Brownian winding and Gaussian multiplicative chaos

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Winding of a planar curve

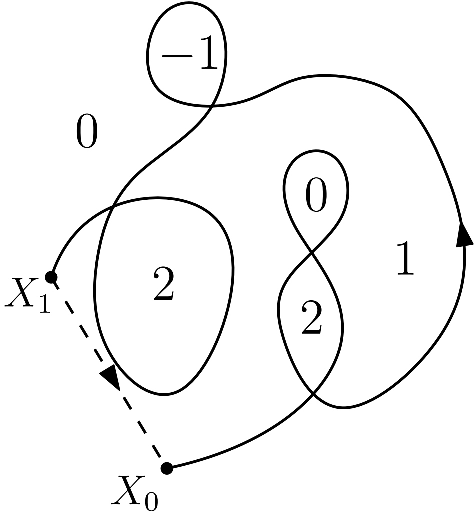

Figure: A curve $X : [0,1] \to \mathbb{R}^2$ on the plane allows to define a **winding function** $\theta^X : \mathbb{R}^2 \to \mathbb{Z}$. For any point $z \in \mathbb{R}^2$, $\theta^X(z)$ describes the number of time X winds around z.

Winding of a planar Brownian motion

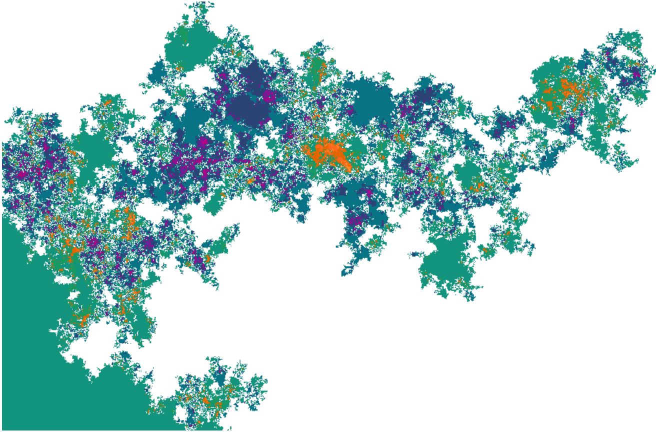

Figure: Coloration of the plane depending on the value of θ^X, when X is a planar Brownian motion.

Winding of a planar Brownian motion

I studied the set of points with winding at least N,

$$\mathcal{D}_N := \{z \in \mathbb{R}^2 : \theta(z) \geq N\}, \tag{1}$$

and its Lebesgue measure

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Theorem

The sequence D_N is equivalent to $\frac{1}{2\pi N}$ in L^2.
That is,

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I obtained the following improvement [2].

Theorem

The sequence D_N is equivalent to $\frac{1}{2\pi N}$, in L^p for all p and in the almost sure sense. The convergence rate is at least $N^{-\frac{1}{2}+o(1)}$.

To be more specific:

- For all $p \in [1+\infty)$, for all $\epsilon > 0$, there exists C such that for all $N \geq 1$,
$$\mathbb{E}[\left|ND_N - \tfrac{1}{2\pi}\right|^p]^{\frac{1}{p}} \leq CN^{-\frac{1}{2}+\epsilon}.$$
- Almost surely, for all $\epsilon > 0$, there exists C such that for all $N \geq 1$,
$$\left|ND_N - \tfrac{1}{2\pi}\right| \leq CN^{-\frac{1}{2}+\epsilon}.$$

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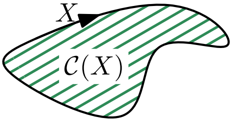
$$\int X^1 \,\mathrm{d}X^2 = \pm \mathrm{Leb}(\mathcal{C}(X)).$$



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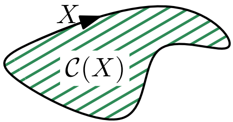

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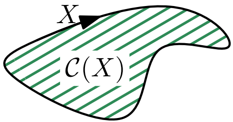

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- Monte–Carlo method: for an homogeneous Poisson process $\mathcal{P}$ with large intensity,

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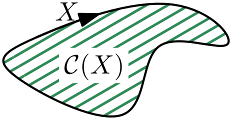

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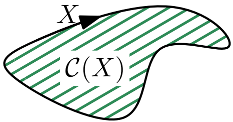

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By averaging the winding of random points, we can compute integrals along smooth curves.

What about the Brownian case?

The estimation on the convergence rate of D_N, in the almost sure sense, allows to prove the following.

Theorem ([2])

For a planar Brownian motion X, the function θ^X is not integrable, but

$$\int X^1 \,\mathrm{d}X^2 = \lim_{K \to +\infty} \int_{\mathbb{R}^2} \max(-K, \min(\theta^X(z), K)) \,\mathrm{d}z.$$

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(the left-hand side is a stochastic integral, either in the sense of Itö or Stratonovich)
The average winding of random points converges in distribution, toward a Cauchy variable centered at this value.

$$\frac{1}{\#\mathcal{P}} \sum_{z \in \mathcal{P}} \theta^X(z) \xrightarrow{(d)} \mathrm{Cauchy}\Big(\int \mathrm{X}^1 \,\mathrm{d}\mathrm{X}^2, \tfrac{1}{2\pi}\Big).$$

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Theorem ([3])

The measure $2\pi N \mathbb{1}_{\mathcal{D}_N}\,\mathrm{d}z$ *converges almost surely, weakly, toward the occupation measure of the Brownian motion.*

Gaussian multiplicative chaos

To compute other integrals, the Lebesgue measure must be replaced with other area elements. For a smooth 1-form η,

$$\int_X \eta = \int_{\mathbb{R}^2} \theta^X \, \mathrm{d}\eta.$$

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What happens when the area measure becomes very irregular?

Theorem ([4])

When the measure M *is a Gaussian multiplicative chaos on the plane with small enough intermittency parameter* γ*, and* X *is a planar Brownian motion, one can still define a 'principal value' for the integral* $\int_{\mathbb{R}^2} \theta^X \, \mathrm{dM}$.

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This defines a notion of Lévy area enclosed by the Brownian motion, but with GMC as underlying area measure.

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Roughly speaking, we are studying a *‘Brownian particle interacting with an extremely weak quantum Yang–Mills field’.*

That's it!

Wendelin Werner.
Sur les points autour desquels le mouvement brownien plan tourne beaucoup.
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