The problem of tropical homomorphism in vertex-colored graph

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An homomorphism from graph \( G \) to graph \( H \) is a function \( f \) from vertices of \( G \) to vertices of \( H \) such that for all edge \( uv \) in \( G \), \( h(u)h(v) \) is an edge in \( H \). Consequently, the problem of existence of a proper colouring is easily generalised to the problem of existence of an homomorphism to a fixed graph \( H \) :

\[
\text{\( H \)-COLORING}
\]

\textbf{Input:} A graph \( G \).

\textbf{Question:} Does \( G \) admit an homomorphism to \( H \) ?

A theorem of dichotomy for this set of problem is known : If \( H \) is bipartite, then \( H \)-COLORING is polynomial. If \( H \) is not bipartite, then \( H \)-COLORING is NP-Complete.

In our work, we study a generalisation of \( H \)-COLORING. A tropical homomorphism from vertex-coloured graph \( (G, c) \) to vertex-coloured graph \( (H, c') \) is an homomorphism \( h \) from \( G \) to \( H \) such that for each vertex \( v \) in \( G \), \( h(v) \) has the same colour than \( v \). Thus, \( H \)-COLORING is generalised as :

\[
\text{\( (H, c) \)-COLORING}
\]

\textbf{Input:} A vertex-coloured graph \( (G, c') \).

\textbf{Question:} Does \( (G, c') \) admit a tropical homomorphism to \( (H, c) \) ?

This talk aims to present our results in the search of a dichotomy for the \( (H, c) \)-COLORING problems. The main result is that the existence of such a dichotomy is equivalent to the existence of a dichotomy for the Constraint Satisfaction Problems (CSP), a largely studied and still unresolved conjecture. We have also studied the complexity of this problem for some families of graph.

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