|T\|_{S_2} = \operatorname{Tr}(T^*T)^{\frac{1}{2}}$. Again, this does not depend on the choice of the orthonormal basis (e_i). Using the polar decomposition, one can prove the following.

Proposition B.1. *Let $T \in S_1(\mathcal{H})$. Then there exist $A, B \in S_2(\mathcal{H})$ satisfying $T = A^*B$ and*

$$\|A\|_{S_2} = \|B\|_{S_2} = \|T\|_{S_1}^{\frac{1}{2}}.$$

Appendix C

Besov spaces and generalised Hankel matrices

We provide here the results concerning Besov spaces that are used in this thesis. For a more detailed treatment, see [Pel03, Appendix 2.6]. For $n \geq 1$, let $W_n : \mathbb{T} \to \mathbb{C}$ be the polynomial whose Fourier coefficients are given by

$$\hat{W}_n(k) = \begin{cases} 2^{-n+1}(k - 2^{n-1}), & \text{if } k \in [2^{n-1}, 2^n] \\ 2^{-n}(2^{n+1} - k), & \text{if } k \in [2^n, 2^{n+1}] \\ 0, & \text{otherwise,} \end{cases}$$

and put $W_0(z) = 1 + z$. For $s \in \mathbb{R}$, we define the Besov space of analytic functions $B_1^s(\mathbb{T})$ as the space of all series $\varphi(z) = \sum_{n \geq 0} a_n z^n$ such that

$$\|\varphi\|_{B_1^s} = \sum_{n \geq 0} 2^{ns} \|W_n * \varphi\|_{L_1(\mathbb{T})} < \infty.$$

Observe that the notation is not the same as that of [Pel03]. Since we only deal with the particular case of analytic functions, the space $B_1^s(\mathbb{T})$ corresponds to $(B_1^s)_+$ in [Pel03]. The following result relates Besov spaces to generalised Hankel matrices in $S_1(\ell_2(\mathbb{N}))$. It is a particular case of [Pel03, Theorem 6.8.9].

Theorem C.1 (Peller)**.** *Let $\varphi : \mathbb{D} \to \mathbb{C}$ be an analytic function, and $\alpha, \beta > -\frac{1}{2}$. Then the generalised Hankel matrix*

$$((1+i)^\alpha (1+j)^\beta \hat{\varphi}(i+j))_{i,j \geq 0}$$

defines an element of $S_1(\ell_2(\mathbb{N}))$ if and only if $\varphi \in B_1^{1+\alpha+\beta}(\mathbb{T})$.

This result allows us to show that Besov spaces are invariant under the shift operator.

Proposition C.2. *Let $s > 0$ and $\sum_{n \geq 0} a_n z^n \in B_1^s(\mathbb{T})$. Then the series*

$$\sum_{n \geq 0} a_{n+1} z^n \quad \textit{and} \quad \sum_{n \geq 1} a_{n-1} z^n$$

define elements of $B_1^s(\mathbb{T})$ as well.

Proof. Take $\alpha = \frac{s-1}{2}$. By Theorem C.1, the matrix $H_{i,j} = (1+i)^\alpha (1+j)^\alpha a_{i+j}$ belongs to $S_1(\ell_2(\mathbb{N}))$. Let S be the forward shift operator on $\ell_2(\mathbb{N})$ and let $D \in \mathcal{B}(\ell_2(\mathbb{N}))$ be the diagonal operator given by $D_{i,i} = \left(\frac{1+i}{2+i}\right)^\alpha$. Then $\tilde{H} = DS^*H$ belongs to $S_1(\ell_2(\mathbb{N}))$, and

$$\tilde{H}_{i,j} = \langle H\delta_j, SD\delta_i\rangle = \left(\frac{1+i}{2+i}\right)^\alpha \langle H\delta_j, \delta_{i+1}\rangle = (1+i)^\alpha (1+j)^\alpha a_{i+j+1}.$$

Again by Theorem C.1, we conclude that $\sum_{n\geq 0} a_{n+1} z^n \in B_1^s(\mathbb{T})$.
Now observe that D^{-1} is a bounded operator as well, so $H' = SD^{-1}H \in S_1(\ell_2(\mathbb{N}))$ and

$$H'_{i,j} = \begin{cases} 0, & \text{if } i = 0, \\ (1+i)^\alpha (1+j)^\alpha a_{i+j-1}, & \text{if } i > 0. \end{cases}$$

Consider also the rank-1 operator R defined by

$$R_{i,j} = \begin{cases} (1+j)^\alpha a_{j-1}, & \text{if } i = 0, j > 0, \\ 0, & \text{otherwise,} \end{cases}$$

and observe that

$$\|R\|_{S_1} = \left(\sum_{j\geq 1} (1+j)^{2\alpha} |a_{j-1}|^2\right)^{\frac{1}{2}} = \|H'e_0\|_2 < \infty.$$

Hence $H' + R \in S_1(\ell_2(\mathbb{N}))$, and (putting $a_{-1} = 0$),

$$H'_{i,j} + R_{i,j} = (1+i)^\alpha (1+j)^\alpha a_{i+j-1}.$$

By Theorem C.1, we have $\sum_{n\geq 1} a_{n-1} z^n \in B_1^s(\mathbb{T})$. □

We will also need to make use of the operators I_α of fractional integration. For all $\alpha, s \in \mathbb{R}$, and $\varphi \in B_1^s$, define $I_\alpha \varphi$ by

$$I_\alpha \varphi(z) = \sum_{n\geq 0} (1+n)^{-\alpha} \hat{\varphi}(n) z^n. \tag{C.1}$$

This operator satisfies $I_\alpha B_1^s = B_1^{s+\alpha}$. For a proof, see e.g. [dlS09, Theorem 2.4], where this is done in the more general context of vector-valued Besov spaces.

Lemma C.3. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers and $\alpha, \beta > -\frac{1}{2}$. Consider the matrices*

$$H_1 = ((1+i)^\alpha (1+j)^\beta a_{i+j})_{i,j\in\mathbb{N}},$$
$$H_2 = ((1+i+j)^{\alpha+\beta} a_{i+j})_{i,j\in\mathbb{N}}.$$

Then $H_1 \in S_1(\ell_2(\mathbb{N}))$ if and only if $H_2 \in S_1(\ell_2(\mathbb{N}))$. Moreover, when $\alpha+\beta \in \mathbb{N}$, these conditions are equivalent to

$$H_3 = \left(\binom{\alpha+\beta+i}{\alpha+\beta}^{\frac{1}{2}} \binom{\alpha+\beta+j}{\alpha+\beta}^{\frac{1}{2}} a_{i+j}\right)_{i,j\in\mathbb{N}} \in S_1(\ell_2(\mathbb{N})). \tag{C.2}$$

Proof. By Theorem C.1, H_1 belongs to $S_1(\ell_2(\mathbb{N}))$ if and only if the analytic function $\varphi(z) = \sum_{n\geq 0} a_n z^n$ belongs to the Besov space $B_1^{1+\alpha+\beta}(\mathbb{T})$. This is equivalent to the fact that the analytic function

$$I_{-\alpha-\beta}\varphi(z) = \sum_{n\geq 0}(1+n)^{\alpha+\beta} a_n z^n$$

belongs to $B_1^1(\mathbb{T})$, where $I_{-\alpha-\beta}$ is defined as in (C.1). And again by Theorem C.1, this is equivalent to $H_2 \in S_1(\ell_2(\mathbb{N}))$. Now suppose that $m = \alpha+\beta$ is a natural number. Then

$$\binom{m+k}{m}^{\frac{1}{2}} \sim (1+k)^{\frac{m}{2}},$$

which implies that the diagonal matrix D given by $D_{ii} = \binom{m+i}{m}^{-\frac{1}{2}}(1+i)^{\frac{m}{2}}$ defines a bounded operator on $\ell_2(\mathbb{N})$, and so does D^{-1}. Put $\tilde{H}_1 = ((1+i)^{\frac{m}{2}}(1+j)^{\frac{m}{2}} a_{i+j})_{i,j\in\mathbb{N}}$. Observing that $\tilde{H}_1 = DH_3D$ and $\alpha+\beta = \frac{m}{2}+\frac{m}{2}$, we get

$$\begin{aligned} H_3 \in S_1(\ell_2(\mathbb{N})) &\iff \tilde{H}_1 \in S_1(\ell_2(\mathbb{N})) \\ &\iff H_2 \in S_1(\ell_2(\mathbb{N})). \end{aligned}$$

□

Appendix D

A general criterion for weak amenability

We explain here the procedure mentioned in Section 4.4.2, that allows us to prove weak amenability by means of a proper isometric action. Let us state first a characterisation of weak amenability for discrete groups, which is usually given as a definition in this context.

Proposition D.1. *Let Γ be a countable discrete group and $C \geq 1$. Then Γ is weakly amenable with $\Lambda(\Gamma) \leq C$ if and only if there exists a sequence of finitely supported functions $\varphi_n : \Gamma \to \mathbb{C}$ converging pointwise to 1, and such that $\|\varphi_n\|_{M_0(\Gamma)} \leq C$ for all $n \in \mathbb{N}$.*

For a proof of this result, see [BO08, Theorem 12.3.10] where this is taken as the definition of weak amenability. Then the equivalence follows from Theorem 1.18.

Proposition D.2. *Let X be a connected graph satisfying the following conditions.*

(i) There exists a constant $C \geq 1$ such that, for all $r \in (0,1)$, the function $n \mapsto r^n$ defines a radial Schur multiplier on X of norm at most C.

(ii) There exists a polynomial p such that the functions $\chi_n : X \times X \to \mathbb{C}$ defined by

$$\chi_n(x,y) = \begin{cases} 1 & \text{if } d(x,y) = n \\ 0 & \text{otherwise} \end{cases}$$

are Schur multipliers on X satisfying $\|\chi_n\|_{cb} \leq p(n)$ for all $n \in \mathbb{N}$.

Let Γ be a countable discrete group acting properly by isometries on X. Then Γ is weakly amenable with $\Lambda(\Gamma) \leq C$.

Proof. The key point is to observe that, for all $r \in (0,1)$ and all $x, y \in X$,

$$r^{d(x,y)} = \sum_{j \geq 0} \chi_j(x,y) r^j.$$

Take $\varepsilon > 0$. For every $n \in \mathbb{N}$, choose $k_n \in \mathbb{N}$ such that

$$\sum_{j>k_n} p(j)e^{-\frac{j}{n}} < \varepsilon.$$

This can be made in such a way that the sequence $(k_n)_{n\in\mathbb{N}}$ is increasing. Then define $\psi_n : X \times X \to \mathbb{C}$ by

$$\begin{aligned}\psi_n(x, y) &= \sum_{j\leq k_n} \chi_j(x, y)e^{-\frac{j}{n}} \\ &= e^{-\frac{1}{n}d(x,y)} - \sum_{j>k_n} \chi_j(x, y)e^{-\frac{j}{n}}, \quad \forall x, y \in X.\end{aligned}$$

Thus, by (i) and (ii),

$$\begin{aligned}\|\psi_n\|_{cb} &\leq C + \sum_{j>k_n} p(j)e^{-\frac{j}{n}} \\ &\leq C + \varepsilon.\end{aligned}$$

Now fix $x_0 \in X$ and define $\varphi_n : \Gamma \to \mathbb{C}$ by $\varphi_n(s) = \psi_n(x_0, s \cdot x_0)$. Since Γ acts properly on X, these functions are finitely supported. Now fix momentarily $n \in \mathbb{N}$. By Theorem 1.3, there is a Hilbert space $\mathcal{H}$ and functions $P, Q : X \to \mathcal{H}$ such that

$$\psi_n(x, y) = \langle P(x), Q(y)\rangle, \quad \forall x, y \in X,$$

and $\|P\|_\infty\|Q\|_\infty \leq C + \varepsilon$. Since ψ_n is radial and Γ acts isometrically on X,

$$\begin{aligned}\varphi_n(st^{-1}) &= \psi_n(x_0, st^{-1} \cdot x_0) \\ &= \psi_n(s^{-1} \cdot x_0, t^{-1} \cdot x_0) \\ &= \left\langle P(s^{-1} \cdot x_0), Q(t^{-1} \cdot x_0)\right\rangle,\end{aligned}$$

which shows by Theorem 1.6 that $\|\varphi_n\|_{M_0(\Gamma)} \leq C + \varepsilon$. Moreover, by construction, the sequence (φ_n) converges pointwise to 1. Hence, by Proposition D.1, Γ is weakly amenable with $\Lambda(\Gamma) \leq C + \varepsilon$. Since ε is arbitrary, we conclude that $\Lambda(\Gamma) \leq C$. □

Bibliography

[ALR10] Guimei An, Jung-Jin Lee, and Zhong-Jin Ruan. On p-approximation properties for p-operator spaces. *J. Funct. Anal.*, 259(4):933–974, 2010.

[BF84] Marek Bożejko and Gero Fendler. Herz-Schur multipliers and completely bounded multipliers of the Fourier algebra of a locally compact group. *Boll. Un. Mat. Ital. A (6)*, 3(2):297–302, 1984.

[Bla06] Bruce Blackadar. *Operator algebras*, volume 122 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2006. Theory of C^*-algebras and von Neumann algebras, Operator Algebras and Non-commutative Geometry, III.

[BO08] Nathanial P. Brown and Narutaka Ozawa. *C^*-algebras and finite-dimensional approximations*, volume 88 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.

[Bon86] Frank F. Bonsall. Some nuclear Hankel operators. In *Aspects of mathematics and its applications*, volume 34 of *North-Holland Math. Library*, pages 227–238. North-Holland, Amsterdam, 1986.

[CE78] Man Duen Choi and Edward G. Effros. Nuclear C^*-algebras and the approximation property. *Amer. J. Math.*, 100(1):61–79, 1978.

[CH89] Michael Cowling and Uffe Haagerup. Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one. *Invent. Math.*, 96(3):507–549, 1989.

[CH13] Victor Chepoi and Mark F. Hagen. On embeddings of CAT(0) cube complexes into products of trees via colouring their hyperplanes. *J. Combin. Theory Ser. B*, 103(4):428–467, 2013.

[Che00] Victor Chepoi. Graphs of some CAT(0) complexes. *Adv. in Appl. Math.*, 24(2):125–179, 2000.

[Cow98] Michael Cowling. The predual of the space of convolutors on a locally compact group. *Bull. Austral. Math. Soc.*, 57(3):409–414, 1998.

[Daw10] Matthew Daws. p-operator spaces and Figà-Talamanca–Herz algebras. *J. Operator Theory*, 63(1):47–83, 2010.

[dCH85] Jean de Cannière and Uffe Haagerup. Multipliers of the Fourier algebras of some simple Lie groups and their discrete subgroups. *Amer. J. Math.*, 107(2):455–500, 1985.

[Der11] Antoine Derighetti. *Convolution operators on groups*, volume 11 of *Lecture Notes of the Unione Matematica Italiana*. Springer, Heidelberg; UMI, Bologna, 2011.

[DF93] Andreas Defant and Klaus Floret. *Tensor norms and operator ideals*, volume 176 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1993.

[dLdlS15] Tim de Laat and Mikael de la Salle. Strong property (T) for higher-rank simple Lie groups. *Proc. Lond. Math. Soc. (3)*, 111(4):936–966, 2015.

[dlS09] Mikael de la Salle. Operator space valued Hankel matrices. *ArXiv e-prints*, September 2009.

[Dor96] Brian Dorofaeff. Weak amenability and semidirect products in simple Lie groups. *Math. Ann.*, 306(4):737–742, 1996.

[DS19] Matthew Daws and Nico Spronk. On convoluters on L^p-spaces. *Studia Math.*, 245(1):15–31, 2019.

[EL77] Edward G. Effros and E. Christopher Lance. Tensor products of operator algebras. *Adv. Math.*, 25(1):1–34, 1977.

[ER00] Edward G. Effros and Zhong-Jin Ruan. *Operator spaces*, volume 23 of *London Mathematical Society Monographs. New Series*. The Clarendon Press, Oxford University Press, New York, 2000.

[Eym64] Pierre Eymard. L'algèbre de Fourier d'un groupe localement compact. *Bull. Soc. Math. France*, 92:181–236, 1964.

[Fol95] Gerald B. Folland. *A course in abstract harmonic analysis*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995.

[FT65] Alessandro Figà-Talamanca. Translation invariant operators in L^p. *Duke Math. J.*, 32:495–501, 1965.

[GH10] Erik Guentner and Nigel Higson. Weak amenability of CAT(0)-cubical groups. *Geom. Dedicata*, 148:137–156, 2010.

[GN43] Israel Gelfand and Mark A. Naimark. On the imbedding of normed rings into the ring of operators in Hilbert space. *Rec. Math. [Mat. Sbornik] N.S.*, 12(54):197–213, 1943.

[Gro53] Alexander Grothendieck. Résumé de la théorie métrique des produits tensoriels topologiques. *Bol. Soc. Mat. São Paulo*, 8:1–79, 1953.

[Gro87] Mikhael Gromov. Hyperbolic groups. In *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987.

[Haa16] Uffe Haagerup. Group C^*-algebras without the completely bounded approximation property. *J. Lie Theory*, 26(3):861–887, 2016.

[Haa79] Uffe Haagerup. An example of a nonnuclear C^*-algebra, which has the metric approximation property. *Invent. Math.*, 50(3):279–293, 1978/79.

[HdL13] Uffe Haagerup and Tim de Laat. Simple Lie groups without the approximation property. *Duke Math. J.*, 162(5):925–964, 2013.

[HdL16] Uffe Haagerup and Tim de Laat. Simple Lie groups without the approximation property II. *Trans. Amer. Math. Soc.*, 368(6):3777–3809, 2016.

[Hel78] Sigurdur Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume 80 of *Pure and Applied Mathematics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1978.

[Her71] Carl Herz. The theory of p-spaces with an application to convolution operators. *Trans. Amer. Math. Soc.*, 154:69–82, 1971.

[Her73] Carl Herz. Harmonic synthesis for subgroups. *Ann. Inst. Fourier (Grenoble)*, 23(3):91–123, 1973.

[HK94] Uffe Haagerup and Jon Kraus. Approximation properties for group C^*-algebras and group von Neumann algebras. *Trans. Amer. Math. Soc.*, 344(2):667–699, 1994.

[HKdL16] Uffe Haagerup, Søren Knudby, and Tim de Laat. A complete characterization of connected Lie groups with the approximation property. *Ann. Sci. Éc. Norm. Supér. (4)*, 49(4):927–946, 2016.

[HSS10] Uffe Haagerup, Troels Steenstrup, and Ryszard Szwarc. Schur multipliers and spherical functions on homogeneous trees. *Internat. J. Math.*, 21(10):1337–1382, 2010.

[Jol92] Paul Jolissaint. A characterization of completely bounded multipliers of Fourier algebras. *Colloq. Math.*, 63(2):311–313, 1992.

[Kir77] Eberhard Kirchberg. C^*-nuclearity implies CPAP. *Math. Nachr.*, 76:203–212, 1977.

[Kna02] Anthony W. Knapp. *Lie groups beyond an introduction*, volume 140 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, second edition, 2002.

[KR97] Richard V. Kadison and John R. Ringrose. *Fundamentals of the theory of operator algebras. Vol. I*, volume 15 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1997. Elementary theory, Reprint of the 1983 original.

[Laf08] Vincent Lafforgue. Un renforcement de la propriété (T). *Duke Math. J.*, 143(3):559–602, 2008.

[Lan73] Christopher Lance. On nuclear C^*-algebras. *J. Functional Analysis*, 12:157–176, 1973.

[LdlS11] Vincent Lafforgue and Mikael de la Salle. Noncommutative L^p-spaces without the completely bounded approximation property. *Duke Math. J.*, 160(1):71–116, 2011.

[Lep68] Horst Leptin. Sur l'algèbre de Fourier d'un groupe localement compact. *C. R. Acad. Sci. Paris Sér. A-B*, 266:A1180–A1182, 1968.

[LM96] Christian Le Merdy. Factorization of p-completely bounded multilinear maps. *Pacific J. Math.*, 172(1):187–213, 1996.

[Mar91] Gregory A. Margulis. *Discrete subgroups of semisimple Lie groups*, volume 17 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1991.

[MdlS17] Tao Mei and Mikael de la Salle. Complete boundedness of heat semigroups on the von Neumann algebra of hyperbolic groups. *Trans. Amer. Math. Soc.*, 369(8):5601–5622, 2017.

[Miz08] Naokazu Mizuta. A Bożejko-Picardello type inequality for finite-dimensional CAT(0) cube complexes. *J. Funct. Anal.*, 254(3):760–772, 2008.

[Mur90] Gerard J. Murphy. *C^*-algebras and operator theory*. Academic Press, Inc., Boston, MA, 1990.

[Oza08] Narutaka Ozawa. Weak amenability of hyperbolic groups. *Groups Geom. Dyn.*, 2(2):271–280, 2008.

[Pel03] Vladimir V. Peller. *Hankel operators and their applications*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2003.

[Pis90] Gilles Pisier. Completely bounded maps between sets of Banach space operators. *Indiana Univ. Math. J.*, 39(1):249–277, 1990.

[Pis01] Gilles Pisier. *Similarity problems and completely bounded maps*, volume 1618 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, expanded edition, 2001. Includes the solution to "The Halmos problem".

[Pis03] Gilles Pisier. *Introduction to operator space theory*, volume 294 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2003.

[RS00] Hans Reiter and Jan D. Stegeman. *Classical harmonic analysis and locally compact groups*, volume 22 of *London Mathematical Society Monographs. New Series*. The Clarendon Press, Oxford University Press, New York, second edition, 2000.

[Rua88] Zhong-Jin Ruan. Subspaces of C^*-algebras. *J. Funct. Anal.*, 76(1):217–230, 1988.

[Sag95] Michah Sageev. Ends of group pairs and non-positively curved cube complexes. *Proc. London Math. Soc. (3)*, 71(3):585–617, 1995.

[Ser80] Jean-Pierre Serre. *Trees*. Springer-Verlag, Berlin-New York, 1980. Translated from the French by John Stillwell.

[Sł80] Marek Słociński. On the Wold-type decomposition of a pair of commuting isometries. *Ann. Polon. Math.*, 37(3):255–262, 1980.

[Tak64] Masamichi Takesaki. On the cross-norm of the direct product of C^*-algebras. *Tôhoku Math. J. (2)*, 16:111–122, 1964.

[Ver17] Ignacio Vergara. The p-approximation property for simple Lie groups with finite center. *J. Funct. Anal.*, 273(11):3463–3503, 2017.

[Ver18] Ignacio Vergara. Radial Schur multipliers on some generalisations of trees. *Studia Math.*, (to appear), 2018.

[vN30] John von Neumann. Zur Algebra der Funktionaloperationen und Theorie der normalen Operatoren. *Math. Ann.*, 102(1):370–427, 1930.

[Wys95] Janusz Wysoczański. A characterization of radial Herz-Schur multipliers on free products of discrete groups. *J. Funct. Anal.*, 129(2):268–292, 1995.