RESEARCH STATEMENT

1. Summary

My research is concerned with stochastic processes and Riemannian geometry, more specifically with the winding around points of random curves in 2-dimensional Riemannian manifolds.

Consider a smooth curve γ and a smooth 1-form α on the plane $\mathbb{R}^2$. The integral of α along γ is related to the winding function θ_γ of γ by the following Stokes, or Green formula :

$$\int_\gamma \alpha = \int_{\mathbb{R}^2} \theta_\gamma \, \mathrm{d}\alpha. \tag{1}$$

If γ is not a smooth curve anymore, but a random curve, that is, the trajectory of a 2-dimensional stochastic process, then in many situations of interest, the left-hand side of (1) is well defined as a Stratonovich integral. The function θ_γ on the other hand, usually fails shortly from being integrable.

The question that we address is to determine when it is possible to define a 'principal value' for the right-hand side of (1), by cutting-off the largest values of $|\theta_\gamma|$. We prove in [2] that when γ is a Brownian motion and the 1-form α is $x\,\mathrm{d}y$, it is indeed possible to define such a principal value. In this case, it relates the Lévy area $\int_0^1 X_t \,\mathrm{d}Y_t$ with the 'average' windings of the curve (X, Y) around the points of a Poisson process on the plane.

This may sound like a very special case, but it is robust with respect to a change of coordinates, and we expect, in a forthcoming work, to show that it extends naturally to a wide class of Riemannian surfaces and 1-forms.

2. The Young case

Let $\gamma = (x, y) : [0, 1] \to \mathbb{R}^2$ be a continuous curve with finite p-variation for some $p < 2$. The theory of Young integration allows one to define the integral

$$\int_0^1 x_t \,\mathrm{d}y_t \tag{2}$$

Let us consider a smooth 1-form $\alpha = \alpha_1 \,\mathrm{d}x + \alpha_2 \,\mathrm{d}y$. Then the functions $t \mapsto \alpha_i(x_t, y_t)$ also have finite p-variation and the integral

$$\int_\gamma \alpha = \int_0^1 \alpha_1(x_t, y_t) \,\mathrm{d}x_t + \int_0^1 \alpha_2(x_t, y_t) \,\mathrm{d}y_t \tag{3}$$

is well defined.

In the case where x and y have different regularities – for example, if x has finite p-variation for some $p \in [2, 3)$ and y has finite q-variation for some $q \in [1, 2]$, with $\frac{1}{p} + \frac{1}{q} > 1$, then the definition of $\int_\gamma \alpha$ is less simple. One way to proceed is to remark that the quantities $\left(\int_0^t x_s \,\mathrm{d}y_s\right)_{t\in[0,1]}$ determine a unique step-2 geometric rough path extension of γ. One can then wonder whether the quantity $\int_\gamma \alpha$ depends on the choice of the coordinate functions. The answer to that question is unfortunately that *it does*, since even the regularity condition depends on the coordinate system. A simple rotation leads to a new system of coordinates in which, in general, both components have the worst regularity, namely finite p-variation, and the procedure that we just described fails.

We propose a way to circumvent this difficulty by a new definition of the integral through Green's formula (1). In [2], we show that, as soon as Young integration is possible, the right-hand side of (1) is also well defined, and both sides agree (up to an explicit boundary term when the curve is not closed).

3. The Brownian case

One can wonder what happens when the curve γ is replaced with a Brownian motion B. Werner [3] showed in that case that, although the right-hand side of (1) is ill defined, because the function θ_γ is not integrable, it is possible to take approximations of it and prove convergence in probability towards the left-hand side, interpreted as a Stratonovich integral. By using a different approximation, we were able to achieve two goals in [2].

1. We obtain a convergence in the almost sure sense.
2. When γ is a smooth curve, and from a probabilistic point of view, the right-hand side of (1) can be understood as the quantity

$$\frac{1}{K}\mathbf{E}\Big[\sum_{z\in\mathcal{P}}\theta_\gamma(z)\Big]$$

in which $\mathcal{P}$ is a Poisson process with intensity $K\,\mathrm{d}\alpha$. In particular, it is the limit in distribution, as K goes to infinity, of the quantity

$$\frac{1}{K}\sum_{z\in\mathcal{P}}\theta_\gamma(z).$$

When the curve γ is replaced with a planar Brownian motion, we are able to show that the convergence still holds, but the limiting distribution is non-degenerate. Instead, it is a Cauchy distribution, the position parameter of which is equal to the left-hand side of (1) (up to an explicit boundary term when the curve is not closed).

We are also able to show that similar quantities can be used to defined new integrals when α is very irregular. Such irregular 1-forms do arise in theoretical physics.

4. Further questions

To a class of integrals (usual, Itô, rough) is usually associated a class of differential equations (ODEs, SDEs, rough differential equations). I would like to investigate which class of differential equations can be defined and studied using the approach to Young integrals described in Section 2 above.

I also would like to extend the results described in Section 3 to other processes than the Brownian motion. In particular, I expect to be able to define, in a similar way, the holonomy of a Yang–Mills field along a planar α-stable process for any $\alpha < 2$, something that is not possible using just the p-variation regularity of the curve. This is a work in progress.

Finally, I would also like to extend my results to higher dimensions: this includes higher ambient dimensions, but also integration of forms of higher degree along sub-manifolds of higher dimension.

References

[1] Thierry Lévy. Yang-Mills measure on compact surfaces. *Mem. Amer. Math. Soc.*, 166(790):xiv+122, 2003.

[2] Isao Sauzedde. Lévy area without approximation. 2021. arXiv:2101.03992.

[3] W. Werner. Formule de Green, lacet brownien plan et aire de Lévy. *Stochastic Processes Appl.*, 57(2):225–245, 1995.