RENORMALISED AMPEREAN AREA OF BROWNIAN LOOPS AND SYMANZIK REPRESENTATION OF THE 2D HIGGS-YANG–MILLS FIELDS

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Abstract. The Amperean area of a Brownian motion is to its Lévy area what its self-intersection measure is to its occupation measure. It plays an important role in the study of the displacement of a particle surrounded by a random magnetic field.

We prove that the Amperean area of a planar Brownian motion admits a renormalisation. We also explain the central role played by this Amperean area in the Symanzik loop representation of the continuous Abelian Higgs–Yang–Mills field in 2 dimensions. As much as the self-intersection measure is related to the self-interaction φ^4 term, the Amperean area is related to the quartic interaction term between the Higgs and the Yang–Mills fields.

As the renormalisation method uses a mollification and a counterterm that we can asymptotically estimate up to a $o(1)$, we can rigorously link the mollified Higgs–Yang–Mills fields with the mollified Amperean area.

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1. Introduction

1.1. **Amperean Area and Higgs–Yang–Mills Field.** Let W, W' two Brownian motions in the Euclidean plane, each concatenated with a straight line segment between its endpoints. Consider their winding functions

$$\mathbf{n}_W : \mathbb{R}^2 \setminus \mathrm{Range}(W) \to \mathbb{Z}, \qquad \mathbf{n}_{W'} : \mathbb{R}^2 \setminus \mathrm{Range}(W) \to \mathbb{Z},$$

which to a point z maps the number of time the corresponding loop winds around z.

These functions are unbounded in the vicinity of the corresponding path, and in fact are not even locally integrable on $\mathbb{R}^2$. The integrals[1]

$$I_W(f) := \int_{\mathbb{R}^2} f \mathbf{n}_W \, \mathrm{d}\lambda \qquad \text{and} \qquad \bar{Y}_{W,W'}(f) := \int_{\mathbb{R}^2} f \mathbf{n}_W \mathbf{n}_{W'} \, \mathrm{d}\lambda$$

with respect to the Lebesgue measure $\mathrm{d}\lambda$ are thus ill-defined, yet plays important roles in both physics and mathematics.

The first one can in fact be defined by several different regularisation methods which give the same limit (see [8, 11]). This limiting value defining $I_W(f)$ is equal to the Stratonovich stochastic integral along W of any vector field A such that $\mathrm{curl}\, A = f$. Such an equality can be

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[1]The bar over Y is to emphasize the straight line segments concatenated to the Brownian motions

understood as a formal application of Stokes' theorem. In particular for $f = 1$, corresponding to $A = \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{2}$, $I_W(f)$ is merely the *Lévy area* of W.

The second integral $\bar{Y}_{W,W'}(f)$, in the case $W = W'$ and $f = 1$, has been coined the *Amperean area* of the loop. Wendelin Werner showed in [10] that upon deleting from the integration the points at distance less than ϵ from W, one gets a finite random variable, which diverges logarithmically fast as ϵ goes to 0.[2]

In physics, such an integral arises in the study of particle displacement in random magnetic flux (see e.g. [4]), as well as in the Symanzik's polymer representation of the Abelian Higgs-Yang–Mills (HYM) field in two dimensions (also known as Ginzburg–Landau field), which informally is a random couple (Φ, A), where Φ is a complex valued function (truly, a distribution) , and A is a vector field (truly, distributional) with probability distribution given by

$$\mathrm{d}\mathbb{P}(\Phi, A) := \frac{1}{Z}\exp\Big(-\frac{\|\operatorname{grad}\Phi + i\alpha A\Phi\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^2)} + \|\operatorname{curl} A\|^2_{L^2(\mathbb{R}^2,\mathbb{R})}}{2} + c.t.\Big)\,\mathcal{D}\Phi\mathcal{D}A,$$

where *c.t.* are formally infinite counterterms which prevent Φ from collapsing to 0.

This relation goes as follows. For a given A, let Z_A be the formally defined normalising constant such that

$$\frac{1}{Z_A}\exp\Big(-\frac{\|\operatorname{grad}\Phi + i\alpha A\Phi\|^2}{2}\Big)\mathcal{D}\Phi$$

is a probability measure, and consider γ a straight line segment from $y \in \mathbb{R}^2$ to $x \in \mathbb{R}^2$, and look at the gauge-invariant 2-points observable

$$S_{x,y} := \big\langle \Phi(x), \exp\big(i\alpha\int_\gamma A\big)\Phi(y)\big\rangle.$$

Then, it formally holds that

$$\mathbb{E}\Big[\Big(\frac{Z_A}{ZZ'}\Big)^{-1} S_{x,y}\Big] = \int_0^\infty p_t(x,y)\mathbb{E}_{t,x,y}\big[\exp\big(-\frac{\alpha^2}{2}\bar{Y}_{W,W}(1)\big)\big]\,\mathrm{d}t, \tag{1}$$

where W is a Brownian bridge from x to y with duration t under $\mathbb{P}_{t,x,y}$. Details of the computation (which we cannot emphasize enough, is purely formal and in no way rigorous) are given in Appendix A. Similar but more complex expressions hold formally for the moments

$$\mathbb{E}\Big[\Big(\prod_{i=1}^{j}\int_{\ell_i} A\Big)\Big(\prod_{k=1}^{l} S_{x_k,y_k}\Big)\Big],$$

where the ℓ_j are smooth enough loops. These general moments are believed to characterise the probability distribution of $\mathbb{P}$ (up to gauge equivalence).

Symanzik's polymer representation turns terms in the action defining a field into terms associated to Brownian trajectories, as follows,

Mass or potential term $\langle\phi, m\phi\rangle$
$\leftrightarrow$ occupation measure μ_W, $\int m(z)\mu_W(\,\mathrm{d}z) = \int m(W_s)\,\mathrm{d}s$
Interaction with external magnetic potential $(\cdots + i\alpha B\phi)$
$\leftrightarrow$ winding function $\mathbf{n}_W$, $\int \mathbf{n}_W(z)\operatorname{curl}(B)\,\mathrm{d}z = \int B(W_s)\circ\mathrm{d}W_s$
renormalised self-interaction potential $\langle\phi, \lambda\phi\rangle^2$
$\leftrightarrow$ renormalised self-intersection measure ν_W, $\int \lambda(z)\nu_W(\,\mathrm{d}z)$
Interaction with renormalised internal magnetic potential $(\cdots + i\alpha A\phi) + \|\operatorname{curl} A\|^2$
$\leftrightarrow$ squared winding function $\mathbf{n}_W^2$, $\int \mathbf{n}_W^2(z)\operatorname{curl}(B)\,\mathrm{d}z = Y_{W,W}(\operatorname{curl} B)$.

It is the fourth link (which I am not aware is documented) which makes the study of the Amperean area relevant.

[2] In [10], a Brownian loop is considered rather than a "free" Brownian motion. Our results here only consider the case of the free Brownian motion, although part of them can easily be transferred to bridges. The two situations are qualitatively the same, and the difficulties to transfer results between them purely technical.

The main goal of this paper is to rigorously define the random variables $\bar{Y}_{W,W'}(f)$ through a regularisation procedure when W and W' are independent, and through a renormalisation procedure when $W = W'$.

These procedures use mollification, so that at fixed $\epsilon > 0$ the approximations $\bar{Y}^{\epsilon}_{W,W'}(f)$ can rigorously be related to the *mollified Higgs–Yang–Mills field*, in which first A is build, then mollified into A^ϵ, and then for a given and smooth A^ϵ, Φ is built as a Gaussian field interacting with the now external magnetic potential A^ϵ.

1.2. **Regularisation procedure and main results.** From now on, let W and W' be independent. As the index functions $\mathbf{n}_W, \mathbf{n}_{W'}$ are not even integrable, it might look like we cannot event mollify them in the first place. We consider a symmetric, positive, compactly supported, smooth enough, mollifier φ with integral 1, set φ^ϵ the rescaled function $x \mapsto \epsilon^{-2}\varphi(\epsilon^{-1}x)$ Then we *define* the mollified winding $\mathbf{n}^\epsilon(z)$ as the stochastic integral of $\tau_z A^\epsilon$ along W, where $\operatorname{curl} A^\epsilon = \varphi^\epsilon$, and τ_z is a shift operator. This is to be thought as a rigorous way to define the ill-defined convolution $\mathbf{n} \star \varphi^\epsilon$, and does not depend on the specific choice of A^ϵ, provided we consider Stratonovich integrals. We will always choose A^ϵ such that $\operatorname{div} A^\epsilon = 0$, so that Ito and Stratonovich integrals agree with each other. We then show that the random variables

$$\bar{Y}^{\epsilon}_{W,W'}(f) := \int_{\mathbb{R}^2} f \mathbf{n}^\epsilon_W \mathbf{n}^\epsilon_{W'} \,\mathrm{d}\lambda \qquad \text{and} \qquad :\bar{X}^\epsilon_W(1): := \bar{X}^\epsilon_W(1) - \mathbb{E}[\bar{X}^\epsilon_W(1)],$$

both converge in L^2 in the limit $\epsilon \to 0$, where $\bar{X}^\epsilon_W = \bar{Y}^\epsilon_{W,W}$. We will refer to the limits, noted $\bar{Y}_{W,W'}(f)$ and $:\bar{X}_W(1):$, as the *Amperean area between W and W'* and the *renormalised Amperean area of W*.

Remark 1. We do not construct the renormalised variables $:\bar{X}^\epsilon_W(f):$ for more general functions f, but it should be noted that the correct way to normalise these random variables is by subtracting the **random** counterterm

$$\frac{\log(\epsilon^{-1})}{2\pi} \int f(W_t)\,\mathrm{d}t = \frac{\log(\epsilon^{-1})}{2\pi} \int_{\mathbb{R}^2} f(z)\mu_W(\,\mathrm{d}z),$$

which is the main order estimation for the non-renormalised variables $\bar{Y}^\epsilon_{W,W}(f)$ as we show in the companion paper []. The analogy with the construction of the self-intersection measure, which requires an identical normalisation, will float around during the paper.

We will also show that there exists a constant C_φ, which depend on the mollifier φ, such that

$$\mathbb{E}[\bar{X}^\epsilon_W(1)] = \frac{T}{2\pi}\log(\epsilon^{-1}) + \frac{T\log T}{4\pi} + C_\varphi T + O(\epsilon), \tag{2}$$

where T is the duration of the loop. The first term in this expansion is not surprising, as it matches the main result in [10]. The regularisation methods (i.e. the choice of the functions $\mathbf{n}^\epsilon$ used to approximate $\mathbf{n}$) used here and there are different, but one can convince ourself both methods should have the same asymptotic expansion up to order 1 (excluded).

We are interested in this asymptotic expansion mostly because it tells us which negative, asymptotically divergent, mass to use as a counterterm in the definition of Higgs–Yang–Mills field, and which effective mass results. In this regard, the expansion is relevant up to order $o(1)$.

Remark 2. Combining this with Jay Rosen's estimation for the renormalised self-intersection local time γ_W of W in [6], it tells us that the counterterms used to define $:\bar{X}_W(1): -\frac{1}{4}\gamma_W$ cancel each other not only at the main order, but up to order 1.[3] This means that for Higgs-Yang–Mills fields with a self-interaction φ^4 term, there exists a specific tuning of the self-interaction constant λ (or more precisely, of $\lambda\alpha^{-2}$, where as above α is the electric charge) for which some of the counterterms cancel each other.

[3]The factor $\frac{1}{4}$ is to be replaced by another factor for different ways to normalise γ_W.

Stretch-exponential moments of the limit $\bar{Y}_{W,W'}(f)$ are also investigated: we prove that for all $\eta < \eta_0 = 1$ and sufficiently nice function f,

$$\mathbb{E}[\exp |\bar{Y}_{W,W'}(f)|^{\eta}] < \infty. \tag{3}$$

Although we do not make the full computation, we believe the same holds for $:\bar{X}_W(1):$ and that this value of η_0 is optimal for the following reason: neglecting the divergences that come from the fact that the domain of integration is unbounded, the only possibility for $:\bar{X}_W(1):$ to be exceptionally large (resp. small) is that there exist a large value k such that the set of points with windings larger than k (in absolute value) is exceptionally large (resp. small) compared to its typical size. Yet, for large values of k, the fluctuations of this size should comes in part from fluctuations of the renormalized self-intersection local time.[4] It is thus very likely that the variable $:\bar{X}_W(1):$ do not admit better moments than the ones of the renormalized self-intersection local time, which are known to be exactly exponential moments [5].

Remark 3. The question of which exponential moments do exist, which is crucial for further study of Symanzik's loop representation of the Higgs–Yang–Mills fields, seems to be much subtler. The relevant observables are then given by

$$\mathbb{E}\Big[\exp\Big(\sum_{i,j}(c\bar{Y}_{W_i,W_j}(1) + c'\gamma_{W_i,W_j}\Big)\Big],$$

where γ_{W_i,W_j} is the intersection local time between the Brownian paths W_i and W_j, and the diagonal terms are renormalised. A credible possibility is that this expectation is finite if and only if some affine combination of c and c' is non-negative, in which case it is finite *for all possible total duration of the Brownian paths* if and only if some linear combination of c and c' is non-negative. Study of such moments might require a much better understanding of the relations between the self-intersection local time and the Amperean area, in particular the relation between their large deviations.

Remark 4. Such exponential moments being infinite would indicate that additional counterterms are necessary to define the Higgs-Yang–Mills field. This would not be very surprising, considering its construction through lattice approximation [1] does rely on the introduction of such additional counterterms.

In summary, the main results we obtain are the following.

Theorem 1. *Let $f \in L^p(\mathbb{R}^2)$ for some $p \in (1,\infty]$. The mollified random variables $\bar{Y}^{\epsilon}_{W,W'}(f)$ and $:\bar{X}^{\epsilon}_W(1):$ both converge in L^2, as $\epsilon \to 0$. The limits do not depend on the choice of the mollifier.*

For $f = 1$, the asymptotic estimation on average (2) holds.

For $f \in L^{\infty}_c(\mathbb{R}^2)$, $\bar{Y}^{\epsilon}_{W,W'}(f)$ has finite stretch-exponential moment up to order 1, i.e. Equation (3) holds for all $\eta < 1$.

1.3. **Different possible construction of the Amperean areas.** There are other natural candidates to define the Amperean areas which we now present (only in the case $f = 1$ for simplicity, although more complex formulas for general f are also likely to hold). The construction we make here has two specific advantages. The first is that the approximations can directly be linked to the mollified Higgs–Yang–Mills field, when some other constructions only make formal links between the limiting objects. The other is that it allows to compute the divergence rate (2) of the counterterm, when other constructions would not give any value to this counterterm, or make it impossible to compute practically.

The following definition *directly at the limit* has the strong advantage that it should allow for accurate numerical estimation, while being fairly easy to manipulate with straightforward probabilistic tools.

[4]In [7], the author proved a result corresponding to such a statement, but for a pair of Brownian paths. In such a case, corresponding to the variable $\bar{Y}_{W,W'}(1)$, the contributions coming from the fluctuations of the intersection local time actually cancel each other, which is not the case for the contributions coming from the fluctuations of the self-intersection local time.

Conjecture 1. Let W, W' be independent, with respective durations S and T. Up to an additive boundary term that depends only on the endpoints of W and W', the random variables $\bar{Y}_{W,W'}(1)$ and $:\bar{X}_W(1):$ are given by

$$\bar{Y}_{W,W'}(1) = \int_0^S \int_0^T \log(|W_s - W'_t|^{-1})\, \mathrm{d}W'_t\, \mathrm{d}W_s, \ :\bar{X}_W(1):= 2\int_0^T \int_0^t \log(|W_s - W_t|^{-1})\, \mathrm{d}W_s\, \mathrm{d}W_t.$$

I believe such a formula was first proposed in [4], however the analysis made there is not mathematically rigorous, and involves infinite integrals in the computation, even when W and W' are smooth loops. In this smoother case, one can turn the arguments in [4] into a rigorous proof, but this is already non-trivial. Were such a formula to hold, it might lead to some simplifications in our proofs. However, the most technical steps rely on integral estimations which would probably appear identically even with such a formula in hand.

Conjecture 2. For relative integers k, j, let A_k and $A_{k,j}$ be the measure of the sets

$$\mathcal{A}_k(W) \coloneqq \{z : \mathbf{n}_W(z) = k\}, \qquad \mathcal{A}_{k,j}(W, W') \coloneqq \mathcal{A}_k(W) \cap \mathcal{A}_j(W').$$

The random variables $\bar{Y}_{W,W'}(1)$ and $:\bar{X}_W(1):$ are given by

$$\bar{Y}_{W,W'}(1) = \lim_{k_0, j_0 \to \infty} \sum_{k=-k_0}^{k_0} \sum_{j=-j_0}^{j_0} kjA_{k,j}, \ :\bar{X}_W(1):= \lim_{k_0 \to \infty} \sum_{k=-k_0}^{k_0} k^2(A_k - \mathbb{E}[A_k]).$$

The sums $\sum_{k\in\mathbb{Z}} k^2|A_k - \mathbb{E}[A_k]|$ and $\sum_{(k,j)\in\mathbb{Z}^2} |kj|A_{k,j}$ are both infinite.

One advantage of such a normalisation is that it is geometry independent: it would allow to compute the Amperean areas with the data of the curve and the Lebesgue measure on the plane, without the detail of the underlying Riemmanian (indeed Euclidean) metrics.

2. Notations and preliminary remarks

2.1. **General notations.** The Euclidean norm in $\mathbb{R}^2$ is written $|\cdot|$. The ball centred at 0 with radius R is written B_R, and the ball centred at z with radius R is written $B_R(z)$. For a point $z \in \mathbb{R}^2$, we write $(z^1, z^2) = z$ its coordinates. For a function $f : \mathbb{R}^2 \to \mathbb{R}$ (resp. a vector field $V : \mathbb{R}^2 \to \mathbb{R}^2$), we write either f_z or $f(z)$ (resp. V_z) for the value of f (resp. V) at z. We write (V^1, V^2) the coordinates of V, so for example, $V^1_z \coloneqq (V_z)^1 = (V^1)_z$. For three points x, y, z in the plane, $\widehat{xyz}$ is the angle at y in the interval $(-\pi, \pi]$. When x, y, z are given by longer expressions, we will prefer instead the notation $\langle x, y, z\rangle \coloneqq \widehat{xyz}$.

We write $\int_W V$ for the Stratonovich integral of the vector field V along the Brownian path $W : [0, t] \to \mathbb{R}^2$,

$$\int_W V \coloneqq \sum_{i\in\{1,2\}} \int_0^t V^i_{W_s} \circ \mathrm{d}W^i_s.$$

Where necessary, we set $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space on which W is a Brownian motion from $\mathbb{R}_+$ to $\mathbb{R}^2$ started from 0, with $(\mathcal{F}_t)_{t\geq 0}$ the canonical filtration. Furthermore we assume this probability space is also endowed with measures $\mathbb{P}_x$, $\mathbb{P}_{t,x}$ and $\mathbb{P}_{t,x,y}$, for all $t > 0$ and $x, y \in \mathbb{R}^2$, such that W is a Brownian motion with infinite duration started from x under $\mathbb{P}_x$ (in particular $\mathbb{P}_0 = \mathbb{P}$), a Brownian motion with duration t started from x under $\mathbb{P}_{t,x}$, and a Brownian bridge with duration t from x to y under $\mathbb{P}_{t,x,y}$, and such that for all $t > 0$ we have the disintegration formula

$$\mathbb{P}_{|\mathcal{F}_t} = \int p_t(0, y)\mathbb{P}_{t,0,y}\, \mathrm{d}y,$$

where p_t is the whole-plane heat kernel. We emphasize that p_t always designate the 2 dimensional heat kernel. Sometimes, for $r \geq 0$, we use the shortcut notation $p_t(0, r)$ for the common value $p_t(0, x)$ of any x with $|x| = r$.

2.2. **Winding notations.** A smooth *mollifying* function $\varphi : \mathbb{R}^2 \to \mathbb{R}_+$, rotationally invariant and which integrates to 1, is fixed for the whole paper. We further assume that φ is compactly supported and $\mathcal{C}^\infty$. Through the paper we let K_φ the smallest constant K such that $\mathrm{Supp}(\varphi) \subseteq B_{K_\varphi}$. We will often write C_φ for a constant that depend only on the mollifier φ, but we can change from proof to proof (although it is fixed in any given computation). Let also $\rho_\varphi := \sup_z |z|^3\varphi(z) < +\infty$. Given $\epsilon > 0$, we set $\varphi^\epsilon : z \mapsto \epsilon^{-2}\varphi(\epsilon^{-1})$, which also has integral 1.

We define a vector field $\theta : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2$, by

$$\theta_x = \frac{1}{2\pi|x|^2}(-x^2, x^1).$$

For $z \in \mathbb{R}^2$, we also define $\theta^z : \mathbb{R}^2 \setminus \{z\} \to \mathbb{R}^2$ the vector field θ shifted to be centred around z, i.e. $\theta^z_x := \theta_{x-z}$.

This vector field θ^z has the following properties. When integrated along a loop γ which is smooth enough ($\gamma \in \mathcal{C}^{\frac{1}{2}+\epsilon}$) and which avoids the point z (i.e. $z \notin \mathrm{Range}\,\gamma$), the integral is equal to a relative integer, which is the winding index $\mathbf{n}_\gamma(z)$ of γ around z. The distributional divergence of θ^z is equal to 0, whilst its distributional curl is equal to the Dirac measure δ_z.

Since $\theta^z \in L^1 + L^\infty$, i.e. it decomposes as the sum of a vector field in L^1 and one in L^∞, e.g. $\theta^z = \theta^z \mathbb{1}_{B_1(z)} + \theta^z \mathbb{1}_{\mathbb{R}^2 \setminus B_1(z)} =: \theta^z_0 + \theta^z_\infty$, the convolution $\theta^z \star f$ is a well-defined vector field in L^∞ as soon as the function f is in $L^1 \cap L^\infty$. We can thus defined the mollified vector field

$$\psi^{z,\epsilon} := \theta^z \star \varphi^\epsilon. \tag{4}$$

We omit the superscript z when $z = 0$ and the superscript ϵ when $\epsilon = 1$.

For a Brownian motion $W : [0,t] \to \mathbb{R}^2$, we define $\mathbf{n}^\epsilon_W(z)$ the *ϵ-mollified winding index of x at z* as

$$\mathbf{n}^\epsilon_W(z) := \int_W \psi^{z,\epsilon}. \tag{5}$$

We define $\bar{W}$ the loop obtained by concatenation of $W : [0,t] \to \mathbb{R}^2$ with the straight line segment $[W_t, W_0]$ between its endpoints. Since we are only interested in integrals of vector fields along that curve, and since these integrals are invariant by orientation-preserving reparametrisation, the specifics of time parametrisation of this line segment is irrelevant. We then define

$$\mathbf{n}^\epsilon_{\bar{W}}(z) := \int_{\bar{W}} \psi^{z,\epsilon} = \int_W \psi^{z,\epsilon} + \int_{[W_t,W_0]} \psi^{z,\epsilon} = \mathbf{n}^\epsilon_W(z) - (2\pi)^{-1} \int_{\mathbb{R}^2} \varphi(v)\langle W_0, z + \epsilon v, W_t\rangle \,\mathrm{d}v.$$

In some cases, we will prefer to $\mathbf{n}^\epsilon_{\bar{W}}$ the function $\hat{\mathbf{n}}^\epsilon_W$ defined by

$$\hat{\mathbf{n}}^\epsilon_W(z) := \mathbf{n}^\epsilon_W(z) - (2\pi)^{-1}\langle W_0, z, W_t\rangle.$$

For $t > 0$, we define functions K, K^*, K_t, K^*_t from $(\mathbb{R}^2)^2$ to $\mathbb{R} \cup \{+\infty\}$, all equal to $+\infty$ on the diagonal $x = y$, and given for $x \neq y$ by the formulas

$$K(x,y) := \int_0^1 K_t(x,y)\,\mathrm{d}t, \qquad K_t(x,y) := \int_{\mathbb{R}^2} p_t(0,z)\langle \theta^x_z, \theta^y_z\rangle \,\mathrm{d}z,$$

and

$$K^*(x,y) := \int_0^1 K^*_t(x,y)\,\mathrm{d}t, \qquad K^*_t(x,y) := \int_{\mathbb{R}^2} \frac{p_t(0,z)}{|x-z||y-z|}\,\mathrm{d}z = 4\pi^2 \int_{\mathbb{R}^2} p_t(0,z)|\theta^x_z||\theta^y_z|\,\mathrm{d}z.$$

We further define

$$K^\dagger(v) := \sup\{K^*(x,y) : x - y = v\}, \qquad K^\dagger_t(v) := \sup\{K^*_t(x,y) : x - y = v\}.$$

By looking at the behaviour of these integrals near $+\infty$ (sub-Gaussian decay), near 0 (logarithmic divergence for K^*), near x and near y (integrable power-like divergences), $K^*(x,y)$ and $K^*_t(x,y)$ are seen to be finite as soon as $x \neq y$, so that $K(x,y)$ and $K_t(x,y)$ are well-defined. The two cases $x = 0$ and $y = 0$ are to be treated separately from the case when $x \neq 0$ and $y \neq 0$, since then two of the singularities are merged together, but the integral is still finite in these specific cases. An artefact of these merging singularities will remain in later computation:

for fixed $x \neq 0$, these functions diverge logarithmically in $|x-y|^{-1}$ as $y \to x$, but in a way that is not uniform in x near 0.

For $x > 0$, we denote by $\Gamma(0,x)$ the value of the incomplete Gamma function at $(0,x)$: for $z \in \mathbb{R}^2 \setminus \{0\}$,

$$\Gamma(0, \frac{|z|^2}{2}) = 2\pi \int_0^1 p_t(0,z)\,\mathrm{d}t.$$

2.3. **Preliminary properties and remarks.** Let us explain further why we call $\mathbf{n}^\epsilon_W$ a mollified winding index. For a loop γ and $z \in \mathbb{R}^2 \setminus \mathrm{Range}(\gamma)$, let $\mathbf{n}_\gamma(z) \in \mathbb{Z}$ be the winding index of γ of around z —which roughly speaking counts how many times γ winds around z. As long as $\gamma \in \mathcal{C}^\alpha$ for some $\alpha > \frac{1}{2}$, one can also define $\mathbf{n}^\epsilon_\gamma(z)$ by a formula analogous to (5), except the Stratonovich integrals are replaced with the corresponding Young integrals (or Riemann–Stieltjes integrals, if γ has bounded variation). Then, it does hold that

$$\mathbf{n}^\epsilon_\gamma = \mathbf{n}_\gamma \star \varphi^\epsilon. \tag{6}$$

Indeed, for any $z \in \mathbb{R}^2 \setminus \mathrm{Range}(\gamma)$,

$$\int_\gamma \theta^z = \mathbf{n}_\gamma(z)$$

by residue theorem. The formula (6) then follows from swapping the order of integration, which crucially one *cannot* do when W is say a Brownian loop, for then the winding index $\mathbf{n}_W$ is not an integrable function any longer, and the convolution $\mathbf{n}_W \star \varphi^\epsilon$ is ill-defined. As we mentioned already, there are nonetheless ways to define rigorously the 'integral' of $\mathbf{n}_W$, or even the integral of the product $f\mathbf{n}_W$, as long as f is smooth enough. Interpreting the convolution in the right-hand side of (6) in these ways, (6) becomes a true formula also for $\gamma = \bar{W}$ with W a Brownian motion.

From a mathematical point of view, we do not rely on such formula in this paper (although they would allow for slight simplification at some specific places). Yet, keeping in mind this point of view that $\mathbf{n}^\epsilon_{\bar{W}}$ is the 'convolution' of $\mathbf{n}_{\bar{W}}$ with φ^ϵ has fruitful consequences. For example, one quickly sees that $\mathbf{n}^\epsilon_{\bar{W}}$ is compactly supported on $B_{\|W\|_\infty + \epsilon K}$. This can then be checked easily: for z outside this ball, $\mathbf{n}^\epsilon_W(z)$ is the integral along the loop $\bar{W}$ of a function that is holomorphic on $B_{\|W\|_\infty}$. We conclude by relying on the fact that meromorphic integral and Stratonovich integral agree.

We will have to go back and forth between the three functions $\mathbf{n}^\epsilon_W$, $\mathbf{n}^\epsilon_{\bar{W}}$, and $\hat{\mathbf{n}}^\epsilon_W$, mostly for technical reasons and because each has a unique property that distinguishes it from the other two, but also some drawbacks: the first, as a stochastic integral along a martingale, has explicit second moment through Ito isometry. However, it decays very slowly as $|z| \to \infty$, which prevents it from having good integrability properties. As we have just seen, the second is compactly supported (although its support is random), which avoid some very cumbersome estimation. It is to be understood as the mollification of $\mathbf{n}_{\bar{W}}$. The last one often serves as an intermediate between the other two. It is not compactly supported but still has better integrability properties than $\mathbf{n}^\epsilon_W$, as it decays a bit faster near $+\infty$. It often has a simpler usage than $\mathbf{n}^\epsilon_{\bar{W}}$, as the corrective term $(2\pi)^{-1}\langle W_0, z, W_t\rangle$ do not depend on ϵ.

Setting x and y the endpoints of W and using complex coordinates $z = re^{i\theta}$, it holds that $\mathbf{n}^\epsilon_W(z)$ decays as $\widehat{xzy}$, which is of order r^{-1} when $r \to \infty$, and in particular $\mathbf{n}^\epsilon_W \notin L^2(\mathbb{R}^2)$ although it belongs in $L^2_{loc}(\mathbb{R}^2)$. This lack of global square-integrability will in fact be the cause of many troubles, which are superficial in nature but rather tricky to deal with.

Since $\langle W_0, z+\epsilon v, W_t\rangle \in (-\pi, \pi]$ and $\|\varphi^\epsilon\|_{L^1} = 1$, it follows that for any point z and independently from ϵ,

$$|\mathbf{n}^\epsilon_{\bar{W}}(z) - \mathbf{n}^\epsilon_W(z)| \leq \frac{1}{2} \quad \text{and} \quad |\hat{\mathbf{n}}^\epsilon_W(z) - \mathbf{n}^\epsilon_W(z)| \leq \frac{1}{2}. \tag{7}$$

Locally, these are good approximations since $\mathbf{n}^\epsilon_W$ is unbounded as a function of z and ϵ.

Certainly the function $\hat{\mathbf{n}}^\epsilon_W$ is slightly less natural than $\mathbf{n}^\epsilon_{\bar{W}}$ to consider, from a geometric point of view. For example, it cannot be written as one line integral along a loop, as opposed to $\mathbf{n}^\epsilon_{\bar{W}}$, and it also does not enjoy the same property as $\mathbf{n}^\epsilon_{\bar{W}}$ of being compactly supported. Yet, it is computationally slightly more pleasant to deal with. Furthermore, we will see by comparing these two functions that

$$\hat{\mathbf{n}}^\epsilon_W(z) \underset{z\to\infty}{=} O(|z|^{-2}). \tag{8}$$

In particular it does belong in $L^2(\mathbb{R}^2)$. Local square-integrability will follow from the fact that $\mathbf{n}^\epsilon_W$ is continuous, which we are about to see. Notice we can easily deduce from this that $\mathbf{n}^\epsilon_{\bar{W}}$ is continuous, but that $\hat{\mathbf{n}}^\epsilon_W(z)$ admits exactly two discontinuity points, at the endpoints of W, but where the "jumps" remain bounded.

The three families of functions $\mathbf{n}^\epsilon_W$, $\hat{\mathbf{n}}^\epsilon_W$ and $\mathbf{n}^\epsilon_{\bar{W}}$ are exactly invariant by translation and scaling, in the sense that for f^ϵ_W any one of these three functions, for all $v \in \mathbb{R}^2$ and for all $\lambda > 0$, for all $z \in \mathbb{R}^2$,

$$f^\epsilon_W(z) = f^\epsilon_{W+v}(z+v) = f^{\lambda\epsilon}_{\lambda W}(\lambda z).$$

Lemma 2.1. *For all $z \in \mathbb{R}^2$, $\psi^{z,\epsilon} \in \mathcal{C}^3(\mathbb{R}^2, \mathbb{R}^2)$, and the map $z \mapsto \psi^{z,\epsilon}$ is continuous from $\mathbb{R}$ to $\mathcal{C}^3(\mathbb{R}^2, \mathbb{R}^2)$.*

Proof. Here we only assume that $\varphi \in W^{3,\infty}$, since this is sufficient to conclude.

First we show that $\psi^{z,\epsilon} \in W^{3,\infty}$ for all z, then that $z \mapsto \psi^{z,\epsilon}$ is continuous from $\mathbb{R}^2$ to $W^{3,\infty}$. This would be enough to conclude: for an arbitrary multiindex $I \in \{1,2\}^3$, we would deduce

$$|\partial^3_I \psi^{z,\epsilon}(x+h) - \partial^3_I \psi^{z,\epsilon}(x) = |\partial^3_I \psi^{z-h,\epsilon}(x) - \partial^3_I \psi^{z,\epsilon}(x)| \le \|\partial^3_I \psi^{z-h,\epsilon} - \partial^3_I \psi^{z,\epsilon}\|_\infty \underset{h\to 0}{\longrightarrow} 0,$$

hence $\psi^{z,\epsilon} \in \mathcal{C}^3$. Since the topology on $\mathcal{C}^3$ is that of $W^{3,\infty}$, it would follow that $z \mapsto \psi^{z,\epsilon}$ is continuous indeed as a $\mathcal{C}^3$-valued function.

The fact $\psi^{z,\epsilon} \in W^{3,\infty}$ follows from

$$\|\partial^3_I \psi^{z,\epsilon}\|_\infty = \|(\theta^z_0 + \theta^z_\infty) \star \partial^3_I \psi^\epsilon\|_\infty \le \|\theta^z_0\|_{L^1} \|\psi^\epsilon\|_{L^\infty} + \|\theta^z_\infty\|_{L^\infty} \|\psi^\epsilon\|_{L^1} < \infty.$$

To prove the continuity of $z \mapsto \psi^{z,\epsilon}$ from $\mathbb{R}^2$ to $W^{3,\infty}$, at a given point z, we decompose $\theta^z = \theta^z_0 + \theta^z_\infty$ as above, and we decompose $\theta^w = \theta^w \mathbb{1}_{B_1(z)} + \theta^w \mathbb{1}_{\mathbb{R}^2 \setminus B_1(z)} = \theta^w_0 + \theta^w_\infty$. Then, it holds

$$\|\partial^3_I \psi^{w,\epsilon} - \partial^3_I \psi^{z,\epsilon}\|_\infty = \|(\theta^w - \theta^z) \star \partial^3_I \varphi^\epsilon\|_\infty \le \|\theta^w_0 - \theta^z_0\|_{L^1} \|\partial^3_I \varphi^\epsilon\|_\infty + \|\theta^w_\infty - \theta^z_\infty\|_{L^\infty} \|\partial^3_I \varphi^\epsilon\|_{L^1}.$$

The first term then converges toward 0 as $w \to z$ by Scheffé's lemma, whilst the second term converges toward 0 as θ^w_∞ converges uniformly toward θ^z_∞ on $\mathbb{R}^2 \setminus B_1(z)$, and both function are vanishing on $B_1(z)$. This concludes the proof. □

Corollary 2.2. *For all $\epsilon > 0$, $\mathbf{n}^\epsilon_W(z)$ is defined jointly for all z and the function $z \mapsto \mathbf{n}^\epsilon_W(z)$ is continuous.*

Proof. Recall there are ways to define stochastic integals in such a way that the integration map $V \mapsto \int_W V$ is almost surely defined on the whole space $\mathcal{C}^3(\mathbb{R}^2, \mathbb{R}^2)$, and is continuous in the $\mathcal{C}^3$ topology. This can be done for example through rough path theory. Since $z \mapsto \psi^{z,\epsilon}$ is continuous as well, the result follows. □

Corollary 2.3. *The functions $\mathbf{n}^\epsilon_{\bar{W}}$ and $\hat{\mathbf{n}}^\epsilon_W$ are both square integrable.*

Remark 5. For all $z \in \mathbb{R}^2$, it holds $\operatorname{div}(\theta^z) = 0$ in the distributional sense. Therefore, $\operatorname{div}(\psi^{z,\epsilon}) = 0$ for all $z \in \mathbb{R}^2$ and $\epsilon > 0$. As the Stratonovich-to-Ito correction term for the integral of a vector field V along a Brownian motion takes the form of an integral of $\operatorname{div}(V)$, we deduce that $\int_W \psi^{z,\epsilon}$ has the same value whether it is interpreted as an Ito or as a Stratonovich integral, so in practice we will evaluate it as an Ito integral and use Ito's formula in particular, although in principle it should really be thought of as a Stratonovich integral.

Remark 6. In the following, when we compare some random variable defined from $\mathbf{n}^\epsilon_W$ with the same random variable but for ϵ' instead of ϵ **we do not assume that the same mollifier is necessarily used to define $\mathbf{n}^\epsilon_W$ and $\mathbf{n}^{\epsilon'}_W$**, and the same goes when we have four different ϵ's. It would just be too cumbersome to keep track of the mollifier in the notation.

For example, when we will prove that $Y^{\epsilon_1,\epsilon_2}(f) - Y^{\epsilon_3,\epsilon_4}(f)$ goes to 0 as the ϵ_i go to 0, it should be understood that four, possibly all different, arbitrary but fixed, mollifiers are used, and in particular the limit then automatically does not depend on the chosen mollifiers (although the convergence is of course **not** uniform over all possible choice of mollifier).

3. Some upper bounds

In this section obtain some upper-bounds of both analytic and probabilistic nature to be used through the paper.

Lemma 3.1. *The following properties hold.*

1. *For all $x \in \mathbb{R}^2$ and $\epsilon > 0$, $|\psi^\epsilon(x)|^2 = \epsilon^{-2}|\psi(\epsilon^{-1}x)|^2$.*
2. *There exists a constant C_φ such that for all $x \in \mathbb{R}^2$, $|\psi(x)|^2 \leq \frac{C_\varphi}{(1+|x|)^2} \leq \frac{C_\varphi}{1+|x|^2}$.*
3. *For all $x \in \mathbb{R}^2$ and $\epsilon > 0$, $|\psi^{z,\epsilon}(x)|^2 \leq \frac{C_\varphi}{\epsilon^2+|z-x|^2}$.*
4. *As $r \to \infty$,*
$$\int_{B_r} |\psi(x)|^2 \,\mathrm{d}x = \frac{1}{2\pi}\log(r) + O(1).$$

Proof. Here we do not assume φ is compactly supported.

Item (1) is a simple change of variable in this convolution defining ψ^ϵ.

For Item (2), it suffices to show that ψ is bounded and that for C, R large enough, $|\psi(x)| \leq C|x|^{-1}$ for all x with $|x| \geq R$. On the one hand, we have

$$|\psi|(x) \leq \frac{1}{2\pi}\int_{B_1(x)} \frac{1}{|z-x|}\varphi(z)\,\mathrm{d}z + \frac{1}{2\pi}\int_{\mathbb{R}^2\setminus B_1(x)} \frac{1}{|z-x|}\varphi(z)\,\mathrm{d}z \leq \|\varphi\|_\infty + \frac{\|\varphi\|_{L^1}}{2\pi}.$$

On the other hand,

$$\begin{aligned}|\psi^i(x)| = \Big|\int_{\mathbb{R}^2}\varphi(x-v)\theta^i(v)\,\mathrm{d}v\Big| &\leq \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{\varphi(x-v)}{|v|}\,\mathrm{d}v\\ &\leq \rho_\varphi\int_{B_{\frac{|x|}{2}}}\frac{4}{|v||x|^3}\,\mathrm{d}v + \int_{\mathbb{R}^2\setminus B_{\frac{|x|}{2}}}\frac{2\varphi(x-v)}{|x|}\,\mathrm{d}v\\ &\leq \frac{4\rho_\varphi+2}{2\pi|x|}, \qquad (\text{for } |x|\geq 1)\end{aligned}$$

which concludes the proof of the second item.

Item 3 follows immediately from the previous 2 with elementary computation.

For the last estimation we need to be a bit more precise. Let $R = |x|^\alpha$, $\alpha \in (0,1)$. Notice for $|x| \geq 1$,

$$\Big|\int_{B_R(x)}\varphi(x-v)\Big(\frac{v^i}{|v|^2}-\frac{x^i}{|x|^2}\Big)\,\mathrm{d}v\Big| \leq \frac{\max_{v\in B_R(x)}|v^i|x|^2-x^i|v|^2|}{|x|^2(|x|-R)^2} \leq \frac{3|x|^{2+\alpha}}{|x|^4} = O(|x|^{\alpha-2}) \quad (9)$$

Besides,

$$\int_{\mathbb{R}^2\setminus B_R}\varphi(u)\,\mathrm{d}u \leq 2\pi\rho_\varphi\int_R^\infty\frac{\mathrm{d}r}{r^3} \leq \frac{\pi\rho_\varphi}{R^2},$$

thus

$$\Big|\int_{\mathbb{R}^2\setminus B_R(x)}\varphi(x-v)\,\mathrm{d}v\frac{x^i}{|x|^2}\Big| = O(|x|^{-1-2\alpha}). \quad (10)$$

Furthermore,

$$\Big|\int_{B_{\frac{|x|}{2}}}\varphi(x-v)\frac{v^i}{|v|^2}\,\mathrm{d}v\Big| \leq \frac{8\rho_\varphi}{|x|^3}\int_{B_{\frac{|x|}{2}}}\frac{v^i}{|v|^2}\,\mathrm{d}v = \frac{4\pi\rho_\varphi}{|x|^2}, \quad (11)$$

and

$$\Big|\int_{\mathbb{R}^2\setminus(B_{\frac{|x|}{2}}\cup B_R(x))}\varphi(x-v)\frac{v^i}{|v|^2}\,\mathrm{d}v\Big| \leq \frac{2}{|x|}\int_{\mathbb{R}^2\setminus B_R(x)}\varphi(x-v)\,\mathrm{d}v = O(|x|^{-1-2\alpha}). \quad (12)$$

Combining (9), (10), (11), and (10), we deduce

$$\int_{\mathbb{R}^2} \varphi(x-v)\frac{v^i}{|v|^2} = \int_{\mathbb{R}^2} \varphi(x-v)\,\mathrm{d}v \frac{x^i}{|x|^2} + O(|x|^{-1-2\alpha} + |x|^{\alpha-2}).$$

The optimal bound is obtained for $\alpha = \frac{1}{3}$ and we get

$$\psi^i(x) = \theta^i(x) + O(|x|^{-\frac{5}{3}}).$$

Notice we would, with similar methods, get a better bound by assuming a quicker decrease of φ.

It follows that $|\psi(x)|^2 = |\theta(x)|^2 + O(|x|^{-\frac{8}{3}})$. Let C be such that $||\psi(x)|^2 - |\theta(x)^2| \leq C|x|^{-\frac{8}{3}}$. Then,

$$\Big| \int_{B_r\setminus B_1} |\psi(x)|^2\,\mathrm{d}x - \int_{B_r\setminus B_1} |\theta(x)|^2\,\mathrm{d}x \Big| \leq C \int_{\mathbb{R}^2\setminus B_1} |x|^{-\frac{8}{3}}\,\mathrm{d}x < \infty.$$

Since $|\theta(x)| = \frac{1}{2\pi|x|}$, we finally get

$$\int_{B_r} |\psi(x)|^2\,\mathrm{d}x = \frac{1}{4\pi^2}\int_{B_r\setminus B_1} |x|^{-2}\,\mathrm{d}x + \int_{B_1} |\psi(x)|^2\,\mathrm{d}x + O(1) = \frac{\log(r)}{2\pi} + O(1),$$

which concludes the proof. □

Lemma 3.2. *Then,*

- *There exists C such that for all $\epsilon, \epsilon' \in (0,1]$, for all $x \neq y$,*

$$|\mathbb{E}[\mathbf{n}_W^{\epsilon}(x)\mathbf{n}_W^{\epsilon'}(y)]| \leq CK^*(x,y). \tag{13}$$

- *As $(\epsilon, \epsilon') \to (0,0)$,*

$$\mathbb{E}[\mathbf{n}_W^{\epsilon}(x)\mathbf{n}_W^{\epsilon'}(y)] \longrightarrow K(x,y). \tag{14}$$

Proof of Lemma 3.2. By Ito isometry,

$$\begin{aligned}\mathbb{E}[\mathbf{n}_W^{\epsilon}(x)\mathbf{n}_W^{\epsilon'}(y)] &= \int_0^1 \mathbb{E}[\langle \psi^{x,\epsilon}(W_t), \psi^{y,\epsilon'}(W_t)\rangle]\,\mathrm{d}t \\ &= \int_0^1 \int_{\mathbb{R}^2} p_t(0,z)\langle \psi^{x,\epsilon}(z), \psi^{y,\epsilon'}(z)\rangle\,\mathrm{d}z\,\mathrm{d}t.\end{aligned}$$

It follows from Lemma 3.1, (3) that

$$\langle \psi^{x,\epsilon}(z), \psi^{y,\epsilon'}(z)\rangle \leq \frac{C}{|x-z||y-z|},$$

from which the first point (13) follows. Furthermore, by properties of the convolution, for all $z \in \mathbb{R}^2 \setminus \{x,y\}$,

$$\langle \psi^{x,\epsilon}(z), \psi^{y,\epsilon'}(z)\rangle \underset{\epsilon\to 0}{\longrightarrow} \langle \theta^x_z, \theta^y_z\rangle.$$

Since

$$\int_0^1 \int_{\mathbb{R}^2} p_t(0,z)|\langle \psi^{x,\epsilon}(z), \psi^{y,\epsilon'}(z)\rangle|\,\mathrm{d}z\,\mathrm{d}t \leq CK^*(x,y),$$

we can apply the dominated convergence theorem (for integrals with respect to z) and conclude to the second point, the convergence (14). □

We will now derive a few upper bounds, most of which are related to these kernels K, K^*. Let us briefly explain what these bounds amount to. It can easily be seen that $K^*(x,y)$ diverges as $x \to y$, for in the limit $x = y$ we have

$$K_t^*(x,x) \geq \int_{B_1(x)} \frac{p_t(0,z)}{|z-x|^2}\,\mathrm{d}z = +\infty.$$

From the shape of the function we integral, one can naturally expect the divergence to be logarithmic. Furthermore as $x \to 0$, the logarithmic divergence of $\int_0^1 p_t(0,z)\,\mathrm{d}t$ comes into play as well. Although it does not prevent $K^*(0,y)$ to be finite, it worsen the divergence of $K^*(0,y)$ as $y \to 0$. In the following we will show that K^* diverges more slowly than $\log^2$, which implies that it is locally integrable to any power. Unfortunately, both K^* and K fails shortly from being square-integrable.

3.1. **Analytic estimations: Pointwise upper bounds for the K's.** For a measurable non-negative function $f:\mathbb{R}^d\to\mathbb{R}_+$, let $f^*:\mathbb{R}^d\to\mathbb{R}_+$ be its symmetric decreasing rearrangement (or *Schwarz symmetrization* of f). In the case when f is given by $f(z)=\bar{f}(|z|)$ for a non-increasing function $\bar{f}:\mathbb{R}_+\to\mathbb{R}_+$, it holds $f^*=f$ and more generally $(x\mapsto f(x+y))^*=f$. Furthermore if $f\leq g$, then $f^*\leq g^*$. We will generally use this inequality with $f=h\mathbb{1}_{\mathbb{R}^2\setminus B_R}$, $h(z)=\bar{h}(|z|)$ for a non-increasing $\bar{h}$, and $g=g^*:z\mapsto\bar{h}(|z|\wedge R)$. These basic facts, together with the three following inequalities, are the only thing we will need to know about these symmetric decreasing rearrangements; which is why we do not provide the precise definition of it. Let us simply say that, as the name suggests, these are rotationally symmetric, non-increasing, and the Lebesgue mass of the superlevel set of f^* is equal to that for f.

The **Hardy–Littlewood inequality** states that for any measurable $f,g:\mathbb{R}^d\to\mathbb{R}_+$,

$$\int_{\mathbb{R}^d} fg\leq\int_{\mathbb{R}^d} f^*g^*.$$

The **Riesz-Sobolev inequality** states that for any measurable $f,g,h:\mathbb{R}^d\to\mathbb{R}_+$,

$$\int_{(\mathbb{R}^d)^2} f(x)g(y)h(x-y)\,\mathrm{d}x\,\mathrm{d}y\leq\int_{(\mathbb{R}^d)^2} f^*(x)g^*(y)h^*(x-y)\,\mathrm{d}x\,\mathrm{d}y.$$

Both these inequalities have later be generalized into the **Brascamp-Lieb-Luttinger inequality** ([, Theorem 3.4]A General Rearrangement Inequality for Multiple Integrals), a special case of which states that for any measurable $f_1,\dots,f_k:\mathbb{R}^d\to\mathbb{R}_+$,

$$\int_{\mathbb{R}^d} f_1\dots f_k\leq\int_{\mathbb{R}^d} f_1^*\dots f_k^*.$$

Lemma 3.3. *For all $t\in(0,1]$ and $v\in\mathbb{R}^2$,*

$$K_t^\dagger(v)\leq\int_{\mathbb{R}^2\setminus B_{|v|/2}}\frac{p_t(0,z)}{|z|^2}\,\mathrm{d}z+6\;|v|^{-1}\int_{B_{|v|/2}}\frac{p_t(0,z)}{|z|}\,\mathrm{d}z.$$

Proof. Let x,y be such that $v=x-y$, set $r=|v|/2$, and define the functions $f_a:z\mapsto|z-a|^{-1}\wedge r^{-1}$ for $a\in\{x,y\}$.

Using Brascamp-Lieb-Luttinger inequality to the functions $p_t(0,\cdot)$, f_x, f_z , we obtain

$$\begin{aligned}\int_{\mathbb{R}^2\setminus(B_r(x)\cup B_r(y))}\frac{p_t(0,z)}{|z-x||z-y|}\,\mathrm{d}z&\leq\int_{\mathbb{R}^2}p_t(0,z)f_x(z)f_y(z)\,\mathrm{d}z\\&\leq\int_{\mathbb{R}^2}p_t(0,z)f_x^*(z)f_y^*(z)\,\mathrm{d}z\\&\leq\int_{\mathbb{R}^2\setminus B_r}\frac{p_t(0,z)}{|z|^2}\,\mathrm{d}z+r^{-1}\int_{B_r}\frac{p_t(0,z)}{|z|}\,\mathrm{d}z.\end{aligned}$$

Furthermore, using the Hardy–Littlewood inequality to the functions $z\mapsto p_t(0,z)\mathbb{1}_{|z|\leq r}$ and f_x, and using the fact that $|z-y|\geq r$ for all $z\in B_r(x)$, we obtain

$$\int_{B_r(x)}\frac{p_t(0,z)}{|z-x||z-y|}\,\mathrm{d}z\leq r^{-1}\int_{B_r(x)}\frac{p_t(0,z)}{|z|}\,\mathrm{d}z,$$

and similarly

$$\int_{B_r(y)}\frac{p_t(0,z)}{|z-x||z-y|}\,\mathrm{d}z\leq r^{-1}\int_{B_r(x)}\frac{p_t(0,z)}{|z|}\,\mathrm{d}z.$$

Combining these estimations, we get

$$\int_{\mathbb{R}^2}\frac{p_t(0,z)}{|z-x||z-y|}\,\mathrm{d}z\leq\int_{\mathbb{R}^2\setminus B_r}\frac{p_t(0,z)}{|z|^2}\,\mathrm{d}z+3\;r^{-1}\int_{B_r}\frac{p_t(0,z)}{|z|}\,\mathrm{d}z.$$

We conclude by taking the supremum over x,y such that $x-y=v$. □

Lemma 3.4. *For all $p > 2$, there exists a constant C such that for all $t \in (0, 1]$,*

$$\int_{\mathbb{R}^2} K_t^\dagger(v)^p \, \mathrm{d}v \le C t^{1-p}.$$

Proof. Let

$$f_1 : v \mapsto \int_{\mathbb{R}^2 \setminus B_{|v|/2}} \frac{p_t(0,z)}{|z|^2} \, \mathrm{d}z \qquad \text{and} \qquad f_2 : \; v \mapsto |v|^{-1} \int_{B_{|v|/2}} \frac{p_t(0,z)}{|z|} \, \mathrm{d}z.$$

The real number $f_1(v)$ is equal to $\frac{1}{2t}\Gamma(0, \frac{|v|^2}{8t})$. Thus, making the change of variables $v = r(\cos(\theta), \sin(\theta))$ and $u = \frac{r}{\sqrt{t}}$ gives

$$\|f_1\|_{L^p}^p = \int_{\mathbb{R}^2} f_1(v)^p \, \mathrm{d}v = \frac{2\pi t}{2^p t^p} \int_{\mathbb{R}} \Gamma(0, \frac{u^2}{8})^p u \, \mathrm{d}u = C_p t^{1-p},$$

for a constant C_p which is finite since $\Gamma(0,\cdot)$ decays subgaussianity fast near infinity and diverges only logarithmically near 0.

On the other hand, with the change of variables $v = r(\cos(\theta), \sin(\theta))$, $z = \rho(\cos(\varphi), \sin(\varphi))$, then $u = t^{-\frac{1}{2}}\rho$, $w = t^{-\frac{1}{2}} r$, we get

$$\begin{aligned} \|f_2\|_{L^p}^p = \int_{\mathbb{R}^2} f_2(v)^p \, \mathrm{d}v &= (2\pi)^{1+p} \int_0^\infty \Big(\int_0^{r/2} p_t(0,\rho) \, \mathrm{d}\rho \Big)^p r^{1-p} \, \mathrm{d}r \\ &= (2\pi)^{1+p} t^{1-p} \int_0^\infty \Big(\int_0^{v/2} p_1(0,u) \, \mathrm{d}u \Big)^p v^{1-p} \, \mathrm{d}v \\ &= C_p' t^{1-p}, \end{aligned}$$

for a constant C_p' which is finite since

$$\Big(\int_0^{v/2} p_1(0,u) \, \mathrm{d}u \Big)^p v^{1-p} \le \min(\frac{1}{(4\pi)^p} v, v^{1-p}),$$

which is integrable in v over $(0, \infty)$ since we assumed $p > 2$.

Using Lemma 3.3 and $\|a+b\|_{L^p}^p \le 2^{p-1}(\|a\|_{L^p}^p + \|b\|_{L^p}^p)$, we deduce

$$\int_{\mathbb{R}^2} K_t^\dagger(v)^p \, \mathrm{d}v \le 2^{p-1}(\|f_1\|_{L^p}^p + 6^p \|f_2\|_{L^p}^p) = C_p'' t^{1-p},$$

as announced. ☐

Lemma 3.5. *For all $v \in \mathbb{R}^2 \setminus \{0\}$, $K^\dagger(v) \le \frac{4\sqrt{2\pi}}{|v|}$.*

Proof. For all $x, y, z \in \mathbb{R}^2$, either or both $|x-z|$ and $|y-z|$ is greater than $|x-y|/2$. It follows that

$$|x-z|^{-1}|y-z|^{-1} = (|x-z| \wedge |y-z|)^{-1}(|x-z| \vee |y-z|)^{-1} \le (|x-z|^{-1} + |y-z|^{-1})(2|x-y|^{-1}).$$

Using Hardy–Littlewood inequality, we deduce

$$\begin{aligned} \int_{\mathbb{R}^2} \frac{p_t(0,z)}{|z-x||z-y|} \, \mathrm{d}z &\le \frac{2}{|x-y|} \Big(\int_{\mathbb{R}^2} \frac{p_t(0,z)}{|z-x|} \, \mathrm{d}z + \int_{\mathbb{R}^2} \frac{p_t(0,z)}{|z-y|} \, \mathrm{d}z \Big) \\ &\le \frac{4}{|x-y|} \int_{\mathbb{R}^2} \frac{p_t(0,z)}{|z|} \, \mathrm{d}z \\ &= \frac{4}{|x-y|} \sqrt{\frac{\pi}{2t}}. \end{aligned}$$

The lemma follows, by integrating over $t \in [0, 1]$ and then taking the supremum over $\{(x, y) : x - y = v\}$. ☐

Lemma 3.6. *There exists a constant C such that for all $v \in \mathbb{R}^2 \setminus \{0\}$,*

$$K^\dagger(v) \le C \max(1, \log(|v|^{-1})^2).$$

Proof. Since we already know by the previous lemma that $K^\dagger(v)$ decreases toward 0 as $|v| \to \infty$, it suffices to show that

$$\liminf_{|v|\to 0} \frac{K^\dagger(v)}{\log(|v|^{-1})^2} < \infty.$$

Recall Lemma 3.3:

$$K_t^\dagger(v) \le \int_{\mathbb{R}^2\setminus B_{|v|/2}} \frac{p_t(0,z)}{|z|^2}\,\mathrm{d}z + 6\,|v|^{-1}\int_{B_{|v|/2}} \frac{p_t(0,z)}{|z|}\,\mathrm{d}z.$$

The exponential decay of $\Gamma(0,r)$ as $r\to\infty$ ensures that

$$\int_0^1\int_{\mathbb{R}^2\setminus B_1} \frac{p_t(0,z)}{|z|^2}\,\mathrm{d}z\,\mathrm{d}t = \int_1^\infty \rho^{-1}\Gamma(0,\frac{\rho^2}{2})\,\mathrm{d}\rho < \infty.$$

Furthermore, the asymptotic estimation

$$\Gamma(0,\frac{\rho^2}{2}) \underset{\rho\to 0}{\sim} \frac{1}{2}\log(\rho^{-1})$$

allows to deduce

$$\int_0^1\int_{\mathbb{R}^2\setminus B_r} \frac{p_t(0,z)}{|z|^2}\,\mathrm{d}z\,\mathrm{d}t = \int_r^\infty \rho^{-1}\Gamma(0,\frac{\rho^2}{2})\,\mathrm{d}\rho \sim 2\int_r^1 \rho^{-1}\log(\rho^{-1})\,\mathrm{d}\rho = \log(r^{-1})^2$$

and

$$r^{-1}\int_0^1\int_{B_r} \frac{p_t(0,z)}{|z|}\,\mathrm{d}z\,\mathrm{d}t = r^{-1}\int_0^r \Gamma(0,\frac{\rho^2}{2})\,\mathrm{d}\rho \sim \frac{1}{2}\log(r^{-1}),$$

hence

$$\liminf_{|y-x|\to 0} \frac{K^*(x,y)}{\log(|x-y|^{-1})^2} \le 1 < \infty,$$

which concludes the proof. ☐

Corollary 3.7. *The function $K^\dagger$ is in $L^p(\mathbb{R}^2)$ for all $p\in(2,\infty)$, hence in $L^p_{loc}(\mathbb{R}^2)$ for all $p\in[1,\infty)$. For all $p\in[1,\infty)$, there exists a constant C_p such that for all measurable subset A of $\mathbb{R}^2$ with finite Lebesgue measure $\mu(A)$,*

$$\|K^\dagger\|^p_{L^p(A)} \le C_p\mu(A)(1\vee\log(\mu(A)^{-1}))^p.$$

Proof. By Lemma 3.5 and Lemma 3.6, there exists a constant C such that for all v,

$$K^\dagger(v) \le \frac{4\sqrt{2\pi}}{|v|} \wedge (C(1\vee\log(|v|^{-1}))),$$

which clearly belongs in $L^p(\mathbb{R}^2)$ if and only if $p\in(2,\infty)$.

To prove the bound on $\|K^\dagger\|^p_{L^p(A)}$, one can restrict to the sets A such that $\mu(A)\le c_p$, where c_p is an arbitrary but fixed positive constant. It then extends to all sets with finite Lebesgue measure, up to eventually replacing the constant C_p with a larger one, using $\|\cdot\|^p_{L^p(\bigcup A_i)} = \sum\|\cdot\|^p_{L^p(A_i)}$, and decomposing a measurable set A with $\mu(A)\ge c_p$ into a disjoint union of sets A_i with $\mu(A_i)\ge c_p/2$.

We take $c_p = \pi e^{-2p}$. This way, for any $\rho>0$ with $\pi\rho^2\le c$, it holds that $\rho<e^{-1}$ and that $r\mapsto r(\log r^{-1})^p$ is increasing on $[0,\rho]$.

Let $f(z) = C(1\vee\log(|z|^{-1}))$ and $g = f\mathbb{1}_A$. Since f is radially decreasing, $g^*\le f\mathbb{1}_{B_\rho}$ where ρ is such that $\pi\rho^2 = \mu(B_\rho) = \mu(A)$. We get

$$\|K^\dagger\|^p_{L^p(A)} \le \int_{B_\rho}(C\log(|z|^{-1}))^p\,\mathrm{d}z = \int_0^\rho C^p\log(r^{-1})^p 2\pi r\,\mathrm{d}r \le 2\pi C^p\rho^2\log(\rho^{-1})^p,$$

which concludes the proof since $\pi\rho^2 = \mu(A)$. ☐

Lemma 3.8. *For all $p>2$,*

$$\int_{(\mathbb{R}^2)^2} K_1^*(x,y)^p\,\mathrm{d}x\,\mathrm{d}y < \infty.$$

Proof. Unfortunately, our proof here is not very elegant. We will decompose the integral into several pieces, and estimate these pieces separately. First, we show that

$$\int_{(\mathbb{R}^2)^2} \mathbb{1}_{|x-y|\geq 1} K_1^*(x,y)^p \,\mathrm{d}x\,\mathrm{d}y$$

is finite. We thus define the following subsets of $(\mathbb{R}^2)^3$, using comas for logical conjunctions.

$$\begin{aligned}
F_0 &:= \{(x,y) : |x| \leq |y|\},\\
F_1 &:= \{(x,y) : |x-y| \geq 1, |x| \leq |y|\},\\
E_0 &:= \{(x,y,z) : |x-y| \geq 1, |x| \leq |y|\},\\
E_1 &:= \{(x,y,z) \in E_0 : |x| \geq 1, |z-x| \geq \tfrac{1}{2}, |z-y| \geq \tfrac{1}{2}, |z| \geq |y|/2\},\\
E_2 &:= \{(x,y,z) \in E_0 : |x| \geq 1, |z-x| \geq \tfrac{1}{2}, |x|/2 \leq |z| < |y|/2\},\\
E_3 &:= \{(x,y,z) \in E_0 : |x| \geq 1, |z| < |x|/2\},\\
.E_4 &:= \{(x,y,z) \in E_0 : |x| \geq 1, |z-x| \geq \tfrac{1}{2}, |z-y| < \tfrac{1}{2}\},\\
E_5 &:= \{(x,y,z) \in E_0 : |x| \geq 1, |z-x| < \tfrac{1}{2}, |y-x| \geq 1, \}\\
E_6 &:= \{(x,y,z) \in E_0 : |x| \geq 1, |z-x| < \tfrac{1}{2}, |y-x| < 1, \}\\
E_7 &:= \{(x,y,z) \in E_0 : |x| \leq 1\},
\end{aligned}$$

For $i \in \{0,\dots,7\}$, let

$$I_i := \int_{(\mathbb{R}^2)^2} \Big(\int_{\mathbb{R}^2} \mathbb{1}_{(x,y,z)\in E_i} \frac{p_1(0,z)}{|x-z||y-z|} \,\mathrm{d}z \Big)^p \,\mathrm{d}x\,\mathrm{d}y.$$

Using the symmetry between x and y, the fact that E_0 is equal to the disjoint union of the $(E_i)_{i\in 1..7}$, and triangle inequality, we get

$$\int_{(\mathbb{R}^2)^2} \mathbb{1}_{|x-y|\geq 1} K_1^*(x,y)^p \,\mathrm{d}x\,\mathrm{d}y = 2I_0 \leq 2\cdot 7^{p-1}(I_1+\cdots+I_7),$$

so that it suffices to show these I_i are finite for $i = 1..7$. For integral involving only the heat kernel, such as

$$\int_{F_0} p_1(0,y)^p \,\mathrm{d}y\,\mathrm{d}x, \qquad \int_{\mathbb{R}^2} \Big(\int_{|z|\geq|x|/2} p_1(0,z)\,\mathrm{d}z \Big)^p \,\mathrm{d}x, \quad \text{or} \quad \int_{\mathbb{R}^2} p_1(0,x)^p \,\mathrm{d}x,$$

we will not systematically prove they are finite, since this is always elementary but tedious (remark however $\int_{F_0} p_1(0,x)^p \,\mathrm{d}y\,\mathrm{d}x$, for example, is infinite, since for all x the integral in y is already infinite).

- First, we have

$$I_1 \leq 4^p \int_{F_0} \Big(\int_{|z|\geq|y|/2} p_1(0,z)\,\mathrm{d}z \Big)^p \,\mathrm{d}x\,\mathrm{d}y < \infty.$$

- On E_2, $|y-z| \geq |y|/2$, hence

$$I_2 \leq 4^p \int_{F_0} \Big(\int_{|z|\geq|x|/2} \frac{p_1(0,z)}{|y|} \,\mathrm{d}z \Big)^p \,\mathrm{d}x\,\mathrm{d}y \leq 4^p \int_{\mathbb{R}^2\setminus B_1} |y|^{-p}\,\mathrm{d}y \int_{\mathbb{R}^2} \Big(\int_{|z|\geq|x|/2} p_1(0,z)\,\mathrm{d}z \Big)^p \,\mathrm{d}x < \infty.$$

- On E_3, $|x-z| \geq |x|/2$ and $|y-z| \geq |y|/2$, hence

$$I_3 \leq 4^p \int_{F_0} |x|^{-p}|y|^{-p} \Big(\int_{\mathbb{R}^2} p_1(0,z)\,\mathrm{d}z \Big)^p \,\mathrm{d}x\,\mathrm{d}y \leq 4^p \Big(\int_{\mathbb{R}^2\setminus B_1} |x|^{-p}\,\mathrm{d}x \Big)^2 < \infty.$$

- On E_4, $|z| \geq |y|/2$, hence

$$I_4 \leq 2^p \int_{F_0} p_1(0,\tfrac{y}{2})^p \Big(\int_{\mathbb{R}^2} \frac{\mathbb{1}_{|z-y|\leq 1/2}}{|y-z|} \,\mathrm{d}z \Big)^p \,\mathrm{d}x\,\mathrm{d}y = 2^p \int_{F_0} p_1(0,\tfrac{y}{2})^p \,\mathrm{d}x\,\mathrm{d}y \Big(\int_{B_{\frac{1}{2}}} \frac{1}{|z|} \,\mathrm{d}z \Big)^p < \infty.$$

- On E_5, $|y-z| \geq |y-x|/2$ and $|z| \geq |x|/2$, hence

$$\begin{aligned} I_5 &\leq 2^p \int_{F_0} \frac{\mathbb{1}_{|y-x|\geq 1} p_1(0,x/2)^p}{|y-x|^p} \Big(\int_{B_{\frac12}} \frac{1}{|z|} \,\mathrm{d}z \Big)^p \,\mathrm{d}x \,\mathrm{d}y \\ &\leq 2^p \int_{\mathbb{R}^2 \setminus B_1} \frac{1}{|y|^p} \,\mathrm{d}y \Big(\int_{B_{\frac12}} \frac{1}{|z|} \,\mathrm{d}z \Big)^p \int_{\mathbb{R}^2} p_1(0,x/2)^p \,\mathrm{d}x < \infty. \end{aligned}$$

- On E_6, $|y-z| \geq 1/2$ and $|z| \geq |x|/2$, hence

$$I_6 \leq 2^p \int_{F_0} \mathbb{1}_{|y-x|\leq 1} \Big(\int_{B_{\frac12}(x)} \frac{p_t(0,z)}{|x-z|} \,\mathrm{d}z \Big)^p \,\mathrm{d}x \,\mathrm{d}y \leq 2^p \int_{\mathbb{R}^2} \pi p_1(0,x/2)^p \,\mathrm{d}x \Big(\int_{B_{\frac12}} \frac{1}{|z|} \,\mathrm{d}z \Big)^p < \infty.$$

- Finally,

$$I_7 \leq \int_{B_1} \int_{\mathbb{R}^2} K^*(x,y)^p \,\mathrm{d}y \,\mathrm{d}x \leq \pi \int_{\mathbb{R}^2} K^*(v)^p \,\mathrm{d}v < \infty$$

  by Lemma 3.4, which concludes the proof that

$$\int_{(\mathbb{R}^2)^2} \mathbb{1}_{|x-y|\geq 1} K_1^*(x,y)^p \,\mathrm{d}x \,\mathrm{d}y < \infty.$$

In order to show that

$$\int_{(\mathbb{R}^2)^2} \mathbb{1}_{|x-y|\leq 1} K_1^*(x,y)^p \,\mathrm{d}x \,\mathrm{d}y < \infty,$$

we now set

$$\begin{aligned} G_0 &\coloneqq \{(x,y,z) \in (\mathbb{R}^2)^3 : |x| \leq |y|, |x-y| \leq 1\}, \\ G_1 &\coloneqq \{(x,y,z) \in G_0 : |x| \leq 4\}, \\ G_2 &\coloneqq \{(x,y,z) \in G_0 : |x| > 4, |z| \leq |x|^{\frac12}\}, \\ G_3 &\coloneqq \{(x,y,z) \in G_0 : |x| > 4, |z| > |x|^{\frac12}, |z-x| \geq |x-y|/2\}, \\ G_4 &\coloneqq \{(x,y,z) \in G_0 : |x| > 4, |z| > |x|^{\frac12}, |z-x| < |x-y|/2\}, \end{aligned}$$

and set

$$J_i \coloneqq \int_{(\mathbb{R}^2)^2} \Big(\int_{\mathbb{R}^2} \mathbb{1}_{(x,y,z)\in G_i} \frac{p_1(0,z)}{|x-z||y-z|} \,\mathrm{d}z \Big)^p \,\mathrm{d}y \,\mathrm{d}x.$$

- On G_1, we have

$$J_1 \leq \int_{B_4} \int_{\mathbb{R}^2} K_1^*(x,y)^p \,\mathrm{d}y \,\mathrm{d}x \leq 16\pi \int_{\mathbb{R}^2} K_1^\dagger(v)^p \,\mathrm{d}v < \infty$$

  by Lemma 3.4.
- On G_2, since $|z| \leq |x|^{\frac12} \leq |x|/2$, both $|x-z|$ and $|y-z|$ are greater than $|x|/2$, hence

$$J_2 \leq \int_{\mathbb{R}^2 \setminus B_4} \int_{B_1(x)} \Big(\int_{B_{|x|^{\frac12}}} \frac{4}{|x|^2} \,\mathrm{d}z \Big)^p \,\mathrm{d}y \,\mathrm{d}x = \pi^{1+p} 4^{2p} \int_{\mathbb{R}^2 \setminus B_1} |x|^{-p} \,\mathrm{d}x < \infty.$$

- On G_3, let $\theta \in (0,1)$ be such that $\theta p < 2$. Using $|x-z| \geq (|x-y|/2)^\theta |x-z|^{1-\theta}$ and then the Riesz-Sobolev inequality, we get we get

$$\begin{aligned} J_3 &\leq \int_{\mathbb{R}^2 \setminus B_4} \sqrt{p_1(0,|x|^{\frac12})} \int_{B_1(x)} \frac{2^{\theta p}}{|x-y|^{\theta p}} \Big(\int_{\mathbb{R}^2} \frac{\sqrt{p_1(0,z)}}{|z|^{2-\theta}} \,\mathrm{d}z \Big)^p \,\mathrm{d}y \,\mathrm{d}x \\ &\leq C_p \int_{\mathbb{R}^2 \setminus B_4} \sqrt{p_1(0,|x|^{\frac12})} \,\mathrm{d}x < \infty. \end{aligned}$$

- On G_4, it holds $|y-z| \geq |x-y|/2$, thus
$$J_4 \leq \int_{\mathbb{R}^2\setminus B_4} p_1(0,|x|^{\frac{1}{2}}) \int_{B_1(x)} \Big(\frac{2}{|x-y|}\int_{B_{\frac{|x-y|}{2}}} \frac{1}{|v|}\,\mathrm{d}v\Big)^p \,\mathrm{d}y\,\mathrm{d}x$$
$$= \int_{\mathbb{R}^2\setminus B_4} p_1(0,|x|^{\frac{1}{2}})\pi(2\pi)^p \,\mathrm{d}x < \infty,$$
which concludes the proof.

□

Lemma 3.9. *For all $p>2$, for all $t>0$,*
$$\int_{(\mathbb{R}^2)^2} K_t^*(x,y)^p \,\mathrm{d}x\,\mathrm{d}y = C_p t^{2-p}, \qquad \textit{where} \qquad C_p = \int_{(\mathbb{R}^2)^2} K_1^*(x,y)^p \,\mathrm{d}x\,\mathrm{d}y < \infty.$$

Proof. We have just prove the finiteness of C_p in the previous lemma. The fact that $\|K_t^*\|_{L^p}^p$ scale as t^{2-p} follows simply from the change of variables $x' = t^{-\frac{1}{2}}x$, $y' = t^{-\frac{1}{2}}y$, $z' = t^{-\frac{1}{2}}z$ and the relation $p_t(0,z) = t^{-1}p_t(0,t^{-\frac{1}{2}}z)$. □

Remark 7. We just 'gained' a factor t compared to $\int_{\mathbb{R}^2} K_t^\dagger(v)^p \,\mathrm{d}v$ that we estimated in Lemma 3.4. This extra factor will allow us later one to obtain better stretch-exponential moments.

3.2. **Trigonometric estimations.**

Lemma 3.10. *Let*
$$\delta_W(z) := \hat{\mathbf{n}}_W^\epsilon(z) - \mathbf{n}_W^\epsilon(z) = -(2\pi)^{-1}\widehat{0zW_t} \quad \textit{and} \quad \delta_W^\epsilon := \mathbf{n}_{\bar{W}}^\epsilon - \mathbf{n}_W^\epsilon = (\delta_W \star \varphi^\epsilon). \tag{15}$$
Then, for all $p>2$, $\delta_W^\epsilon, \delta_W \in L^p(\mathbb{R}^2)$ and
$$\|\delta_W^\epsilon\|_{L^p}^p \leq \|\delta_W\|_{L^p}^p \leq (\pi + \frac{1}{(p-2)})\frac{|W_1|^2}{4}.$$
In particular, $\|\delta_W\|_{L^p} \in L^q(\Omega)$ for all $p>2$ and all $q\in(1,\infty)$.

Proof. The fact that $\delta_W^\epsilon \in L^p(\mathbb{R}^2)$ if $\delta_W \in L^p(\mathbb{R}^2)$ and the inequality $\|\delta_W^\epsilon\|_{L^p}^p \leq \|\delta_W\|_{L^p}^p$ follow from Young's convolution inequality and the fact that $\|\varphi^\epsilon\|_{L^1} = 1$.

The inequality for $\|\delta_W\|_{L^p}^p$ is elementary trigonometry: for $z \in B_{|W_1|}$, we bound $|\delta_W(z)|$ by $\frac{1}{2}$. For z outside this ball, which contains the ball $B_{|W_1|/2}(W_1/2)$, it holds that $|\widehat{0zW_t}| \leq \frac{\pi}{2}$, and it follows from concavity that $|\widehat{0zW_t}| \leq \frac{\pi}{2}|\sin(\widehat{0zW_t})|$, thus, $|\delta_W(z)| \leq \frac{1}{4}\sin(\widehat{0zW_t})$. Let w be the orthogonal projection of 0 on the axis (z,W_t). In particular, the distance between 0 and w is less than $|W_1|$. By looking at the right triangle $0,w,z$, we get
$$\sin(\widehat{0zW_t}) = \frac{|w|}{|z|} \leq \frac{|W_1|}{|z|}.$$
Thus,
$$\|\delta_W\|_{L^p}^p \leq \frac{\pi}{4}|W_1|^2 + \int_{\mathbb{R}^2\setminus B_{W_1}} \frac{1}{4}\frac{|W_1|^p}{|z|^p}\,dz = (\pi + \frac{1}{(2-p)})\frac{|W_1|^2}{4}.$$

□

Lemma 3.11. *For all x,y in the plane with $|y| \leq \frac{|x|}{2}$, the function $\theta_y : x \mapsto \widehat{0xy}$ satisfies*
$$|\operatorname{grad}\theta_y(x)| \leq 24|y||x|^{-2},$$
where the gradient is on the x variable whilst y is fixed.

Proof. We use complex notations. Since $\theta_{zy}(zx) = \theta_y(x)$ for all z, $\operatorname{grad}\theta_{zy}(zx) = |z|^{-1}\theta_y(x)$, and so it suffices to prove the inequality for $y=1$ to deduce the general case. Let $\theta = \theta_1$.

By elementary computations, we get for $x = x_1 + ix_2$
$$\partial_1\theta(x) = \frac{2x_1x_2 + x_1}{|x|^2|x+1|^2}, \partial_2\theta(x) = \frac{x_1^2 - x_2^2 - x_2}{|x|^2|x+1|^2}.$$

Since $|x+1| \geq \frac{|x|}{2}$, we deduce

$$|\partial_1\theta(x) \leq \frac{12}{|x|^2}, |\partial_2\theta(x)| = \frac{12}{|x|^2},$$

hence the result. □

Lemma 3.12. *Let $x, y, z \in \mathbb{R}^2$ with $|x| \wedge |y| \geq 2|z|$, then,*

$$|\widehat{0xz} - \widehat{0yz}| \leq 24\pi|x-y||z|(|x| \wedge |y|)^{-2}.$$

Proof. We assume without lossing generality that $|y| \geq |x|$. Let γ be the path from x to y obtained as follows. If the straight line segment from x to y does not intersect $B_{|y|}$ or intersects it tangentially, then γ is that segment. Otherwise, let x' be the intersection point other than y. We then take γ be the concatenation of the segment $[x, x']$ with the smallest path γ' from y to x' in $\partial B_{|y|}$ (or one the two such paths in the case when x' and y are opposite points of this circle). The path γ' then as a length equal to $\pi|x'-y|$, hence the length of γ is smaller than or equal to $\pi|x-y|$. Thus,

$$|\widehat{0xz} - \widehat{0yz}| \leq \int |\dot{\gamma}_t| |\operatorname{grad}\theta_z(\gamma_t)| \,\mathrm{d}t \leq \pi|x-y| \sup_{w\in\mathrm{Range}(\gamma)} |\operatorname{grad}\theta_z(w)| \leq 24\pi|x-y||z||x|^{-2},$$

where the last inequality follows from Lemma 3.11. □

In the following, for a subset A of $\mathbb{R}^2$, we set

$$d_A \coloneqq \inf\{|x| : x \in A\}$$

Lemma 3.13. *The following holds.*

- *There exists a constant C such that for all $z \in \mathbb{R}^2$,*
$$|\delta^\epsilon_W(z) - \delta_W(z)| \leq 2\mathbb{1}_{|z|\leq 2|W_1|+2K_\varphi} + C(1 \wedge (|W_1||z|^{-2})).$$
- *For all $p \in (1,\infty)$, there exists a constant such that for all $\epsilon \in (0,1]$,*
$$\|\delta^\epsilon_W - \delta_W\|^p_{L^p} \leq C(1 + \|W_1\|^2 + \epsilon^p(1+\|W_1\|)^{2-p}).$$
In particular, $\sup_{\epsilon\in(0,1]} \|\delta^\epsilon_W - \delta_W\|_{L^p}$ is finite and in $L^q(\Omega)$ for all $p \in (1,\infty)$ and all $q \in [1,\infty)$.
- *For all $p, q \in [1,\infty)$, there exists a constant C such that for all $\epsilon \in (0,1]$, for all measurable subset A of $\mathbb{R}^2$,*
$$\mathbb{E}\Big[\Big(\int_A |\delta^\epsilon_W - \delta_W||\delta^\epsilon_{W'} - \delta_W|\Big)^2\Big] \leq C\mu(A)^2(1 \wedge d_A^{-8}).$$

Proof. For z inside the ball $B_{2|W_1|+2K_\varphi}$, we simply bound $|\delta^\epsilon_W - \delta_W|$ by 2.

For z outside this ball, we have

$$|\delta^\epsilon_W(z) - \delta_W(z)| \leq \sup_{v\in B_{\epsilon K_\varphi}} |\widehat{0zW_1} - \widehat{0(z+v)W_1}| \leq \sup_{v\in B_{\epsilon K_\varphi}} 24\pi\epsilon K_\varphi|W_1|(|z|-\epsilon K_\varphi)^{-2} \leq 96\pi\epsilon K_\varphi|W_1||z|^{-2},$$

by lemma 3.12. Thus,

$$\|\delta^\epsilon_W - \delta_W\|^p_{L^p} \leq 2^{p+2}\pi(K_\varphi + |W_1|)^2 + (96\pi\epsilon K_\varphi)^p 2\pi|W_1|^p \int_{2|W_1|+2K_\varphi}^\infty r^{-2p+1}\,\mathrm{d}r,$$

from which the second point follows.

As for the fourth point, the case $d_A \leq 4K_\varphi$ follows from $|\delta^\epsilon_W - \delta_W| \leq 2$ and we focus on the case $d_A \geq 4K_\varphi$. Then, for all $x, y \in A$, the condition $|x| \leq 2|W_1| + 2K_\varphi$ implies $|W_1| \geq d_A/4$, hence

$$\mathbb{E}\big[|\delta^\epsilon_W(x) - \delta_W(x)||\delta^\epsilon_W(y) - \delta_W(y)|\big] \leq 8\mathbb{P}(|W_1| \geq \frac{d_A}{4}) + 2C^2\mathbb{E}[|W_1|^2]d_A^{-4} \leq C' d_A^{-4},$$

from which it follows that

$$\mathbb{E}\Big[\Big(\int_A |\delta^\epsilon_W - \delta_W||\delta^\epsilon_{W'} - \delta_{W'}|\Big)^2\Big] = \int_{A^2} \mathbb{E}\big[|\delta^\epsilon_W(x) - \delta_W(x)||\delta^\epsilon_W(y) - \delta_W(y)|\big]^2 \,\mathrm{d}x\,\mathrm{d}y \leq C'\mu(A)^2 d_A^{-8}.$$

□

3.3. Local and global upper bounds for $\bar{K}$.

Lemma 3.14. *There exists a constant C such that for all measurable subset A, A' of $\mathbb{R}^2$ with finite Lebesgue measures, for all $\epsilon, \epsilon' \in (0,1]$, for all $f, g \in L^\infty(\mathbb{R}^2)$,*

$$\Big|\int_{A\times A'} f(x)g(y)\mathbb{E}[\mathbf{n}^\epsilon_{\bar{W}}(x)\mathbf{n}^\epsilon_{\bar{W}}(y)]\,\mathrm{d}x\,\mathrm{d}y\Big| \leq \|f\|_\infty\|g\|_\infty C E(A)E(A'),$$

where

$$E(A) := \mu(A)(1\vee \log(\mu(A)^{-1}))^{\frac{1}{4}}.$$

The same hold for the 8 other inequalities when $\mathbf{n}^\epsilon_{\bar{W}}(x)$ is replaced with either $\mathbf{n}^\epsilon_W(x)$ or $\hat{\mathbf{n}}^\epsilon_W(x)$ and $\mathbf{n}^\epsilon_{\bar{W}}(y)$ is replaced with either $\mathbf{n}^\epsilon_W(y)$ or $\hat{\mathbf{n}}^\epsilon_W(y)$.

Proof. We decompose $\mathbf{n}^\epsilon_{\bar{W}}(x) = \mathbf{n}^\epsilon_W(x) + \delta^\epsilon(x)$, and we bound the four integrals this give us independently. First, we have

$$\begin{aligned}\Big|\int_{A\times A'} f(x)g(y)\mathbb{E}[\mathbf{n}^\epsilon_W(x)\mathbf{n}^\epsilon_W(y)]\,\mathrm{d}x\,\mathrm{d}y\Big| &= \mathbb{E}\Big[\Big(\int_A f(x)\mathbf{n}^\epsilon_W(x)\,\mathrm{d}x\Big)\Big(\int_{A'} g(y)\mathbf{n}^\epsilon_W(y)\,\mathrm{d}y\Big)\Big]\\ &\leq \mathbb{E}\Big[\Big(\int_A f(x)\mathbf{n}^\epsilon_W(x)\,\mathrm{d}x\Big)^2\Big]^{\frac{1}{2}}\mathbb{E}\Big[\Big(\int_{A'} g(y)\mathbf{n}^\epsilon_W(y)\,\mathrm{d}y\Big)^2\Big]^{\frac{1}{2}}\\ &\leq \|f\|_\infty\|g\|_\infty\Big(\int_{A^2} K^\dagger(x-y)\,\mathrm{d}x\,\mathrm{d}y\Big)^{\frac{1}{2}}\Big(\int_{A'^2} K^\dagger(x-y)\,\mathrm{d}x\,\mathrm{d}y\Big)^{\frac{1}{2}}\\ &\leq C\|f\|_\infty\|g\|_\infty\mu(A)\mu(A')(1\vee\log(\mu(A)^{-1}))^{\frac{1}{4}}(1\vee\log(\mu(A')^{-1}))^{\frac{1}{4}},\end{aligned}$$

where the last inequality follows from Corollary 3.7 applied with $p=1$.

Secondly, we have trivially

$$\Big|\int_{A\times A'} f(x)g(y)\mathbb{E}[\delta^\epsilon_W(x)\delta^\epsilon_W(y)]\,\mathrm{d}x\,\mathrm{d}y\Big| \leq \|f\|_\infty\|g\|_\infty\mu(A)\mu(A').$$

Finally, we have by Hölder inequality in $L^2(\Omega)$ and using the two previous bounds

$$\begin{aligned}\Big|\int_{A\times A'} f(x)g(y)\mathbb{E}[\delta^\epsilon_W(x)\mathbf{n}^\epsilon_W(y)]\,\mathrm{d}x\,\mathrm{d}y\Big| &\leq \mathbb{E}\Big[\Big(\int_A f(x)\delta^\epsilon_W(x)\,\mathrm{d}x\Big)^2\Big]^{\frac{1}{2}}\mathbb{E}\Big[\Big(\int_{A'} g(y)\mathbf{n}^\epsilon_W(y)\,\mathrm{d}y\Big)^2\Big]^{\frac{1}{2}}\\ &\leq \sqrt{C}\|f\|_\infty\|g\|_\infty\mu(A)\mu(A')(1\vee\log(\mu(A')^{-1}))^{\frac{1}{4}},\end{aligned}$$

and by symmetry also

$$\Big|\int_{A\times A'} \mathbb{E}[f(x)g(y)\mathbf{n}^\epsilon_W(x)\delta^\epsilon_W(y)]\,\mathrm{d}x\,\mathrm{d}y\Big|\sqrt{C}\|f\|_\infty\|g\|_\infty\mu(A)\mu(A')(1\vee\log(\mu(A)^{-1}))^{\frac{1}{4}},$$

which concludes the proof (the cases with $\hat{\mathbf{n}}$ are treated similarly). □

Now we will in two steps sharpen this estimation.

Lemma 3.15. *There exists a constant C such that for any two measurable subsets A, A' of $\mathbb{R}^2$ with finite mass, for all $\epsilon, \epsilon' \in (0,1)$, for all $f, g \in L^\infty(\mathbb{R}^2)$,*

$$\int_{A\times A'} f(x)g(y)\mathbb{E}[\mathbf{n}^\epsilon_{\bar{W}}(x)\mathbf{n}^{\epsilon'}_{\bar{W}}(y)]\,\mathrm{d}x\,\mathrm{d}y \leq C\|f\|_\infty\|g\|_\infty e^{-\frac{1}{4}(d_A\wedge d_{A'})^2}E(A)E(A'),$$

where $d_X := \inf\{|x| : x\in X\}$.

Proof. Let τ be the first time when W exits the ball B centered at 0 and with radius $(d_A\wedge d_{A'}) - K_\varphi$, or $\tau = 0$ if $(d_A\wedge d_{A'}) - K_\varphi \leq 0$, or $\tau = 1$ if $\|W\|_{\infty,[0,1]} < (d_A\wedge d_{A'}) - K_\varphi$. This is a stopping time, and it holds that both $\mathbf{n}^\epsilon_{\bar{W}}(x)$ and $\mathbf{n}^{\epsilon'}_{\bar{W}}(y)$ are equal to 0 in the event $\tau = 1$. Let $W' : t\in[1-\tau]\mapsto W_{t+\tau}$, which by strong Markov property is the restriction to $[1-\tau]$ of a Brownian motion $\tilde{W}'$ started from W_τ and independent from τ conditionally on W_τ. For $z, w\in\mathbb{R}^2$, set $r_z(w)\in\{-1,0,1\}$ to be 0 for w outside the triangle with vertices $0, z, W_1$, and equal otherwise to ±1 depending on the orientation of this triangle, so that $\mathbf{n}^\epsilon_{\bar{W}} = \mathbf{n}^\epsilon_{\bar{W}'} + r_{W_\tau}$.

We get

$$\begin{aligned}&\mathbb{E}\Big[\int_{A\times A'} f(x)g(y)\mathbf{n}^{\epsilon}_{\bar W}(x)\mathbf{n}^{\epsilon'}_{\bar W}(y)\,\mathrm{d}x\,\mathrm{d}y\Big]\\&\leq \mathbb{P}(\tau<1)\sup_{t\in[0,1],z\in\partial B}\mathbb{E}_{t,z}\Big[\int_{A\times A'} f(x)g(y)(r_{W_\tau}(x)+\mathbf{n}^{\epsilon}_{\bar W'}(x)))(r_{W_\tau}(y)+\mathbf{n}^{\epsilon'}_{\bar W'}(y))\,\mathrm{d}x\,\mathrm{d}y\Big].\end{aligned}$$

Since r is bounded by 1, it holds

$$\sup_{t\in[0,1],z\in\partial B}\mathbb{E}_{t,z}\Big[\int_{A\times A'} f(x)g(y)r_{W_\tau}(x)r_{W_\tau}(y)\,\mathrm{d}x\,\mathrm{d}y\Big]\leq \|f\|_\infty\|g\|_\infty\mu(A)\mu(A').$$

By Lemma 3.14, and using scale and translation invariance to eliminate the supremum, we get

$$\sup_{t\in[0,1],z\in\partial B}\mathbb{E}_{t,z}\Big[\int_{A\times A'} f(x)g(y)\mathbf{n}^{\epsilon}_{\bar W'}(x)\mathbf{n}^{\epsilon'}_{\bar W'}(y)\,\mathrm{d}x\,\mathrm{d}y\Big]\leq C\|f\|_\infty\|g\|_\infty E(A)E(A').$$

By using Hölder inequality we deduce

$$\sup_{t\in[0,1],z\in\partial B}\mathbb{E}_{t,z}\Big[\int_{A\times A'} f(x)g(y)r_{W_\tau}(x)\mathbf{n}^{\epsilon'}_{\bar W'}(y)\,\mathrm{d}x\,\mathrm{d}y\Big]\leq \sqrt{C}\|f\|_\infty\|g\|_\infty\mu(A)E(A').$$

The lemma then follow from

$$\mathbb{P}(\tau\leq 1)\leq C'e^{-c(d_A\wedge d_{A'})^2},$$

for C' large enough, which follows from a succession of reflexion principle and invariance by reflexion: say $d_A\leq d_{A'}$. Then,

$$\begin{aligned}\mathbb{P}(\tau\leq 1)&=\mathbb{P}(\|W\|_\infty\geq d_A)\leq 2\mathbb{P}(\|W^1\|_\infty\geq \frac{d_A}{\sqrt{2}})\leq 4\mathbb{P}(\sup_{t\in[0,1]}W^1_t\geq \frac{d_A}{\sqrt{2}})=4\mathbb{P}(|W^1_1|\geq\frac{d_A}{\sqrt{2}})\\&=8\mathbb{P}(W^1_1\geq\frac{d_A}{\sqrt{2}})\leq\frac{4\sqrt{2}}{\pi d_A}e^{-\frac{d_A^2}{4}},\end{aligned}$$

from which the lemma follows (bounding instead this probability by 1 for say $d_A\leq 1$). ☐

Lemma 3.16. *Let $c<\frac{1}{8}$. There exists a constant C such that for any two measurable subsets A,A' of $\mathbb{R}^2$ with finite mass, and such that for all $\epsilon,\epsilon'\in(0,1)$, for all $f,g\in L^\infty(\mathbb{R}^2)$,*

$$\int_{A\times A'} f(x)g(y)\mathbb{E}[\mathbf{n}^{\epsilon}_{\bar W}(x)\mathbf{n}^{\epsilon'}_{\bar W}(y)]\,\mathrm{d}x\,\mathrm{d}y\leq C\|f\|_\infty\|g\|_\infty e^{-c(d_A^2+d_{A'}^2)}E(A)E(A').$$

Proof. It suffices to notice

$$\int_{A\times A'} f(x)g(y)\mathbb{E}[\mathbf{n}^{\epsilon}_{\bar W}(x)\mathbf{n}^{\epsilon'}_{\bar W}(y)]\,\mathrm{d}x\,\mathrm{d}y=\mathbb{E}\Big[\Big(\int_A f(x)\mathbf{n}^{\epsilon}_{\bar W}(x)\,\mathrm{d}x\Big)\Big(g(y)\int_{A'}\mathbf{n}^{\epsilon'}_{\bar W}(y)\,\mathrm{d}y\Big)\Big],$$

use Hölder inequality and the previous lemma 3.15. ☐

Remark 8. It is certainly possible to show that the subgaussian decay in $|x|\wedge|y|$ holds directly at the pointwise level, i.e. that

$$\mathbb{E}[\mathbf{n}^{\epsilon}_{\bar W}(x)\mathbf{n}^{\epsilon}_{\bar W}(y)]\leq Ce^{-2c(|x|^2\wedge|y|^2)}(1\vee\log(|x-y|^{-1}))^2.$$

We do believe that the stronger estimation also holds pointwise, i.e. that

$$\mathbb{E}[\mathbf{n}^{\epsilon}_{\bar W}(x)\mathbf{n}^{\epsilon}_{\bar W}(y)]\leq e^{-c(|x|^2+|y|^2)}(1\vee\log(|x-y|^{-1}))^2,$$

but that seem way more complicated to show.

3.4. **Local upper bounds for $\hat{K}$.**

Lemma 3.17. *There exists a constant C such that for any two measurable subsets A, A' of $\mathbb{R}^2$, for all $\epsilon, \epsilon' > 0$, for all $f, g \in L^\infty(\mathbb{R}^2)$,*

$$\int_{A\times A'} f(x)g(y)\mathbb{E}[\hat{\mathbf{n}}^\epsilon_W(x)\hat{\mathbf{n}}^{\epsilon'}_W(y)]\,\mathrm{d}x\,\mathrm{d}y \leq C\|f\|_\infty\|g\|_\infty(1\wedge d_A^{-2})(1\wedge d_{A'}^{-2})E(A)E(A').$$

Proof. We assume without loosing generality that $d_A \leq d'_A$. First notice if we replace $\hat{\mathbf{n}}^\epsilon_W$ with $\mathbf{n}^\epsilon_{\bar{W}}$ then the result follows from Lemma 3.16. Recall $\mathbf{n}^\epsilon_{\bar{W}} - \hat{\mathbf{n}}^\epsilon_W = \delta^\epsilon_W - \delta_W$. By Lemma 3.13,

$$\begin{aligned}
&\int_{A\times A'} f(x)g(y)\mathbb{E}[(\delta^\epsilon_W(x)-\delta_W(x))(\delta^{\epsilon'}_W(y)-\delta_W(y))]\,\mathrm{d}x\,\mathrm{d}y\\
&\leq 4\mathbb{P}(|W_1|\geq d_A)\|f\|_\infty\|g\|_\infty\mu(A)\mu(A') + C\|f\|_\infty\|g\|_\infty\mathbb{E}[|W_1|^2]\int_{A\times A'}|x|^{-2}|y|^{-2}\,\mathrm{d}x\,\mathrm{d}y\\
&\leq C'\|f\|_\infty\|g\|_\infty(e^{-\frac{d_A^2}{4}} + d_A^{-2}d_{A'}^{-2})\mu(A)\mu(A').
\end{aligned}$$

By Hölder inequality,

$$\begin{aligned}
&\int_{A\times A'} f(x)g(y)\mathbb{E}[(\delta^\epsilon_W(x)-\delta_W(x))(\delta^{\epsilon'}_W(y)-\delta_W(y))]\,\mathrm{d}x\,\mathrm{d}y\\
&\leq \mathbb{E}\Big[\Big(\int_A f(x)(\delta^\epsilon_W(x)-\delta_W(x))\,\mathrm{d}x\Big)^2\Big]^{\frac12}\mathbb{E}\Big[\Big(\int_{A'} g(y)(\delta^{\epsilon'}_W(y)-\delta_W(y))\,\mathrm{d}y\Big)^2\Big]^{\frac12}\\
&\leq C'\|f\|_\infty\|g\|_\infty(e^{-\frac{d_A^2}{4}} + d_A^{-4})^{\frac12}(e^{-\frac{d_A^2}{4}} + d_{A'}^{-4})^{\frac12}\mu(A)\mu(A')\\
&\leq C''\|f\|_\infty\|g\|_\infty d_A^{-2}d_{A'}^{-2}\mu(A)\mu(A').
\end{aligned}$$

Furthermore, using Hölder inequality again, we get

$$\begin{aligned}
&\int_{A\times A'} f(x)g(y)\mathbb{E}[(\delta^\epsilon_W(x)-\delta_W(x))\mathbf{n}^\epsilon_{\bar{W}}(y)]\,\mathrm{d}x\,\mathrm{d}y\\
&\leq \mathbb{E}\Big[\Big(\int_A f(x)(\delta^\epsilon_W(x)-\delta_W(x))\,\mathrm{d}x\Big)^2\Big]^{\frac12}\mathbb{E}\Big[\Big(\int_{A'} g(y)\mathbf{n}^\epsilon_{\bar{W}}(y)\,\mathrm{d}y\Big)^2\Big]^{\frac12}\\
&\leq C_3 d_A^{-1}d_{A'}^{-1}\|f\|_\infty\|g\|_\infty E^{-\frac{c}{2}(d_A^2+d_{A'}^2)}\mu(A)E(A') \leq C_4\|f\|_\infty\|g\|_\infty d_A^{-2}d_{A'}^{-2}E(A)E(A'),
\end{aligned}$$

from which we conclude easily. □

Remark 9. With more subtle estimations using the symmetry of the mollifier, it should be possible to improve the factor $d_A^{-2}d_{A'}^{-2}$ into $d_A^{-3}d_{A'}^{-3}$, which in turn would imply that $(x,y)\mapsto \mathbb{E}[\hat{\mathbf{n}}^\epsilon_W(x)\hat{\mathbf{n}}^{\epsilon'}_W(y)]$ is globally integrable whilst our estimations only allows to deduce it is in L^p for all $p>1$.

4. L^2 CONVERGENCE FOR THE VARIABLES Y

In these sections we will consider the following random variables:

$$\begin{aligned}
Y^{\epsilon,\epsilon'}_{W,W'}(f) &:= \int_{\mathbb{R}^2} f(z)\mathbf{n}^\epsilon_W(z)\mathbf{n}^{\epsilon'}_{W'}(z)\,\mathrm{d}z,\\
\hat{Y}^{\epsilon,\epsilon'}_{W,W'}(f) &:= \int_{\mathbb{R}^2} f(z)\hat{\mathbf{n}}^\epsilon_W(z)\hat{\mathbf{n}}^{\epsilon'}_{W'}(z)\,\mathrm{d}z,\\
\bar{Y}^{\epsilon,\epsilon'}_{W,W'}(f) &:= \int_{\mathbb{R}^2} f(z)\mathbf{n}^\epsilon_{\bar{W}}(z)\mathbf{n}^{\epsilon'}_{\bar{W}'}(z)\,\mathrm{d}z.
\end{aligned}$$

As we will show, these are well-defined for W, W' two independent planar Brownian motions with duration 1 and started from 0, $\epsilon,\epsilon'\in(0,1]$, and for f a measurable function which belong in $L^r(\mathbb{R}^2)$, for $r\in(1,\infty)$ in the case of $Y^{\epsilon,\epsilon'}_{W,W'}(f)$ and for $r\in(1,\infty]$ in the case of $\hat{Y}^{\epsilon,\epsilon'}_{W,W'}(f)$ and $\bar{Y}^{\epsilon,\epsilon'}_{W,W'}(f)$, and we will show that these random variables converge in L^2 as $\epsilon,\epsilon'\to 0$.

The fact that $Y^{\epsilon,\epsilon'}_{W,W'}(f)$ is in general ill-defined for $f \in L^\infty(\mathbb{R}^2)$ is directly related to the lack of global square-integrability of the function θ.[5]

4.1. **The case of L^r-functions, $r<2$.**

Proposition 4.1. *Let $r \in (1,2)$. For all $f \in L^r(\mathbb{R}^2)$, $Y^{\epsilon,\epsilon'}_{W,W'}(f)$ converges in $L^2(\Omega)$, as $\epsilon,\epsilon' \to 0$. The limit $Y_{W,W'}(f)$ satisfies*

$$\mathbb{E}[Y_{W,W'}(f)^2] = \int_{(\mathbb{R}^2)^2} f(x)K(x,y)^2 f(y)\,\mathrm{d}x\,\mathrm{d}y.$$

Furthermore, $\hat{Y}^{\epsilon,\epsilon'}_{W,W'}(f)$ and $\bar{Y}^{\epsilon,\epsilon'}_{W,W'}(f)$ also converges in L^2 as $\epsilon,\epsilon' \to 0$ toward a common limit $\hat{Y}_{W,W'}(f)$.

Proof. We will prove convergence in $L^2(\Omega)$ (e.g. for $Y^{\epsilon,\epsilon'}_{W,W'}(f)$) by showing the L^2 distance goes to 0 (e.g. $d_{L_2(\Omega)}(Y^{\epsilon^1,\epsilon^2}_{W,W'}(f), Y^{\epsilon^3,\epsilon^4}_{W,W'}(f)) \underset{\epsilon^i\to 0}{\longrightarrow} 0$), so we first need to show that for any given $\epsilon,\epsilon' \in (0,1]$, $Y^{\epsilon,\epsilon'}_{W,W'}(f) \in L^2(\Omega)$. Let q be such that $q^{-1}+2r^{-1}=2$. The condition $r\in(1,2)$ ensures that $q\in(1,\infty)$. Let $\epsilon^1,\epsilon^2,\epsilon^3,\epsilon^4 \in (0,1]$. First assume f takes non-negative values so we can apply Tonelli's theorem to swap integration in space and in Ω. By (13),

$$\begin{aligned}
\mathbb{E}[Y^{\epsilon^1,\epsilon^2}_{W,W'}(f)Y^{\epsilon^3,\epsilon^4}_{W,W'}(f)] &= \mathbb{E}\Big[\Big(\int_{\mathbb{R}^2} f(x)\mathbf{n}^{\epsilon^1}_W(x)\mathbf{n}^{\epsilon^2}_{W'}(x)\,\mathrm{d}x\Big)\Big(\int_{\mathbb{R}^2} f(y)\mathbf{n}^{\epsilon^3}_W(y)\mathbf{n}^{\epsilon^4}_{W'}(y)\,\mathrm{d}y\Big)\Big] \\
&= \int_{(\mathbb{R}^2)^2} f(x)f(y)\mathbb{E}[\mathbf{n}^{\epsilon^1}_W(x)\mathbf{n}^{\epsilon^3}_W(y)]\mathbb{E}[\mathbf{n}^{\epsilon^2}_{W'}(x)\mathbf{n}^{\epsilon^4}_{W'}(y)]\,\mathrm{d}x\,\mathrm{d}y \\
&\le C^2\int_{(\mathbb{R}^2)^2} f(x)f(y)K^*(x,y)^2\,\mathrm{d}x\,\mathrm{d}y \\
&\le C'\int_{(\mathbb{R}^2)^2} f(x)f(y)K^\dagger(x-y)^2\,\mathrm{d}x\,\mathrm{d}y \\
&\le C''\|f\|^2_{L^r}\|K^\dagger\|^2_{L^{2q}} < \infty.
\end{aligned}$$

At the end we used Young's convolution inequality, and Corollary 3.7 to ensure the finiteness $\|K^\dagger\|_{L^{2q}}$. Since this expectation is finite, we can now repeat the same computation without the assumption that f takes non-negative values, and where the first equality is now justified by Fubini's theorem.

In the process we have also shown that the function

$$(x,y)\mapsto f(x)f(y)\mathbb{E}[\mathbf{n}^{\epsilon^1}_W(x)\mathbf{n}^{\epsilon^3}_W(y)]\mathbb{E}[\mathbf{n}^{\epsilon^2}_{W'}(x)\mathbf{n}^{\epsilon^4}_{W'}(y)]$$

is bounded in absolute value by a function independent from the $\epsilon^i \in (0,1]$ and integrable over $(\mathbb{R}^2)^2$. Since we also know by (14) that $\mathbb{E}[\mathbf{n}^{\epsilon}_W(x)\mathbf{n}^{\epsilon'}_{W'}(y)]$ converges toward $K(x,y)$, for all $x,y\in\mathbb{R}^2$, the dominated convergence theorem ensures that

$$\begin{aligned}
\mathbb{E}[Y^{\epsilon^1,\epsilon^2}_{W,W'}(f)Y^{\epsilon^3,\epsilon^4}_{W,W'}(f)] &= \int_{(\mathbb{R}^2)^2} f(x)f(y)\mathbb{E}[\mathbf{n}^{\epsilon^1}_W(x)\mathbf{n}^{\epsilon^3}_W(y)]\mathbb{E}[\mathbf{n}^{\epsilon^2}_{W'}(x)\mathbf{n}^{\epsilon^4}_{W'}(y)]\,\mathrm{d}x\,\mathrm{d}y \\
&\underset{\epsilon,\epsilon'\to 0}{\longrightarrow} \int_{(\mathbb{R}^2)^2} f(x)f(y)K(x,y)^2\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}$$

from which we easily deduce that

$$\mathbb{E}[(Y^{\epsilon^1,\epsilon^2}_{W,W'}(f)-Y^{\epsilon^3,\epsilon^4}_{W,W'}(f))^2] \underset{\epsilon^1,\epsilon^2,\epsilon^3,\epsilon^4\to 0}{\longrightarrow} 0.$$

[5]This is *not* related to Remark 9.

It follows from completness that $Y^{\epsilon,\epsilon'}_{W,W'}(f)$ converges in $L^2(\Omega)$ as $\epsilon,\epsilon'\to 0$. Furthermore, the limit $Y_{W,W'}(f)$ satisfies

$$\mathbb{E}[Y_{W,W'}(f)^2]=\lim_{\epsilon,\epsilon'\to 0}\mathbb{E}[Y^{\epsilon,\epsilon'}_{W,W'}(f)^2]=\int_{(\mathbb{R}^2)^2} f(x)f(y)K(x,y)^2\,\mathrm{d}x\,\mathrm{d}y.$$

Now we will deduce the convergence of $\hat{Y}^{\epsilon,\epsilon'}_{W,W'}(f)$ and $\bar{Y}^{\epsilon,\epsilon'}_{W,W'}(f)$. First we show they belong in $L^2(\Omega)$. Recall the definition of δ_W and δ^ϵ_W from (15).

It holds

$$\hat{Y}^{\epsilon,\epsilon'}_{W,W'}(f)-Y^{\epsilon,\epsilon'}_{W,W'}(f)=\int_{\mathbb{R}^2} f(x)\big(\delta_W(x)\mathbf{n}^{\epsilon'}_{W'}(x)+\mathbf{n}^{\epsilon}_{W}(x)\delta_{W'}(x)+\delta_W(x)\delta_{W'}(x)\big)\,\mathrm{d}x.$$

To estimate the last term, let $p\in(4,+\infty)$ be such that $2p^{-1}+r^{-1}=1$, and use Hölder inequality,

$$\Big|\int_{\mathbb{R}^2} f(x)\delta_W(x)\delta_{W'}(x)\,\mathrm{d}x\Big|\leq \|f\|_{L^r(\mathbb{R}^2)}\|\delta_W\|_{L^p(\mathbb{R}^2)}\|\delta_{W'}\|_{L^p(\mathbb{R}^2)}.$$

By Lemma 3.10, the right-hand side is finite and in $L^2(\Omega)$.

Furthermore, using first the Cauchy-Schwarz inequality in $L^2((\mathbb{R}^2)^2)$, then Young's convolution inequality, then the Cauchy-Schwarz inequality in $L^2(\mathbb{R}^2)$ with $s : s^{-1}+r^{-1}=1$, we get

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_{\mathbb{R}^2} f(x)\delta_W(x)\mathbf{n}^{\epsilon'}_{W'}(x)\,\mathrm{d}x\Big)^2\Big]^{\frac12}\\
&=\int_{(\mathbb{R}^2)^2} f(x)f(y)\mathbb{E}[\mathbf{n}^{\epsilon'}_{W'}(x)\mathbf{n}^{\epsilon'}_{W'}(y)]\mathbb{E}[\delta_W(x)\delta_W(y)]\,\mathrm{d}x\,\mathrm{d}y\\
&\leq C\Big(\int_{(\mathbb{R}^2)^2} f(x)f(y)K^\dagger(x-y)^2\,\mathrm{d}x\,\mathrm{d}y\Big)^{\frac12}\Big(\int_{(\mathbb{R}^2)^2} f(x)f(y)\mathbb{E}[\delta_W(x)\delta_W(y)]^2\,\mathrm{d}x\,\mathrm{d}y\Big)^{\frac12}\\
&\leq C\|f\|_{L^r}\|K^\dagger\|_{L^{2q}}\mathbb{E}\Big[\Big(\int_{\mathbb{R}^2} f(x)\delta_W\Big)^2\Big]^{\frac12}\\
&\leq C\|f\|^2_{L^r}\|K^\dagger\|_{L^{2q}}\mathbb{E}[\|\delta_W\|^{2s}_{L^s}]^{\frac12}.
\end{aligned}\tag{16}$$

Since $r<2$, $s>2$ and Lemma 3.10 together with Corollary 3.7 ensure this last expression is finite, which finally concludes the proof that $\hat{Y}^{\epsilon,\epsilon'}_{W,W'}(f)-Y^{\epsilon,\epsilon'}_{W,W'}(f)\in L^2(\Omega)$, hence $\hat{Y}^{\epsilon,\epsilon'}_{W,W'}(f)\in L^2(\Omega)$.

The proof that $\bar{Y}^{\epsilon,\epsilon'}_{W,W'}(f)\in L^2(\Omega)$ is identical in every point, except all the δ_W and $\delta_{W'}$ must be replaced with δ^ϵ_W or $\delta^{\epsilon'}_{W'}$.

Furthermore, doing the same computation by with the δ_W and $\delta_{W'}$replaced with $\delta^\epsilon_W-\delta_W$ and $\delta^{\epsilon'}_{W'}-\delta_{W'}$, we deduce

$$\begin{aligned}
\mathbb{E}(\hat{Y}^{\epsilon,\epsilon'}_{W,W'}(f)-\bar{Y}^{\epsilon,\epsilon'}_{W,W'}(f))^2]\leq C'\|f\|^2_{L^r}(&\mathbb{E}[\|\delta_W-\delta^\epsilon_W\|^2_{L^p(\mathbb{R}^2)}\|\delta_{W'}-\delta^{\epsilon'}_{W'}\|^2_{L^p(\mathbb{R}^2)}]\\
&+\|K^\dagger\|_{L^{2q}}(\mathbb{E}[\|\delta_W-\delta^\epsilon_W\|^{2s}_{L^s}]^{\frac12}+\mathbb{E}[\|\delta_{W'}-\delta^{\epsilon'}_{W'}\|^{2s}_{L^s}]^{\frac12}).
\end{aligned}$$

As δ^ϵ_W converges toward δ_W when $\epsilon\to 0$, and as $\|\delta_W-\delta^\epsilon_W\|$ is dominated by $4\|\delta_W\|$, it follows that

$$\hat{Y}^{\epsilon,\epsilon'}_{W,W'}(f)-\bar{Y}^{\epsilon,\epsilon'}_{W,W'}(f)\xrightarrow[\epsilon,\epsilon'\to 0]{L^2(\Omega)}0.$$

It only remains to show that the distance in $L^2(\Omega)$ between $\hat{Y}^{\epsilon^1,\epsilon^2}_{W,W'}(f)$ and $\hat{Y}^{\epsilon^3,\epsilon^4}_{W,W'}(f)$ converges toward 0 as the ϵ^i goes to 0.

An elementary computation, using the fact that $\delta_W(z)$ do not depend on ϵ, shows that

$$\begin{aligned}
\hat{Y}^{\epsilon^1,\epsilon^2}_{W,W'}(f)-\hat{Y}^{\epsilon^3,\epsilon^4}_{W,W'}(f)=Y^{\epsilon^1,\epsilon^2}_{W,W'}(f)-Y^{\epsilon^3,\epsilon^4}_{W,W'}(f)&+\int_{\mathbb{R}^2} f(x)\delta_W(x)(\mathbf{n}^{\epsilon^2}_{W'}(x)-\mathbf{n}^{\epsilon^4}_{W'}(x))\,\mathrm{d}x\\
&-\int_{\mathbb{R}^2} f(x)\delta_{W'}(x)(\mathbf{n}^{\epsilon^1}_{W'}(x)-\mathbf{n}^{\epsilon^3}_{W'}(x))\,\mathrm{d}x.
\end{aligned}$$

Since $d_{L^2(\Omega)}(Y^{\epsilon^1,\epsilon^2}_{W,W'}(f), Y^{\epsilon^3,\epsilon^4}_{W,W'}(f)) \underset{\epsilon^1,\epsilon^2,\epsilon^3,\epsilon^4\to 0}{\longrightarrow} 0$ and by symmetry between W and W', it suffices to show that

$$R^{\epsilon,\epsilon'} := \mathbb{E}\Big[\Big(\int_{\mathbb{R}^2} f(x)\delta_W(x)(\mathbf{n}^{\epsilon}_{W'}(x) - \mathbf{n}^{\epsilon'}_{W'}(x))\,\mathrm{d}x\Big)^2\Big] \underset{\epsilon,\epsilon'\to 0}{\longrightarrow} 0.$$

Using first the independence between W and W', we deduce

$$R^{\epsilon,\epsilon'} \leq \int_{(\mathbb{R}^2)^2} |f(x)f(y)||\mathbb{E}[\delta_W(x)\delta_W(y)|\big|\mathbb{E}[(\mathbf{n}^{\epsilon}_{W'}(x) - \mathbf{n}^{\epsilon'}_{W'}(x))(\mathbf{n}^{\epsilon}_{W'}(y) - \mathbf{n}^{\epsilon'}_{W'}(y))]\big|\,\mathrm{d}x\,\mathrm{d}y.$$

For all x and y,

$$\mathbb{E}[(\mathbf{n}^{\epsilon}_{W'}(x) - \mathbf{n}^{\epsilon'}_{W'}(x))(\mathbf{n}^{\epsilon}_{W'}(y) - \mathbf{n}^{\epsilon'}_{W'}(y))] \underset{\epsilon,\epsilon'\to 0}{\longrightarrow} 0$$

by (14). Besides,

$$\begin{aligned}&|f(x)f(y)||\mathbb{E}[\delta_W(x)\delta_W(y)|\big|\mathbb{E}[(\mathbf{n}^{\epsilon}_{W'}(x) - \mathbf{n}^{\epsilon'}_{W'}(x))(\mathbf{n}^{\epsilon}_{W'}(y) - \mathbf{n}^{\epsilon'}_{W'}(y))]\big|\\ &\leq 4|f(x)f(y)||\mathbb{E}[\delta_W(x)\delta_W(y)|K^\dagger(x-y)^2,\end{aligned}$$

the integral of which is smaller than $\|f\|^2_{L^r}\|K^\dagger\|_{L^{2q}}\mathbb{E}[\|\delta_W\|^{2s}_{L^s}]^{\frac{1}{2}}$ by (16), hence is finite. The dominated convergence thus applies and shows that $R^{\epsilon,\epsilon'} \underset{\epsilon,\epsilon'\to 0}{\longrightarrow} 0$, which concludes the proof of the proposition.

□

4.2. Uniform intergability: the case of L^∞ function.

Lemma 4.2. *For all $c < \frac{1}{2}$, there exists a constant C such that for all measurable subset A of $\mathbb{R}^2$ with finite area, for all $\epsilon, \epsilon' \in (0,1]$, for all $f \in L^\infty(\mathbb{R}^2)$*

$$\mathbb{E}[\bar{Y}^{\epsilon,\epsilon'}_{W,W'}(f\mathbb{1}_A)^2] \leq C\|f\|^2_\infty e^{-cd_A^2}\mu(A)^2(1\vee\log(\mu(A)^{-1}))^2.$$

Proof. In the event that either $\|W\|_\infty < d_A - K_\varphi$ or $\|W'\|_\infty < d_A - K_\varphi$, it holds that either $\mathbf{n}^{\epsilon}_{\bar{W}}(x) = 0$ for all $x \in A$, or $\mathbf{n}^{\epsilon'}_{\bar{W}}(x) = 0$ for all $xy \in A$, and in both case it follows that $\bar{Y}^{\epsilon,\epsilon'}_{W,W'}(A) = 0$.

In the complementary event, let τ (resp. τ') be the first time t when $|W_t| = d_A - K_\varphi$ (resp. $|W'_t| = \rho_A/2$), which are stopping times, or $\tau = \tau' = 0$ in the case $d_A \leq K_\varphi$. Let W_p be the restriction of W to $[0,\tau]$, and W_f be its restriction to $[\tau, 1]$. By strong Markov property and translation invariance of the winding function, we deduce that

$$\mathbf{n}^{\epsilon}_{\bar{W}} = \mathbf{n}^{\epsilon}_{\bar{W}_p} + \mathbf{n}^{\epsilon}_{\bar{W}_f} + \varphi^\epsilon \star T_{0,W_\tau,W_1},$$

where $T_{a,b,c}(x)$ is equal to 0 for x outside the triangle delimited by a, b, c, and equal to ± 1 otherwise, with the sign depending on the orientation of this triangle. In particular, $\|\varphi^\epsilon \star T_{a,b,c}\|_\infty \leq 1$ for any a, b, c . Since $\mathbf{n}^{\epsilon}_{\bar{W}_p}(x) = 0$ for all $x \in A$, we get for $x, y \in A$

$$\begin{aligned}\mathbb{E}[\mathbf{n}^{\epsilon}_{\bar{W}}(x)\mathbf{n}^{\epsilon}_{\bar{W}}(y)] =&\mathbb{P}(\tau\leq 1)\big(\mathbb{E}[\mathbf{n}^{\epsilon}_{\bar{W}_f}(x)\mathbf{n}^{\epsilon}_{\bar{W}_f}(y)|\tau\leq 1] + \mathbb{E}'[\varphi^\epsilon \star T_{0,W_\tau,W_1}(x)\mathbf{n}^{\epsilon}_{\bar{W}_f}(y)|\tau\leq 1]\\ &+ \mathbb{E}'[\mathbf{n}^{\epsilon}_{\bar{W}_f}(x)\varphi^\epsilon \star T_{0,W_\tau,W_1}(y)|\tau\leq 1] + \mathbb{E}'[\varphi^\epsilon \star T_{0,W_\tau,W_1}(x)\varphi^\epsilon \star T_{0,W_\tau,W_1}(y)|\tau\leq 1]\big)\end{aligned}$$

It follows that

$$\begin{aligned}\mathbb{E}[\bar{Y}^{\epsilon,\epsilon'}_{W,W'}(f\mathbb{1}_A)^2] =& \int_{A^2} f(x)f(y)\mathbb{E}[\mathbf{n}^{\epsilon}_{\bar{W}}(x)\mathbf{n}^{\epsilon}_{\bar{W}}(y)]^2\,\mathrm{d}x\,\mathrm{d}y\\ &\leq 4\|f\|^2_\infty\mathbb{P}(\tau\leq 1)^2 \sup_{\substack{t\in(0,1],\\ z\in\partial B_{d_A-K_\varphi}}} \int_{A^2} \big(\mathbb{E}_{t,z}[\mathbf{n}^{\epsilon}_{\bar{W}}(x)\mathbf{n}^{\epsilon}_{\bar{W}}(y)]^2 + \mathbb{E}_{t,z}[\varphi^\epsilon\star T_{0,z,W_t}(x)\mathbf{n}^{\epsilon}_{\bar{W}_f}(y)]^2\\ &+ \mathbb{E}_{t,z}[\mathbf{n}^{\epsilon}_{\bar{W}_f}(x)\varphi^\epsilon\star T_{0,z,W_t}(y)]^2 + \mathbb{E}_{t,z}[\varphi^\epsilon\star T_{0,z,W_t}(x)\varphi^\epsilon\star T_{0,z,W_t}(y)]^2\big)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

By scale invariance and translation invariance, and using Lemma 3.2, we get

$$\sup_{\substack{t\in(0,1],\\ z\in\partial B_{d_A-K_\varphi}}} |\mathbb{E}_{t,z}[\mathbf{n}^\epsilon_{\bar{W}}(x)\mathbf{n}^\epsilon_{\bar{W}}(y)]| \leq CK^\dagger(x-y).$$

Using Corollary 3.7, we deduce

$$\sup_{\substack{t\in(0,1],\\ z\in\partial B_{d_A-K_\varphi}}} \int_{A^2} \mathbb{E}_{t,z}[\mathbf{n}^\epsilon_{\bar{W}}(x)\mathbf{n}^\epsilon_{\bar{W}}(y)]^2 \,\mathrm{d}x\,\mathrm{d}y \leq \mu(A)\sup_{x\in A}\int_A K^\dagger(v+x)^2\,\mathrm{d}x \leq C\mu(A)^2(1\vee\log(\mu(A)^{-1}))^2.$$

Since $|T_{0,z,W_t}| \leq 1$, we also have

$$\sup_{\substack{t\in(0,1],\\ z\in\partial B_{d_A-K_\varphi}}} \int_{A^2} \mathbb{E}[\varphi^\epsilon\star T_{0,z,W_t}(x)\varphi^\epsilon\star T_{0,z,W_t}(y)]^2\,\mathrm{d}x\,\mathrm{d}y \leq \mu(A)^2.$$

The crossed terms requires to juggle a bit in order to uncouple the two functions without creating divergences:

$$\begin{aligned}
&\int_{A^2} \mathbb{E}_{t,z}[\mathbf{n}^\epsilon_{\bar{W}_f}(x)\varphi^\epsilon\star T_{0,z,W_t}(y)]^2\,\mathrm{d}x\mathrm{d}\ y\\
&=\int_{A^2}\mathbb{E}^{\otimes 2}_{t,z}[\mathbf{n}^\epsilon_{\bar{W}_f}(x)\varphi^\epsilon\star T_{0,z,W_t}(y)\mathbf{n}^\epsilon_{\bar{W}'_f}(x)\varphi^\epsilon\star T_{0,z,W'_t}(y)]\,\mathrm{d}x\,\mathrm{d}y\\
&=\mathbb{E}^{\otimes 2}_{t,z}\Big[\int_A \mathbf{n}^\epsilon_{\bar{W}_f}(x)\mathbf{n}^\epsilon_{\bar{W}'_f}(x)\,\mathrm{d}x\int_A\varphi^\epsilon\star T_{0,z,W_t}(y)\varphi^\epsilon\star T_{0,z,W'_t}(y)\,\mathrm{d}y\Big]\\
&\leq \mathbb{E}^{\otimes 2}_{t,z}\Big[\Big(\int_A \mathbf{n}^\epsilon_{\bar{W}_f}(x)\mathbf{n}^\epsilon_{\bar{W}'_f}(x)\,\mathrm{d}x\Big)^2\Big]^{\frac12}\mathbb{E}^{\otimes 2}_{t,z}\Big[\Big(\int_A\varphi^\epsilon\star T_{0,z,W_t}(y)\varphi^\epsilon\star T_{0,z,W'_t}(y)\,\mathrm{d}y\Big)^2\Big]^{\frac12}\\
&\leq C\mu(A)(1\vee\log(\mu(A)^{-1}))\mu(A).
\end{aligned}$$

We conclude the proof with a straightforward computation:

$$\begin{aligned}
\mathbb{P}(\tau\leq 1) &= \mathbb{P}(\|W\|_\infty \geq d_A-K_\varphi) \leq 2\mathbb{P}(\|W^1\|_\infty \geq \frac{d_A-K_\varphi}{\sqrt2}) \leq 4\mathbb{P}(\sup_{t\in[0,1]} W^1_t \geq \frac{d_A-K_\varphi}{\sqrt2})\\
&= 4\mathbb{P}(|W^1_1| \geq \frac{d_A-K_\varphi}{\sqrt2}) = 8\mathbb{P}(W^1_1\geq\frac{d_A-K_\varphi}{\sqrt2}) \leq \frac{4\sqrt2}{\pi(d_A-K_\varphi)}e^{-\frac{(d_A-K_\varphi)^2}{4}}.
\end{aligned}$$

Since we can also bound this probably by 1, we conclude that for any $c<\frac14$, for C large enough,

$$\mathbb{P}(\tau\leq 1)\leq Ce^{-cd_A^2}$$

here the choice $c<\frac14$ rather than $c=\frac14$ allow for the gaussian term to kill the exponential term which arise when we develop the square $(d_A-K_\varphi)^2$. □

Corollary 4.3. *For all $c<\frac14$, there exists a constant C such that for all measurable subsets A, A' of $\mathbb{R}^2$ with finite area, for all $\epsilon^1,\epsilon^2,\epsilon^3,\epsilon^4\in(0,1]$, for all $f,g\in L^\infty(\mathbb{R}^2)$*

$$\mathbb{E}[\bar{Y}^{\epsilon^1,\epsilon^2}_{W,W'}(f\mathbb{1}_A)\bar{Y}^{\epsilon^3,\epsilon^4}_{W,W'}(g\mathbb{1}_{A'})] \leq C\|f\|_\infty\|g\|_\infty e^{-c(d_A^2+d_{A'}^2)}\mu(A)\mu(A')(1\vee\log(\mu(A)^{-1}))(1\vee\log(\mu(A')^{-1})).$$

Proof. This follows directly from Hôlder inequality and Lemma 4.2. □

Lemma 4.4. *There exists a constant C such that for all measurable subset A of $\mathbb{R}^2$ with finite area and such that $d_A>K_\varphi$, for all $\epsilon,\epsilon'\in(0,1]$, for all $f\in L^\infty(\mathbb{R}^2)$*

$$\mathbb{E}[\hat{Y}^{\epsilon,\epsilon'}_{W,W'}(f\mathbb{1}_A)^2] \leq C\|f\|^2_\infty(1\wedge d_A^{-8})\mu(A)^2(1\vee\log(\mu(A)^{-1}))^2.$$

Proof. Using Lemma 4.2 and triangle inequality in $L^2(\Omega)$, it suffices to show that

$$\mathbb{E}\big[\big(\hat{Y}^{\epsilon,\epsilon'}_{W,W'}(f\mathbb{1}_A)-\bar{Y}^{\epsilon,\epsilon'}_{W,W'}(f\mathbb{1}_A)\big)^2\big]\leq C\|f\|^2_\infty\mu(A)^2(1\wedge d_A^{-8})\mu(A)^2(1\vee\log(\mu(A)^{-1}))^2.$$

The right-hand side is equal to

$$\mathbb{E}\Big[\Big(\int_A f(x)\big(\hat{\mathbf{n}}^\epsilon_W(x)\hat{\mathbf{n}}^{\epsilon'}_{W'}(x)-\mathbf{n}^\epsilon_{\bar{W}}(x)\mathbf{n}^{\epsilon'}_{\bar{W}'}(x)\big)\,\mathrm{d}x\Big)^2\Big],$$

which again we can bound by 3 times the sum

$$\mathbb{E}\Big[\Big(\int_A f(\delta_W-\delta_W^\epsilon)\mathbf{n}^{\epsilon'}_{\bar W'}\Big)^2\Big]+\mathbb{E}\Big[\Big(\int_A f\mathbf{n}^\epsilon_{\bar W}(\delta_{W'}-\delta^\epsilon_{W'})\Big)^2\Big]+\mathbb{E}\Big[\Big(\int_A f(\delta_W-\delta^\epsilon_W)(\delta_{W'}-\delta^\epsilon_{W'})\Big)^2\Big].$$

To control the last one, we use the third point in Lemma 3.13, which gives

$$\mathbb{E}\Big[\int_A f(\delta_W-\delta^\epsilon_W)^2\,\mathrm{d}x\Big]\leq C^2\|f\|^2_\infty\mu(A)^2(1\vee d_A^{-8}).$$

The first and second are symmetric so we only treat the first. Using Cauchy-Schwarz inequality, Lemma 3.13 and Lemma 4.2, we get

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_A f(\delta_W-\delta^\epsilon_W)\mathbf{n}^{\epsilon'}_{\bar W'}\Big)^2\Big]\\
&=\int_{A^2} f(x)f(y)\mathbb{E}[(\delta_W(x)-\delta^\epsilon_W(x))(\delta_W(y)-\delta^\epsilon_W(y))]\mathbb{E}[\mathbf{n}^{\epsilon'}_{\bar W'}(x)\mathbf{n}^{\epsilon'}_{\bar W'}(y)]\,\mathrm{d}x\,\mathrm{d}y\\
&\leq C\|f\|^2_\infty\Big(\int_{A^2}\mathbb{E}[(\delta_W(x)-\delta^\epsilon_W(x))(\delta_W(y)-\delta^\epsilon_W(y))]^2\,\mathrm{d}x\,\mathrm{d}y\Big)^{\frac12}\Big(\int_{A^2}\mathbb{E}[[\mathbf{n}^{\epsilon'}_{\bar W'}(x)\mathbf{n}^{\epsilon'}_{\bar W'}(y]^2\,\mathrm{d}x\,\mathrm{d}y\Big)^{\frac12}\\
&\leq C\|f\|^2_\infty(1\wedge d_A^{-4})e^{-cd_A^2}\mu(A)^2(1\vee\log(\mu(A)^{-1}))\\
&\leq C'\|f\|^2_\infty(1\wedge d_A^{-8})\mu(A)^2(1\vee\log(\mu(A)^{-1})),
\end{aligned}$$

which concludes the proof. □

Corollary 4.5. *For all $\delta>0$, there exists $R<\infty$ such that for all $f\in L^\infty(\mathbb{R}^2)$, for all $\epsilon,\epsilon'\in(0,1]$,*

$$\mathbb{E}[\hat Y^{\epsilon,\epsilon'}_{W,W'}(f\mathbb{1}_{\mathbb{R}^2\setminus B_R})^2]\leq\delta\|f\|^2_\infty.$$

and

$$\mathbb{E}[\bar Y^{\epsilon,\epsilon'}_{W,W'}(f\mathbb{1}_{\mathbb{R}^2\setminus B_R})^2]\leq\delta\|f\|^2_\infty.$$

Proof. First note that $\bar Y^{\epsilon,\epsilon'}_{W,W'}(f)$ and $\hat Y^{\epsilon,\epsilon'}_{W,W'}(f)$ are well-defined random variable for all $f\in L^\infty(\mathbb{R}^2)$: indeed both $\mathbf{n}^\epsilon_{\bar W}$ and $\mathbf{n}^\epsilon_{\bar W'}$ are compactly supported and bounded, so there product is integrable, which ensures that $\bar Y^{\epsilon,\epsilon'}_{W,W'}(f)$ is well-defined. As for the fact that $\hat Y^{\epsilon,\epsilon'}_{W,W'}(f)$ is well-defined, it then follows from the fact that for z large enough,

$$\mathbf{n}^\epsilon_{\hat W}(z)\mathbf{n}^\epsilon_{\hat W'}(z)-\mathbf{n}^\epsilon_{\bar W}(z)\mathbf{n}^\epsilon_{\bar W'}(z)=(\mathbf{n}^\epsilon_{\hat W}(z)-\mathbf{n}^\epsilon_{\bar W}(z))(\mathbf{n}^\epsilon_{\hat W'}(z)-\mathbf{n}^\epsilon_{\bar W'}(z))\leq C|z|^{-4},$$

for a constant C which depends on φ,ϵ,W,W'. This is enough to ensure integrability.

For $k\geq0$, set $A_k=B_{k+1}\setminus B_k$, and $f_k=f\mathbb{1}_{A_k}$, so that $f\mathbb{1}_{\mathbb{R}^2\setminus B_n}=\sum_{k=n}^\infty f_k$. Note $d_{A_k}=k$, and $E(A_k)=\mu(A_k)=2\pi(k+1/2)$.

Using the linearity of $\bar Y^{\epsilon,\epsilon'}_{W,W'}$ and Lemma 4.4, we deduce that

$$\begin{aligned}
\mathbb{E}[\bar Y^{\epsilon,\epsilon'}_{W,W'}(f\mathbb{1}_{\mathbb{R}^2\setminus B_n})^2]&\leq\sum_{j,k=n}^\infty\mathbb{E}[\bar Y^{\epsilon,\epsilon'}_{W,W'}(f_k)\bar Y^{\epsilon,\epsilon'}_{W,W'}(f_j)]\\
&\leq\sum_{j,k=n}^\infty\mathbb{E}[\bar Y^{\epsilon,\epsilon'}_{W,W'}(f_k)^2]^{\frac12}\mathbb{E}[\bar Y^{\epsilon,\epsilon'}_{W,W'}(f_j)^2]^{\frac12}\\
&\leq4\pi^2C\|f\|^2_\infty\sum_{k,j=n}^\infty(k+1/2)(j+1/2)k^{-4}j^{-4}.
\end{aligned}$$

The corollary thus follows from the fact this sum is finite and goes to 0 as $n\to\infty$. □

Theorem 4.6. *For all $f\in L^\infty(\mathbb{R}^2)$, both $\bar Y^{\epsilon,\epsilon'}_{W,W'}(f)$ and $\hat Y^{\epsilon,\epsilon'}_{W,W'}(f)$ converges in $L^2(\Omega)$, toward the same limit $\hat Y_{W,W'}(f)$.*

Proof. Let $f \in L^\infty(\mathbb{R}^2)$. Corollary 4.5 ensures that for all $\epsilon, \epsilon' \in (0,1)$, $\hat{Y}^{\epsilon,\epsilon'}_{W,W'}(f) \in L^2(\Omega)$, so we show that $\|\hat{Y}^{\epsilon^1,\epsilon^2}_{W,W'}(f) - \hat{Y}^{\epsilon^3,\epsilon^4}_{W,W'}(f)\|_{L^2(\Omega)}$ goes to 0 as the ϵ^i goes to 0. Let $\delta > 0$. By Corollary 4.5, there exists R such that for all $\epsilon, \epsilon' \in (0,1]$,

$$\|\hat{Y}^{\epsilon,\epsilon'}_{W,W'}(f \mathbb{1}_{\mathbb{R}^2 \setminus B_R})^2\|_{L^2(\Omega)} \leq \frac{\delta}{4}.$$

Since $f\mathbb{1}_{B_R} \in L^r(\mathbb{R}^2)$ for all $r \in (1,2)$, hence for some arbitrary such r, Proposition 4.1 ensures that $\hat{Y}^{\epsilon,\epsilon'}_{W,W'}(f\mathbb{1}_{B_R})$ converges in $L^2(\Omega)$ toward $Y_{W,W'}(f\mathbb{1}_{B_R})$. It follows that there exists ϵ_0 such that for all $\epsilon, \epsilon' in (0, \epsilon_0]$,

$$\|\hat{Y}^{\epsilon,\epsilon'}_{W,W'}(f\mathbb{1}_{B_R}) - Y_{W,W'}(f\mathbb{1}_{B_R})\|_{L^2(\Omega)} \leq \frac{\delta}{4}$$

Thus, for $\epsilon^1, \epsilon^2, \epsilon^3, \epsilon^4 \in (0, \epsilon_0]$,

$$\begin{aligned}\|\hat{Y}^{\epsilon^1,\epsilon^2}_{W,W'}(f) - \hat{Y}^{\epsilon^3,\epsilon^4}_{W,W'}(f)\|_{L^2(\Omega)} &\leq \|\hat{Y}^{\epsilon^1,\epsilon^2}_{W,W'}(f\mathbb{1}_{B_R}) - Y_{W,W'}(f\mathbb{1}_{B_R})\|_{L^2(\Omega)} + \|\hat{Y}^{\epsilon^1,\epsilon^2}_{W,W'}(f\mathbb{1}_{\mathbb{R}^2\setminus B_R})^2\|_{L^2(\Omega)} \\ &+ \|\hat{Y}^{\epsilon^3,\epsilon^4}_{W,W'}(f\mathbb{1}_{\mathbb{R}^2\setminus B_R})^2\|_{L^2(\Omega)} + \|\hat{Y}^{\epsilon^3,\epsilon^4}_{W,W'}(f\mathbb{1}_{B_R}) - Y_{W,W'}(f\mathbb{1}_{B_R})\|_{L^2(\Omega)} \leq \delta,\end{aligned}$$

which ensures indeed the convergence of $\hat{Y}^{\epsilon,\epsilon'}_{W,W'}(f)$. The convergence of $\bar{Y}^{\epsilon,\epsilon'}_{W,W'}(f)$ is shown identically. Furthermore, if the two a priori different limits are denoted $\hat{Y}_{W,W'}(f)$ and $\bar{Y}_{W,W'}(f)$, then it is easily checked that $\|\hat{Y}_{W,W'}(f) - \hat{Y}_{W,W'}(f\mathbb{1}_R)\|_{L^2(\Omega)}$ goes to 0 as $R \to \infty$, but also that $\|\bar{Y}_{W,W'}(f) - \hat{Y}_{W,W'}(f\mathbb{1}_R)\|_{L^2(\Omega)}$ goes to 0 as $R \to \infty$ (recall that the limits in Proposition 4.1 are identical), from which we conclude that $\hat{Y}_{W,W'}(f) = \bar{Y}_{W,W'}(f)$. ☐

5. Stretch-exponential moments

In this section we will prove that for all $f \in L^\infty_c(\mathbb{R}^2)$, the convergence of $Y^{\epsilon,\epsilon'}_{W,W'}(f)$ toward $Y_{W,W'}(f)$ holds with all strech-exponential moment up to order 1, i.e. that for all $\eta < 1$,

$$\mathbb{E}[\exp(|Y_{W,W'}(f)|^\eta)] < \infty \qquad \text{and} \qquad \mathbb{E}[\exp(|Y^{\epsilon,\epsilon'}_{W,W'}(f) - Y_{W,W'}(f)|^\eta)] \underset{\epsilon,\epsilon'\to 0}{\longrightarrow} 0.$$

Our overall strategy is rather standard. Since we have already shown the L^2 convergence of $Y^{\epsilon,\epsilon'}_{W,W'}(f)$, we only need to show the stretch-exponential moments are finite and bounded independently from ϵ, ϵ', i.e. that

$$\forall \eta < 1, f \in L^\infty_c(\mathbb{R}^2), \qquad \sup_{\epsilon,\epsilon'>0} \mathbb{E}[\exp(|Y^{\epsilon,\epsilon'}_{W,W'}(f)|^\eta)] < \infty.$$

To this end, we will show that for all $\delta > 0$ and $f \in L^\infty_c(\mathbb{R}^2)$, there exists a constant C such that

$$\forall k \in \mathbb{N}, \qquad \sup_{\epsilon,\epsilon'>0} \mathbb{E}[Y^{\epsilon,\epsilon'}_{W,W'}(f)^{2k}] \leq C((2k)!)^{1+\delta},$$

which is sufficient to conclude by choosing δ such that $(1+\delta)\eta < 1$.

Setting

$$Y^{\epsilon,\epsilon'}_{s,t}(f) := Y^{\epsilon,\epsilon'}_{W_{|[0,s]}, W'_{|[0,t]}}(f),$$

we will see that $t \mapsto Y^{\epsilon,\epsilon'}_{t,t}(f)$ is a continuous martingale. Through the Burkholder inequality, in its version with the constants depending explicitly on k, the moment of order $2k$ of $Y^{\epsilon,\epsilon'}_{W,W'}(f)$ is thus controlled by the moment of order k of the corresponding quadratic variation.

We write $\langle M \rangle_s$ the value at time s of the quadratic variation of a continuous semimartingale M.

For a positive integer k, we set ξ_k the optimal constant in the Burkholder inequality, i.e. the optimal constant such that for any continuous martingale M,

$$\mathbb{E}[M_t^k] \leq \xi_k^k \mathbb{E}[\langle M \rangle^{\frac{k}{2}}].$$

It has been shown in [2] that ξ_k is the value of the highest zero of the Hermite polynomial of order k. It is known that this value is asymptotically equivalent to $\sqrt{2k}$ as $k \to \infty$, and is in fact smaller than $\sqrt{2k}$ for all k (see e.g. Theorem 2, Equation (1.5) in [3] for a much more accurate estimation which implies this one).

Lemma 5.1. *Let $f \in L^\infty_c(\mathbb{R}^2)$. For all $t, \epsilon, \epsilon' \in (0,1]$, the random variable $Y^{\epsilon,\epsilon'}_{t,t}(f)$ is given by*

$$Y^{\epsilon,\epsilon'}_{t,t}(f) = \sum_{i=1}^{2} \int_0^t \Big(\int_{\mathbb{R}^2} \psi_i^{z,\epsilon}(W_s) \mathbf{n}^{\epsilon'}_{W',s}(z) f(z) \,\mathrm{d}z \Big) \,\mathrm{d}W^i_s + \Big(\int_{\mathbb{R}^2} \mathbf{n}^{\epsilon}_{W,s}(z) \psi_i^{z,\epsilon'}(W'_s) f(z) \,\mathrm{d}z \Big) \,\mathrm{d}W'^i_s.$$

In particular, $t \in (0,1] \mapsto Y^{\epsilon,\epsilon'}_{t,t}(f)$ is an L^2-bounded martingale. Its quadratic variation is given by

$$\begin{aligned}\langle Y^{\epsilon,\epsilon'}_{\cdot,\cdot}(f) \rangle_t = &\int_0^t \int_{(\mathbb{R}^2)^2} \mathbf{n}^{\epsilon}_{W,s}(z) \mathbf{n}^{\epsilon}_{W,s}(w) \langle \psi^{z,\epsilon'}(W'_s), \psi^{w,\epsilon'}(W'_s) \rangle f(z) \,\mathrm{d}z f(w) \,\mathrm{d}w \,\mathrm{d}s \\ &+ \int_0^t \int_{(\mathbb{R}^2)^2} \langle \psi^{z,\epsilon}(W_s), \psi^{w,\epsilon}(W_s) \rangle \mathbf{n}^{\epsilon'}_{W',s}(z) \mathbf{n}^{\epsilon'}_{W',s}(w) f(z) \,\mathrm{d}z f(w) \,\mathrm{d}w \,\mathrm{d}s.\end{aligned}$$

This is merely a question of swapping integrals, for which we will use the stochastic Fubini theorem given by [9, Theorem 2.2]. For convenience we simplify it our situation:

Theorem 5.2 (Special case of [9, Theorem 2.2])**.** *Let W be a Brownian motion with respect to some filtration $\mathcal{F}$, and μ a σ-finite measure on $\mathbb{R}^2$. Let $\psi : \mathbb{R}^2 \times [0,1] \times \Omega$ be $\mathcal{F}$-progressively measurable and such that almost surely,*

$$\int_{\mathbb{R}^2} \Big(\int_0^1 |\psi(x,t)|^2 \,\mathrm{d}t \Big)^{\frac{1}{2}} \,\mathrm{d}\mu < \infty.$$

Then, almost surely, for all $t \in [0,1]$,

$$\int_{\mathbb{R}^2} \int_0^t \psi(x,s) \,\mathrm{d}W_s \mu(\,\mathrm{d}x) = \int_0^t \int_{\mathbb{R}^2} \psi(x,s) \mu(\,\mathrm{d}x) \,\mathrm{d}W_s,$$

where both sides are measurable with respect to $\mathcal{F}_t$.

Proof of Lemma 5.1. For $t \in (0,1]$, let $\mathcal{F}_t$ be the σ-algebra generated by $\mathcal{F}_t(W) \cup \mathcal{F}_t(W')$, where $(\mathcal{F}_t(W'))_t$ (resp. $(\mathcal{F}_t(W))_t$) is the canonical filtration associated with W (resp. W'). Let $\mathcal{F} = (\mathcal{F}_t)_t$.

We apply the stochastic Fubini theorem to the process $\psi(z,t) := \psi_i^{z,\epsilon}(W_s) \mathbf{n}^{\epsilon'}_{W',s}(z)$, the measure $\mu := f \,\mathrm{d}z$, the Brownian motion $t \mapsto W^i_t$ and with the filtration $(\mathcal{F}_{s,u})_{u \in [0,1]}$. For all t, the random function $z \mapsto \psi(z,t)$ is continuous in z. For each z, it is progressively measurable in u. Since $\mathbb{R}^2$ is separable, it follows that $(z,t) \mapsto \psi(z,t)$ is progressively measurable (in the sense of [9]). The integrability condition necessary to apply Theorem 5.2 is

$$\int_{\mathbb{R}^2} \Big(\int_0^1 |\psi_i^{z,\epsilon}(W_s) \mathbf{n}^{\epsilon'}_{W',s}(z)|^2 \,\mathrm{d}t \Big)^{\frac{1}{2}} f(z) \,\mathrm{d}z < \infty,$$

which is easily checked since f is compactly supported and bounded, and $|\psi_i^{z,\epsilon}(W_s) \mathbf{n}^{\epsilon'}_{W',s}(z)|$ is bounded in s and z. It follows that almost surely, for all $t \in [0,1]$,

$$\int_0^t \Big(\int_{\mathbb{R}^2} \psi_i^{z,\epsilon}(W_s) \mathbf{n}^{\epsilon'}_{W',s}(z) f(z) \,\mathrm{d}z \Big) \,\mathrm{d}W^i_u = \int_{\mathbb{R}^2} \Big(\int_0^t \psi_i^{z,\epsilon}(W_s) \mathbf{n}^{\epsilon'}_{W',s}(z) \,\mathrm{d}W^i_u \Big) f(z) \,\mathrm{d}z, \qquad (17)$$

and both sides are measurable with respect to $\mathcal{F}_t$.

A similar computation, with the role of W and W' inverted, gives

$$\int_0^t \Big(\int_{\mathbb{R}^2} \mathbf{n}^{\epsilon}_{W,s}(z) \psi_i^{z,\epsilon'}(W'_s) f(z) \,\mathrm{d}z \Big) \,\mathrm{d}W'^i_u = \int_{\mathbb{R}^2} \Big(\int_0^t \mathbf{n}^{\epsilon}_{W,s}(z) \psi_i^{z,\epsilon'}(W'_s) \,\mathrm{d}W'^i_u \Big) f(z) \,\mathrm{d}z. \qquad (18)$$

By Ito's formula for a product of martingales, for all $z \in \mathbb{R}^2$, the martingale $M^z : t \mapsto \mathbf{n}^{\epsilon}_{W,t}(z) \mathbf{n}^{\epsilon'}_{W,t}(z)$ satisfies

$$M^z_t = \sum_{i=1}^{2} \int_0^t \big(\psi_i^{z,\epsilon}(W_s) \mathbf{n}^{\epsilon'}_{W',s}(z) \,\mathrm{d}W^i_s + \mathbf{n}^{\epsilon}_{W,s}(z) \psi_i^{z,\epsilon'}(W'_s) \,\mathrm{d}W'^i_s \big).$$

Integrating over z with respect to $f\,\mathrm{d}z$ gives

$$Y^{\epsilon,\epsilon'}_{t,t}(f)=\sum_{i=1}^{2}\Big(\int_{\mathbb{R}^2}\Big(\int_0^t\psi_i^{z,\epsilon}(W_s)\mathbf{n}^{\epsilon'}_{W',s}(z)\,\mathrm{d}W^i_u\Big)f(z)\,\mathrm{d}z+\int_{\mathbb{R}^2}\Big(\int_0^t\mathbf{n}^{\epsilon}_{W,s}(z)\psi_i^{z,\epsilon'}(W'_s)\,\mathrm{d}W'^i_u\Big)f(z)\,\mathrm{d}z\Big),$$

which together with (17) and (18) gives the equality we were searching for.

The second formula in the lemma follows easily:

$$\begin{aligned}\langle Y^{\epsilon,\epsilon'}_{\cdot,\cdot}(f)\rangle_t&=\sum_{j=1}^{2}\int_0^t\Big(\int_{\mathbb{R}^2}\mathbf{n}^{\epsilon}_{W,s}(x)\psi_j^{x,\epsilon'}(W'_s)f(x)\,\mathrm{d}x\Big)^2\mathrm{d}s+\sum_{i=1}^{2}\int_0^t\Big(\int_{\mathbb{R}^2}\psi_j^{x,\epsilon}(W_s)\mathbf{n}^{\epsilon'}_{W',s}(x)f(x)\,\mathrm{d}x\Big)^2\mathrm{d}s\\&=\int_0^t\int_{(\mathbb{R}^2)^2}\mathbf{n}^{\epsilon}_{W,s}(x)\mathbf{n}^{\epsilon}_{W,s}(y)\langle\psi^{x,\epsilon'}(W'_s)|\psi^{y,\epsilon'}(W'_s)\rangle f(x)f(y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\&+\int_0^t\int_{(\mathbb{R}^2)^2}\langle\psi^{x,\epsilon}(W_s)|\psi^{y,\epsilon}(W_s)\rangle\mathbf{n}^{\epsilon'}_{W',s}(x)\mathbf{n}^{\epsilon'}_{W',s}(y)f(x)f(y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s,\end{aligned}$$

which concludes the proof. □

Remark 10. The computation we do below actually shows that the integrability condition to apply Theorem 5.2 is in fact satisfied for all $f\in L^p(\mathbb{R}^2)$, and so Lemma 5.1, as well as Lemma 5.5 below, extend to such functions. We simply thought it would be odd to dive into estimations of a rather complicated functional without showing first that this functional is in fact the quadratic variation of the random variable we are interested in in the first place.

In order to simplify some expressions, we use the notations $t_0=0_{\mathbb{R}}$, and $z_0=0_{\mathbb{R}^2}$.

Lemma 5.3. *Let $q\in(1,2)$. There exists $C<\infty$ such that for all $\epsilon,\epsilon'\in(0,1]$, for all $k\in\mathbb{N}$, for all $0=t_0\le t_1\le\dots\le t_k\le 1$, for all $f\in L^q(\mathbb{R}^2)$,*

$$\int_{(\mathbb{R}^2)^{2k}}\mathbb{E}\Big[\prod_{i=1}^{k}\langle\psi^{x_i,\epsilon}(W_{t_i}),\psi^{y_i,\epsilon'}(W_{t_i})\rangle\Big]^2\Big(\prod_{i=1}^{k}f(x_i)f(y_i)\,\mathrm{d}x_i\,\mathrm{d}y_i\Big)\le C^k\|f\|^{2k}_{L^q(\mathbb{R}^2)}\prod_{i=1}^{k}(t_i-t_{i-1})^{1-\frac{q}{2(q-1)}}.$$

Proof. Let $p>2$ be such that $p^{-1}+q^{-1}=1$. For $s>0$, $a\in\mathbb{R}^2$, let

$$R_{s,a}(x,y,w)=\frac{p_s(0,w)}{|x-a-w||y-a-w|}f(x)f(y).$$

Then, by L^p-L^q inequality in $(\mathbb{R}^2)^2$,

$$\int_{\mathbb{R}^3}R_{s,a}(x,y,w)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}w=\int_{\mathbb{R}^2}K^*_s(x',y')f(x'+a)f(y'+a)\,\mathrm{d}x'\,\mathrm{d}y'\le\|K^*_s\|_{L^p((\mathbb{R}^2)^2)}\|f\|^2_{L^q(\mathbb{R}^2)}.$$

By disintegration, then Lemma 3.1, then making the change of variables $w_i=z_i-z_{i-1}$ and writing $a_i:=\sum_{j<i}w_j$ to shorten notations, we get

$$\begin{aligned}\mathbb{E}\Big[\prod_{i=1}^{k}\langle\psi^{x_i,\epsilon}(W_{t_i}),\psi^{y_i,\epsilon}(W_{t_i})\rangle\Big]&=\int_{(\mathbb{R}^2)^k}\prod_{i=1}^{k}\Big(p_{t_i-t_{i-1}}(z_{i-1},z_i)\langle\psi^{x_i,\epsilon}(z_i),\psi^{y_i,\epsilon}(z_i)\rangle\,\mathrm{d}z_i\Big)\\&\le C_\varphi^{2k}\int_{(\mathbb{R}^2)^k}\prod_{i=1}^{k}\Big(\frac{p_{t_i-t_{i-1}}(z_{i-1},z_i)}{|x_i-z_i||y_i-z_i|}\,\mathrm{d}z_i\Big)\\&=C_\varphi^{2k}\int_{(\mathbb{R}^2)^k}\prod_{i=1}^{k}\Big(\frac{p_{t_i-t_{i-1}}(0,w_i)}{|x_i-a_i-w_i||y_i-a_i-w_i|}\,\mathrm{d}w_i\Big),\end{aligned}$$

thus

$$\mathbb{E}\Big[\prod_{i=1}^{k}\langle\psi^{x_i,\epsilon}(W_{t_i}),\psi^{y_i,\epsilon}(W_{t_i})\rangle\Big]\Big(\prod_{i=1}^{k}f(x_i)f(y_i)\Big)\le C_\varphi^{2k}\int_{(\mathbb{R}^2)^k}\prod_{i=1}^{k}\Big(R_{t_i-t_{i-1},a_i}(x_i,y_i,w_i)\,\mathrm{d}w_i\Big).$$

Integrating over the x_i's and y_i's, we get

$$\begin{aligned}\int_{(\mathbb{R}^2)^{2k}} \mathbb{E}\Big[\prod_{i=1}^k \langle \psi^{x_i,\epsilon}(W_{t_i}), \psi^{y_i,\epsilon}(W_{t_i})\rangle\Big]\Big(\prod_{i=1}^k f(x_i)f(y_i)\,\mathrm{d}x_i\,\mathrm{d}y_i\Big)\\ \leq C_\varphi^{2k}\int_{(\mathbb{R}^2)^{3k}}\prod_{i=1}^k\Big(R_{t_i-t_{i-1},a_i}(x_i,y_i,w_i)\,\mathrm{d}x_i\,\mathrm{d}y_i\,\mathrm{d}w_i\Big)\\ \leq C_\varphi^{2k}\|f\|_{L^q(\mathbb{R}^2)}^{2k}\prod_{i=1}^k\|K^*_{t_i-t_{i-1}}\|_{L^p((\mathbb{R}^2)^2)}\\ \leq C_pC_\varphi^{2k}\|f\|_{L^q(\mathbb{R}^2)}^{2k}\prod_{i=1}^k(t_i-t_{i-1})^{1-\frac{p}{2}} \qquad \text{(by Lemma 3.9)}.\end{aligned}$$

which concludes the proof. □

For a non-negative integer k, define the tetrahedra

$$\mathbb{T}^k=\{(t_1,\dots,t_k)\in[0,1]^k : t_0\leq t_1\leq\cdots\leq t_k\},\quad\text{and}\quad \tilde{\mathbb{T}}^k=\{(u_1,\dots,u_k)\in[0,1]^k:\sum_{i=1}^k u_i\leq 1\}.$$

Lemma 5.4. *Let $s>-1$. There exists $C_s<\infty$ such that for all $k\in\mathbb{N}\setminus\{0\}$,*

$$\int_{\mathbb{T}^k}\prod_{i=1}^k(t_i-t_{i-1})^s\,\mathrm{d}t_1\dots\mathrm{d}t_k=\int_{\tilde{\mathbb{T}}^k}\prod_{i=1}^k u_i^s\,\mathrm{d}u_1\dots\mathrm{d}u_k=\frac{\Gamma(1+s)^k}{\Gamma(1+k(1+s))}\leq C_s^kk^{-k(1+s)}.$$

Proof. The first equality is a simple change of variable $u_i=t_i-t_{i-1}$.

For the bound, we set

$$f(k,t)\coloneqq\int_{t\tilde{\mathbb{T}}^k}\prod_{i=1}^k u_i^s\,\mathrm{d}u_1\dots\mathrm{d}u_k$$

for $t>0$ and $k\in\mathbb{N}$. The change of variables $u_i'=t^{-1}u_i$ gives $f(k,t)=t^{(1+s)k}f(k,1)$. Isolating the last variable $u=u_{k+1}$ in $\tilde{\mathbb{T}}^{k+1}$, we get

$$f(k+1,1)=\int_0^1 u^sf(k,1-u)\,\mathrm{d}u=f(k,1)\int_0^1u^s(1-u)^{(1+s)k}\,\mathrm{d}u=f(k,1)\frac{\Gamma(1+s)\Gamma(1+k(1+s)}{\Gamma(1+(k+1)(1+s))},$$

from which we deduce

$$f(k,1)=\prod_{j=0}^{k-1}\frac{\Gamma(1+s)\Gamma(1+j(1+s)}{\Gamma(1+(j+1)(1+s))}=\frac{\Gamma(1+s)^k}{\Gamma(1+k(1+s))}.$$

The upper bound at the end follows from Stirling's approximation. □

Lemma 5.5. *Let $q\in(\frac{6}{5},2)$. There exists $C<\infty$ such that for all $\epsilon,\epsilon'\in(0,1]$, for all $k\in\mathbb{N}$, for all $f\in L^\infty_c(\mathbb{R}^2)$,*

$$\mathbb{E}[Y^{\epsilon,\epsilon'}_{W,W'}(f)^{2k}]\leq C^k\|f\|^{2k}_{L^q(\mathbb{R}^2)}k^{k(1+\frac{q}{2(q-1)})}.$$

Proof. Using Lemma 5.1 to see $Y^{\epsilon,\epsilon'}_{W,W'}(f)$ as the value at time 1 of a martingale, and using Burkholder inequality, we get

$$\mathbb{E}[Y^{\epsilon,\epsilon'}_{W,W'}(f)^{2k}]\leq(2k)^k\mathbb{E}[\langle Y^{\epsilon,\epsilon'}_{\cdot,\cdot}(f)\rangle_1^k].$$

The quadratic variation on the right-hand side is given as the sum of two terms, symmetric with respect to each other,

$$\langle Y^{\epsilon,\epsilon'}_{\cdot,\cdot}(f)\rangle=A^{\epsilon,\epsilon'}_{W,W'}(f)+A^{\epsilon',\epsilon}_{W',W}(f),\qquad \mathbb{E}[Y^{\epsilon,\epsilon'}_{W,W'}(f)^{2k}]\leq 2^{k-1}(2k)^k(\mathbb{E}[A^{\epsilon,\epsilon'}_{W,W'}(f)^k]+\mathbb{E}[A^{\epsilon',\epsilon}_{W',W}(f)^k]).$$

This terms are given by Lemma 5.1. In particular,

$$\mathbb{E}[A^{\epsilon,\epsilon'}_{W,W'}(f)^k] = \mathbb{E}\Big[\Big(\int_0^1\int_{(\mathbb{R}^2)^2}\mathbf{n}^\epsilon_{W,s}(x)\mathbf{n}^\epsilon_{W,s}(y)\langle\psi^{x,\epsilon'}(W'_s),\psi^{y,\epsilon'}(W'_s)\rangle f(x)f(y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\Big)^k\Big]$$
$$= k!\int_{\mathbb{T}^k}\int_{(\mathbb{R}^2)^{2k}}\mathbb{E}\Big[\prod_{i=1}^k\mathbf{n}^\epsilon_{W,t_i}(x_i)\mathbf{n}^\epsilon_{W,t_i}(y_i)\Big]\mathbb{E}\Big[\prod_{i=1}^k\langle\psi^{x_i,\epsilon'}(W'_{t_i}),\psi^{y_i,\epsilon'}(W'_{t_i})\rangle\Big]$$
$$\prod_{i=1}^k(f(x_i)f(y_i)\,\mathrm{d}x_i\,\mathrm{d}y_i)\,\mathrm{d}t_1\dots\mathrm{d}t_k$$
$$\le k!\int_{\mathbb{T}^k}\Big(\int_{(\mathbb{R}^2)^{2k}}\mathbb{E}\Big[\prod_{i=1}^k\mathbf{n}^\epsilon_{W,t_i}(x_i)\mathbf{n}^\epsilon_{W,t_i}(y_i)\Big]^2\prod_{i=1}^k(f(x_i)f(y_i)\,\mathrm{d}x_i\,\mathrm{d}y_i)\Big)^{\frac12}$$
$$\Big(\int_{(\mathbb{R}^2)^{2k}}\mathbb{E}\Big[\prod_{i=1}^k\langle\psi^{x_i,\epsilon'}(W'_{t_i}),\psi^{y_i,\epsilon'}(W'_{t_i})\rangle\Big]^2\prod_{i=1}^k(f(x_i)f(y_i)\,\mathrm{d}x_i\,\mathrm{d}y_i)\Big)^{\frac12}\mathrm{d}t_1\dots\mathrm{d}t_k.$$

The second equality has been obtained by simply developing the power and then ordering the time variables (hence the factor $k!$). We already control the second part in this last equation with Lemma 5.3. For the first part, we remark that

$$\int_{(\mathbb{R}^2)^{2k}}\mathbb{E}\Big[\prod_{i=1}^k\mathbf{n}^\epsilon_{W,t_i}(x_i)\mathbf{n}^\epsilon_{W,t_i}(y_i)\Big]^2\prod_{i=1}^k(f(x_i)f(y_i)\,\mathrm{d}x_i\,\mathrm{d}y_i) = \mathbb{E}[\prod_{i=1}^k Y^{\epsilon,\epsilon}_{t_i,t_i}(f)^2] \le \prod_{i=1}^k\mathbb{E}[Y^{\epsilon,\epsilon}_{t_i,t_i}(f)^{2k}]^{\frac1k}$$
$$\le \mathbb{E}[Y^{\epsilon,\epsilon}_{W,W'}(f)^{2k}].$$

The last inequality follows for example from the fact $t\mapsto Y^{\epsilon,\epsilon}_{t,t}(f)$ is a martingale. Using now Lemma 5.3, we end up with

$$\mathbb{E}[A^{\epsilon,\epsilon'}_{W,W'}(f)^k] \le k!\int_{\mathbb{T}^k}C^k\|f\|^k_{L^q(\mathbb{R}^2)}\prod_{i=1}^k(t_i-t_{i-1})^{\frac12-\frac{q}{4(q-1)}}\mathbb{E}[Y^{\epsilon,\epsilon}_{W,W'}(f)^{2k}]^{\frac12}. \tag{19}$$

Applying this to $\epsilon=\epsilon'$, we get

$$\mathbb{E}[Y^{\epsilon,\epsilon}_{W,W'}(f)^{2k}] \le (4C)^k\|f\|^k_{L^q(\mathbb{R}^2)}k^k(k!)\mathbb{E}[Y^{\epsilon,\epsilon}_{W,W'}(f)^{2k}]^{\frac12}\int_{\mathbb{T}^k}\prod_{i=1}^k(t_i-t_{i-1})^{\frac12-\frac{q}{4(q-1)}}\,\mathrm{d}t_1\dots\mathrm{d}t_k$$
$$\le C_q^k\|f\|^k_{L^q(\mathbb{R}^2)}k^kk!\mathbb{E}[Y^{\epsilon,\epsilon}_{W,W'}(f)^{2k}]^{\frac12}k^{-k(\frac32-\frac{q}{4(q-1)})}\qquad\text{(by Lemma 5.4)}.$$

and it follows that

$$\mathbb{E}[Y^{\epsilon,\epsilon}_{W,W'}(f)^{2k}] \le {C'_q}^k\|f\|^{2k}_{L^q(\mathbb{R}^2)}k^{k(1+\frac{q}{2(q-1)})}.$$

We can now plug this bound (and the identical one with both ϵ replaced by ϵ') into (19), and deduce that $\mathbb{E}[Y^{\epsilon,\epsilon'}_{W,W'}(f)^{2k}]$ satisfies the same type of estimation (with now $\epsilon\neq\epsilon'$). □

Corollary 5.6. *For all $\eta<1$ and $f\in L^\infty_c(\mathbb{R}^2)$,*

$$\mathbb{E}[\exp(|Y_{W,W'}(f)|^\eta)]<\infty\qquad\textit{and}\qquad\mathbb{E}[\exp(|Y^{\epsilon,\epsilon'}_{W,W'}(f)-Y_{W,W'}(f)|^\eta)]\underset{\epsilon,\epsilon'\to0}{\longrightarrow}0.$$

Proof. Since we have proved that $Y^{\epsilon,\epsilon'}_{W,W'}(f)$ converges in $L^2(\Omega)$ toward $Y^{\epsilon,\epsilon'}_{W,W'}(f)$, it suffices to show that

$$\sup_{\epsilon,\epsilon'\in(0,1]}\mathbb{E}[\exp(|Y^{\epsilon,\epsilon'}_{W,W'}(f)|^\eta)]<\infty$$

for all η,f. It then suffices to expand the exponential into a sum, swap the sum with the expectation, and use the estimation provided by Lemma 5.5 for the polynomial moments, taking $q<2$ sufficiently close to 2 so that

$$1+\frac{q}{2(q-1)}<\frac{2}{\eta},$$

which is possible for $\eta<1$. □

6. Convergence for the recentred variable X

In this section we study the random variables

$$X_W^\epsilon := \int_{\mathbb{R}^2} \hat{\mathbf{n}}_W^\epsilon(z)^2 \,\mathrm{d}z \qquad \text{and} \qquad \tilde{X}_W^\epsilon := X_W^\epsilon - \mathbb{E}[X_W^\epsilon].$$

We will show that the latter converges in $L^2(\Omega)$, as $\epsilon \to 0$, with a limit independent of the initial mollifier. We also make an asymptotical estimation of $\mathbb{E}[X_W^\epsilon]$.

To this end, we will decompose X_W^ϵ in a way reminiscent of Paul Lévy's construction of the Lévy area or Varadhan's construction of the renormalised self-intersection local time. Because of the infrared divergences, we must work with the integer-valued winding functions $\hat{\mathbf{n}}$ rather than $\mathbf{n}$. Since these functions $\hat{\mathbf{n}}$ are not *exactly* additive under concatenation of curves (we must add a part corresponding to the triangle between the 3 endpoints of the 2 concatenated curves), the iterative decomposition of X_W^ϵ contains more terms than the iterative decomposition of the Lévy area or the renormalised self-intersection local time: these are the terms $T_{k,j}$ and $Z_{k,j}^\epsilon$ we will soon define.

Lemma 6.1. *For all $\epsilon > 0$, the random variable X_W^ϵ is well-defined and finite almost surely, and lies in $L^2(\Omega)$.*

Proof. The fact it is well-defined and finite is given by Corollary 2.3. Its $L^2(\Omega)$ norm is equal to

$$\int_{(\mathbb{R}^2)^2} \mathbb{E}[\hat{\mathbf{n}}^\epsilon(z)^2 \hat{\mathbf{n}}^\epsilon(w)^2] \,\mathrm{d}z \,\mathrm{d}w.$$

By Burkholder inequality and Ito isometry, we get

$$\mathbb{E}[\mathbf{n}_W^\epsilon(z)^4] \leq C\mathbb{E}\Big[\Big(\int_0^1 |\psi_{W_t}^{x,\epsilon}|^2 \,\mathrm{d}t\Big)^2\Big]$$

Using Hardy–Littlewood inequality, and Lemma 3.1, Item (3), we deduce that for a constant C which only depends on the mollifier ϕ

$$\begin{aligned}\int_0^t \int_{\mathbb{R}^2} p_u(z,w) |\psi_w^{x,\epsilon}|^2 \,\mathrm{d}w \,\mathrm{d}u &\leq C \int_0^t \int_{\mathbb{R}^2} \frac{p_u(0,w)}{\epsilon^2 + |w|^2} \,\mathrm{d}w \,\mathrm{d}u \\ &= \frac{C}{2} \int_0^{2\epsilon^{-2}t} \frac{e^{\frac{1}{v}}\Gamma(0,\frac{1}{v})}{v} \,\mathrm{d}v.\end{aligned}$$

The integrand is smaller than 1 on $\mathbb{R}^+$, and thus

$$\begin{aligned}\mathbb{E}_0[\mathbf{n}_W^\epsilon(x)^4] &\leq 2C \int_0^1 \int_0^t \mathbb{E}\Big[|\psi_{W_s}^{x,\epsilon}|^2 |\psi_{W_t}^{x,\epsilon}|^2\Big] \,\mathrm{d}s \,\mathrm{d}t \\ &\leq C' \int_0^{2\epsilon^{-2}} \int_0^{2\epsilon^{-2}t} \frac{e^{\frac{1}{s}}\Gamma(0,\frac{1}{s})}{s} \frac{e^{\frac{1}{t}}\Gamma(0,\frac{1}{t})}{t} \,\mathrm{d}s \,\mathrm{d}t \\ &\leq C''\epsilon^{-4}.\end{aligned}$$

It follows that $\mathbb{E}_0[\hat{\mathbf{n}}_W^\epsilon(x)^4] \leq C^{(3)}(1+\epsilon^{-4})$, for a constant $C^{(3)}$ independent from x, ensuring that for any given compact K,

$$\int_{K^2} \mathbb{E}[\hat{\mathbf{n}}^\epsilon(z)^2 \hat{\mathbf{n}}^\epsilon(w)^2] \,\mathrm{d}z \,\mathrm{d}w < C_\epsilon \operatorname{vol}(K)^2.$$

One passes to $(\mathbb{R}^2)^2$ with technics similar to the one we have used to prove Lemma 3.17. □

Remark 11. With a lot of refinement revolving around the previous computation, it is in fact possible to show that

$$\mathbb{E}[(X_W^\epsilon)^2] \leq \frac{\log(\epsilon^{-1})^2}{4\pi^2}(1+o(1)),$$

and even to estimate the error term rather nicely, which would be sufficient (together we first moment estimation) to show that $X_W^\epsilon \sim \frac{\log(\epsilon^{-1})}{2\pi}$. However, the Varadhan-type decomposition of X_W^ϵ we are about to prove gives in fact an even better estimation.

6.1. **Iterative decomposition of X^ϵ_W.** We use a parameter s which always varies in $[1,2]$. We define $\mathbb{M}$ the set of couples (k,j), where k is a non-negative integer, j is a positive integer and $j\leq 2^k$. For such a couple, we set $W_{k,j}$ the restriction of W to $[2^{-k}(j-1),2^{-k}j]$. For all $(k,j)\in\mathbb{M}$, $W_{k,j}$ is equal to the concatenation of $W_{k+1,2j-1}$ and $W_{k+1,2j}$.

For four points $x,y,z,o\in\mathbb{R}^2$, let $\delta_{x,y,z}(o)\in\{-2\pi,0,2\pi\}$ be defined by

$$\delta_{x,y,z}(o) \coloneqq \widehat{xoy}+\widehat{yoz}+\widehat{zox},$$

or equivalently

$$\widehat{xoz}=\widehat{xoy}+\widehat{yoz}-\delta_{x,y,z}(o)$$

Let then $\delta_{k,j}$ be the function

$$\delta_{k,j}\coloneqq \delta_{W_{2^{-k}(j-1)},W_{2^{-k-1}(2j-1)},W_{2^{-k}j}}.$$

We then define

$$\begin{aligned}
X^\epsilon_{k,j} &\coloneqq \int_{\mathbb{R}^2}\hat{\mathbf{n}}^\epsilon_{W_{k,j}}(z)^2\,\mathrm{d}z, \qquad X^\epsilon_W = X^\epsilon_{0,1}\\
Y^\epsilon_{k,j} &\coloneqq \hat{Y}^{\epsilon,\epsilon}_{W_{k+1,2j-1},W_{k+1,2j}}(1)=\int_{\mathbb{R}^2}\hat{\mathbf{n}}^\epsilon_{W_{k+1,2j-1}}(z)\hat{\mathbf{n}}^\epsilon_{W_{k+1,2j}}(z)\,\mathrm{d}z,\\
T_{k,j} &\coloneqq \int_{\mathbb{R}^2}\delta_{k,j}(z)^2\,\mathrm{d}z,\\
Z^{1,\epsilon}_{k,j} &\coloneqq \int_{\mathbb{R}^2}\delta_{k,j}(z)\hat{\mathbf{n}}^\epsilon_{W_{k+1,2j-1}}(z)\,\mathrm{d}z, \qquad Z^{2,\epsilon}_{k,j}\coloneqq\int_{\mathbb{R}^2}\delta_{k,j}(z)\hat{\mathbf{n}}^\epsilon_{W_{k+1,2j}}(z)\,\mathrm{d}z,\\
Y^{++,\epsilon}_{k,j} &\coloneqq \int_{\mathbb{R}^2}|\hat{\mathbf{n}}^\epsilon_{W_{k+1,2j-1}}(z)\hat{\mathbf{n}}^\epsilon_{W_{k+1,2j}}(z)|\,\mathrm{d}z,\\
Z^{++,\epsilon}_{k,j} &\coloneqq \int_{\mathbb{R}^2}|\delta_{k,j}(z)(\hat{\mathbf{n}}^\epsilon_{W_{k+1,2j-1}}(z)+\hat{\mathbf{n}}^\epsilon_{W_{k+1,2j}}(z))|\,\mathrm{d}z,\\
Z^\epsilon_{k,j} &\coloneqq Z^{1,\epsilon}_{k,j}+Z^{2,\epsilon}_{k,j}, \qquad Z^{1,\epsilon}_W=Z^{1,\epsilon}_{0,1}, \qquad Z^{2,\epsilon}_W=Z^{2,\epsilon}_{0,1}, \qquad Z^\epsilon_W=Z^\epsilon_{0,1}.
\end{aligned}$$

For each of these quantities $A_{k,j}$, we also set $\tilde{A}_{k,j}=A_{k,j}-\mathbb{E}[A_{k,j}]$.

Lemma 6.2. *For all $\epsilon>0$ and $(k,j)\in\mathbb{M}$, all these quantities are well-defined, and lies in $L^2(\Omega)$.*

Proof. The fact $T_{k,j}$ is well-defined is clear. It lies in $L^2(\Omega)$ by subexponential decay, as $|T_{k,j}|\leq |N_1||N_2|$ for two Gaussian random variables N_1,N_2.

The fact $X^\epsilon_{k,j}$ is well-defined and in $L^2(\Omega)$ follows from Lemma 6.1 by a scaling argument.

The fact $Y^{++,\epsilon}_{k,j}$ and $Z^{++,\epsilon}_{k,j}$ are well-defined and in $L^2(\Omega)$ then follows from Cauchy–Schwarz inequality in $L^2(\mathbb{R}^2\times\Omega)$:

$$\mathbb{E}[(Y^{++,\epsilon}_{k,j})^2]\leq \mathbb{E}[(X^\epsilon_{k+1,2j-1})^2]^{\frac12}\mathbb{E}[(X^\epsilon_{k+1,2j})^2]^{\frac12}<\infty,$$

$$\mathbb{E}[(Z^{++,\epsilon}_{k,j})^2]\leq 2\mathbb{E}[T^2_{k,j}]^{\frac12}(\mathbb{E}[(X^\epsilon_{k+1,2j-1})^2]^{\frac12}+\mathbb{E}[(X^\epsilon_{k+1,2j})^2]^{\frac12})<\infty.$$

The fact $Y^\epsilon_{k,j}$ and $Z^\epsilon_{k,j}$ are well-defined and in $L^2(\Omega)$ follows directly. □

The point of these definitions is that the relation of concatenation $W_{k,j}=W_{k+1,2j-1}\cdot W_{k+1,2j}$, gives the pointwise equality

$$\hat{\mathbf{n}}^\epsilon_{W_{k,j}}(z)=\hat{\mathbf{n}}^\epsilon_{W_{k+1,2j-1}}(z)+\hat{\mathbf{n}}^\epsilon_{W_{k+1,2j}}(z)-\delta_{k,j},$$

from which it follows that

$$X^\epsilon_{k,j}=X^\epsilon_{k+1,2j-1}+X^\epsilon_{k+1,2j}+2Y^\epsilon_{k,j}+T_{k,j}-2Z^\epsilon_{k,j}.$$

Iterating over k, we deduce that for any integer n,

$$X^\epsilon_W=\sum_{j=1}^{2^n}X^\epsilon_{n,j}+\sum_{k=0}^{n-1}\sum_{j=1}^{2^k}T_{k,j}+2\sum_{k=0}^{n-1}\sum_{j=1}^{2^k}Y^\epsilon_{k,j}-2\sum_{k=0}^{n-1}\sum_{j=1}^{2^k}Z^\epsilon_{k,j}. \tag{20}$$

In the following, we use this decomposition with $n \propto \log(\epsilon^{-1})$ in order to deduce the L^2 convergence of X_W^ϵ. As we will now prove, the contribution of the first sum will vanish in the limit $\epsilon \to 0$, but the other three do contribute to the limit.

6.2. L^2 **convergence.**

Lemma 6.3. *As* $n \to \infty$,

$$\sup_{s\in[1,2]} \Big\| \sum_{j=1}^{2^n} \tilde{X}_{n,j}^{s2^{-n/2}} \Big\|_{L^2(\mathbb{P})} \longrightarrow 0.$$

Furthermore, $\sum_{k=0}^{n-1} \sum_{j=1}^{2^n} \tilde{T}_{k,j}$ *converges in* $L^2(\mathbb{P})$.

Proof. By Brownian scaling, we easily obtain the equality in distribution

$$X_{n,j}^\epsilon = 2^{-n} X_W^{2^{n/2}\epsilon}.$$

Taking $\epsilon = s2^{-n/2}$, we get in particular $\operatorname{Var}(X_{n,j}^{s2^{-n/2}}) = 2^{-2n} \operatorname{Var}(X_W^s) < \infty$. Furthermore, by Markov property of the Brownian motion, the family $(X_{n,j}^{s2^{-n/2}})_j$ is a family of i.i.d. random variables which varies continuously in s. Setting $\tilde{X}_n^s := \sum_{j=1}^{2^n} \tilde{X}_{n,j}^{s2^{-n/2}}$, we deduce that,

$$\sup_{s\in[1,2]} \operatorname{Var}(\tilde{X}_n^s) = O(2^{-n}),$$

hence the first point.

As for the second point, let $T_k := \sum_{j=1}^{2^k} T_{k,j}$. We similarly obtain that for any given k, the family $(T_{k,j})_j$ is a family of i.i.d. random variables, with $T_{k,j} \overset{(d)}{=} 2^{-k} T_{0,1}$, hence

$$\operatorname{Var}\big(T_k\big) \le C2^{-k}.$$

Since this is summable over k, we deduce the second convergence. □

Lemma 6.4. *For all* $\epsilon > 0$ *and* $(k,j) \in \mathbb{M}$, $\mathbb{E}[Y_{k,j}^\epsilon] = 0$.

Proof. Let

$$f(x) := \hat{\mathbf{n}}_{W_{k+1,2j-1}}^\epsilon(x) \hat{\mathbf{n}}_{W_{k+1,2j}}^\epsilon(x).$$

Since $Y_{k,j}^{++,\epsilon}$ is in $L^1(\Omega)$, we can apply Fubini's theorem,

$$\mathbb{E}[Y_{k,j}^\epsilon] = \int_{\mathbb{R}^2} \mathbb{E}[f(z)]\, \mathrm{d}z,$$

so it suffices indeed to show that for all $x \in \mathbb{R}^2$, $f(x) \overset{(d)}{=} -f(x)$. Assume $(k,j) = (0,1)$ for simplicity. Otherwise, replace the times $0, \frac{1}{2}, 1$ respectively with $2^{-(k+1)}(2j-2)$, $2^{-(k+1)}(2j-1)$ and $2^{-(k+1)}(2j)$. It suffices then to apply the reflection principle to the Brownian motion $\tilde{W} : t \in [0,1/2] \mapsto W(1/2+t)$, with reflection on the axis $(\tilde{W}(1/2), x)$. Letting $\tilde{W}^\dagger$ be the reflected Brownian motion, we have $(W_{1,1}, \tilde{W}) \overset{(d)}{=} (W_{1,1}, \tilde{W}^\dagger)$, but $\hat{\mathbf{n}}_{\tilde{W}^\dagger}^\epsilon(x) = -\hat{\mathbf{n}}_{\tilde{W}}^\epsilon(x)$, which suffices to conclude. □

Remark 12. Let us emphasize that the symmetry $f(x) \overset{(d)}{=} -f(x)$ is true for each x individually but **not** jointly. The couple $(f(x), f(y))$, for example, is not symmetric in distribution when $x \neq y$. Consequently, the random variables $Y_{k,j}^\epsilon$ themself are **not** symmetric in distribution.

By Theorem 4.6, for any fixed $(k,j) \in \mathbb{M}$, the random variable $Y_{k,j}^\epsilon$ converges in $L^2(\mathbb{P})$, as $\epsilon \to 0$, and we define $Y_{k,j}$ as the limit, which has a vanishing mean. Applying translation and scale invariance, as well as the Markov property, as fixed $\epsilon > 0$ before taking the limit, we deduce that the family $(Y_{k,j})_j$ is also a family of i.i.d. random variables, for each fixed k, with $Y_{k,j} \overset{(d)}{=} 2^{-k} Y_{0,1}$, and it follows from summability of the variances that the sum $Y := \sum_{k=0}^\infty \sum_{j=1}^{2^k} Y_{k,j}$ is well-defined in $L^2(\mathbb{P})$.

Lemma 6.5. *As $n \to \infty$, $Y_n^s := \sum_{k=0}^{n-1} \sum_{j=1}^{2^k} Y_{k,j}^{s2^{-n/2}}$ converges toward Y, and the convergence is uniform over $s \in [1,2]$ in the sense that*

$$\sup_{s\in[1,2]} \|Y_n^s - Y\|_{L^2(\mathbb{P})} \underset{n\to\infty}{\longrightarrow} 0.$$

Proof. The convergence in $L^2(\mathbb{P})$ of $Y_{k,j}^\epsilon$ toward $Y_{k,j}$ as $\epsilon \to 0$ is equivalent to the convergence in $L^2(\mathbb{P})$, uniformly over $s \in [1,2]$, of $Y_{k,j}^{s2^{-n/2}}$ toward $Y_{k,j}$ as $n \to \infty$.

Hence, for any given integer n_0, uniformly over $s \in [0,1]$, as $n \to \infty$, the finite sum $\sum_{k=0}^{n_0-1} \sum_{j=1}^{2^k} Y_{k,j}^{s2^{-n/2}}$ converges in $L^2(\mathbb{P})$ toward $\sum_{k=0}^{n_0-1} \sum_{j=1}^{2^k} Y_{k,j}$.

It only remains to show that for all $\delta > 0$, there exists an integer n_0 such that for all $s \in [1,2]$ and for all $n \geq n_0$,

$$\| \sum_{k=n_0}^{n-1} \sum_{j=1}^{2^k} Y_{k,j}^{s2^{-n/2}} \|_{L^2(\mathbb{P})}^2 \leq \delta.$$

As $Y_{W,W'}^{\epsilon,\epsilon'}$ converges in $L^2(\mathbb{P})$ when $\epsilon,\epsilon' \to 0$, the supremum C of $\|Y_{W,W'}^{\epsilon,\epsilon'}\|_{L^2(\mathbb{P})}$ over $\epsilon,\epsilon' \in (0,2]$ is finite. Using the independence between the $Y_{k,j}^{s2^{-n/2}}$ (where k is fixed and smaller than n), and using the fact the $Y_{k,j}^{s2^{-n/2}}$ are centred, we deduce

$$\mathbb{E}\Big[\Big(\sum_{j=1}^{2^k} Y_{k,j}^{s2^{-n/2}}\Big)^2\Big] \leq \sum_{j=1}^{2^k} \mathbb{E}[(Y_{k,j}^{s2^{-n/2}})^2] \leq C^2 2^{-k}.$$

Hence, for n_0 large enough, it does hold that

$$\| \sum_{k=n_0}^{n-1} \sum_{j=1}^{2^k} Y_{k,j}^{s2^{-n/2}} \|_{L^2(\mathbb{P})}^2 \leq C2^{-n_0} \leq \delta,$$

which concludes the proof. □

Lemma 6.6. *The random variables $Z_W^{1,\epsilon}$ and $Z_W^{2,\epsilon}$ converge in $L^2(\mathbb{P})$ as $\epsilon \to 0$. Hence, Z_W^ϵ also converges in $L^2(\mathbb{P})$ as $\epsilon \to 0$.*

Proof. First we consider $Z_W'^\epsilon := \int_{\mathbb{R}^2} \delta_{0,1}(x) \hat{\mathbf{n}}_{W_{[0,1/4]}}^\epsilon(x)\,\mathrm{d}x$. By disintegration with respect to $W_{1/4}$, $W_{1/2}$ and W_1, and setting

$$f_W^{\epsilon,\epsilon'}(x,y) := (\hat{\mathbf{n}}_{W_{[0,1/4]}}^\epsilon(x) - \hat{\mathbf{n}}_{W_{[0,1/4]}}^{\epsilon'}(x))(\hat{\mathbf{n}}_{W_{[0,1/4]}}^\epsilon(y) - \hat{\mathbf{n}}_{W_{[0,1/4]}}^{\epsilon'}(y)),$$

and $q(z,W) := p_{1/4}(0,W_{1/4})p_{1/4}(W_{1/4},z)$, we get, for an arbitrary positive measurable function g which we will take to be $g(z) := p_{\frac{1}{4}}(0,\frac{z}{2})$,

$$\begin{aligned}
\mathbb{E}[(Z_W'^\epsilon - Z_W'^{\epsilon'})^2] &= \int_{(\mathbb{R}^2)^4} \delta_{0,z,w}(x)\delta_{0,z,w}(y)p_{1/2}(0,z)p_{1/2}(z,w)\mathbb{E}_{1/2,0,z}[f^{\epsilon,\epsilon'}(x,y)]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w \\
&= \int_{(\mathbb{R}^2)^4} \delta_{0,z,w}(x)\delta_{0,z,w}(y)p_{1/2}(z,w)\,\mathrm{d}w\mathbb{E}[q(z,W)f^{\epsilon,\epsilon'}(x,y)]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&\leq \Big(\int_{(\mathbb{R}^2)^3} g(z)\Big(\int_{\mathbb{R}^2} \delta_{0,z,w}(x)\delta_{0,z,w}(y)p_{1/2}(z,w)\,\mathrm{d}w\Big)^2 \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\Big)^{\frac{1}{2}} \\
&\times \Big(\int_{(\mathbb{R}^2)^3} g(z)^{-1}\mathbb{E}[q(z,W)f^{\epsilon,\epsilon'}(x,y)]^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\Big)^{\frac{1}{2}}.
\end{aligned}$$

We recognize

$$\begin{aligned}
&\int_{(\mathbb{R}^2)^2} \mathbb{E}[q(z,W)f^{\epsilon,\epsilon'}(x,y)]^2 \,\mathrm{d}x\,\mathrm{d}y \\
&= \mathbb{E}^{\otimes 2}[q(z,W)q(z,W')\Big(\int_{(\mathbb{R}^2)^2} (\hat{\mathbf{n}}^{\epsilon}_{W_{[0,\frac14]}}(x) - \hat{\mathbf{n}}^{\epsilon'}_{W_{[0,\frac14]}}(x))(\hat{\mathbf{n}}^{\epsilon}_{W'_{[0,\frac14]}}(x) - \hat{\mathbf{n}}^{\epsilon'}_{W'_{[0,\frac14]}}(x))\,\mathrm{d}x\Big)^2] \\
&= \mathbb{E}^{\otimes 2}[q(z,W)q(z,W')(\hat{Y}^{\epsilon,\epsilon}_{W,W'}(1) - \hat{Y}^{\epsilon',\epsilon}_{W,W'}(1) - \hat{Y}^{\epsilon,\epsilon'}_{W,W'}(1) + \hat{Y}^{\epsilon',\epsilon'}_{W,W'}(1))] \\
&\leq \sup_{w\in\mathbb{R}^2} q(z,w)^2 E_{\epsilon,\epsilon'}, \qquad E_{\epsilon,\epsilon'} := \mathbb{E}^{\otimes 2}[|(\hat{Y}^{\epsilon,\epsilon}_{W,W'}(1) - \hat{Y}^{\epsilon',\epsilon}_{W,W'}(1) - \hat{Y}^{\epsilon,\epsilon'}_{W,W'}(1) + \hat{Y}^{\epsilon',\epsilon'}_{W,W'}(1)|].
\end{aligned}$$

The expectation $E_{\epsilon,\epsilon'}$ goes to 0 as $\epsilon,\epsilon'\to 0$, by L^2 convergence of $\hat{Y}^{\epsilon',\epsilon}_{W,W'}(1)$.

The supremum of $q(z,w)$ over w is achieved for $w = z/2$, as it is seen by noting that $|w|^2 + |w-z|^2 = |z|^2/2 + 2|w - z/2|^2$, and is equal to $p_{\frac14}(0,\frac{z}{2})^2$. It follows that

$$\int_{(\mathbb{R}^2)^3} g(z)^{-1}\mathbb{E}[q(z,W)f^{\epsilon,\epsilon'}(x,y)]^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \leq E_{\epsilon,\epsilon'}\int_{\mathbb{R}^2} p_{\frac14}(0,\frac{z}{2})\,\mathrm{d}z \underset{\epsilon,\epsilon'\to 0}{\longrightarrow} 0.$$

On the other hand,

$$\begin{aligned}
C &:= \int_{(\mathbb{R}^2)^3} p_{\frac14}(0,\tfrac{z}{2})\Big(\int_{\mathbb{R}^2} \delta_{0,z,w}(x)\delta_{0,z,w}(y)p_{\frac12}(z,w)\,\mathrm{d}w\Big)^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&\leq \int_{(\mathbb{R}^2)^5} p_{\frac14}(0,\tfrac{z}{2})\delta_{0,z,w}(x)\delta_{0,z,w'}(x)\delta_{0,z,w}(y)\delta_{0,z,w'}(y)p_{\frac12}(z,w)p_{\frac12}(z,w')\,\mathrm{d}w\,\mathrm{d}w'\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.
\end{aligned}$$

Using for example

$$\int_{\mathbb{R}^2} \delta_{0,z,w}(x)\delta_{0,z,w'}(x)\,\mathrm{d}x \leq |z||w|, \qquad \int_{\mathbb{R}^2} \delta_{0,z,w}(y)\delta_{0,z,w'}(y)\,\mathrm{d}y \leq |z||w|,$$

we get

$$\begin{aligned}
C &\leq \int_{\mathbb{R}^2} |z|^2 p_{\frac14}(0,\tfrac{z}{2})\int_{\mathbb{R}^2} p_{\frac12}(z,w)|w|^2\,\mathrm{d}w\,\mathrm{d}z \\
&= \int_{\mathbb{R}^2} |z|^2 p_{\frac14}(0,\tfrac{z}{2})\int_{\mathbb{R}^2} p_{\frac12}(z,w)(|z|^2 + |w-z|^2)\,\mathrm{d}w\,\mathrm{d}z \\
&= \int_{\mathbb{R}^2} |z|^2 p_{\frac14}(0,\tfrac{z}{2})(|z|^2+1)\,\mathrm{d}z \\
&< \infty,
\end{aligned}$$

which concludes the proof that

$$\|Z'^{\epsilon}_W - Z'^{\epsilon'}_W\|_{L^2(\mathbb{P})} \underset{\epsilon,\epsilon'\to 0}{\longrightarrow} 0.$$

A similar but simpler computation, replacing $f^{\epsilon,\epsilon'}(x,y)$ with $\hat{n}^{\epsilon}_{[0,1/4]}(x)\hat{n}^{\epsilon}_{[0,1/4]}(y)$, shows that $Z'^{\epsilon}_W \in L^2(\mathbb{P})$. These two properties together ensure that Z'^{ϵ}_W converges in $L^2(\mathbb{P})$. The random variable $Z''^{\epsilon}_W := \int_{\mathbb{R}^2}\delta_{0,1}(x)\hat{\mathbf{n}}^{\epsilon}_{W_{[1/4.1/2]}}(x)\,\mathrm{d}x$, although *not* identical in distribution to Z'^{ϵ}_W, it shown to converge in a very similar fashion, and we deduce that $Z^{1,\epsilon}_W = Z'^{\epsilon}_W + Z''^{\epsilon}_W + \int_{\mathbb{R}^2}\delta_{0,1}(z)\delta_{1,2}(z)\,\mathrm{d}z$ converges in $L^2(\mathbb{P})$.

Using the properties of invariance of the Brownian motion and the Lebesgue measure in the plane, we see that $Z^{1,\epsilon}_W$ and $Z^{2,\epsilon}_W$ *are* identical in distribution, so $Z^{2,\epsilon}_W$ also converges in $L^2(\mathbb{P})$. □

Corollary 6.7. *The sum* $\sum_{k=0}^{n-1}\sum_{j=1}^{2^k}\tilde{Z}^{s2^{-n/2}}_{k,j}$ *converges in* $L^2(\Omega)$ *as* $n\to\infty$*, and the convergence is uniform over* $s\in[1,2]$*.*

Proof. This is identical to Lemma 6.5, except Y is replaced with $\tilde{Z}$ and that we use Lemma 6.6 instead of Theorem 4.6 to ensure the convergence, uniformly over $s\in[1,2]$, of the individual $Z^{s2^{-n/2}}_{k,j}$, as $n\to\infty$. □

We can now conclude:

Theorem 6.8. *As $\epsilon \to 0$, $\tilde{X}^\epsilon_W$ converges in $L^2(\mathbb{P})$.*

Proof. Since for a given $\epsilon \in (0,1)$, there exists a couple $(s,n) \in (1,2] \times \mathbb{N}$ such that $s2^{-n} = \epsilon$, proving the theorem amounts to show that $\tilde{X}^{s2^{-n}}_W$ converges as $n \to \infty$, uniformly over $s \in [1,2]$ and toward a limit independent from s. This follows from the decomposition (20), Lemma 6.3, Lemma 6.5 and Corollary 6.7. □

6.3. **Estimation of the average $\mathbb{E}[X^\epsilon_W]$.** We now obtain asymptotic estimation for the average $\mathbb{E}[X^\epsilon_W]$, which is important in relation to Higgs-Yang–Mills theory as it tells us about the infinite negative mass term used for renormalisation.

We let Z_W (resp. Z^1_W, Z^2_W, $Z_{k,j}$) be the L^2-limit of Z^ϵ_W (resp. $Z^{1,\epsilon}_W$, $Z^{2,\epsilon}_W$, $Z^\epsilon_{k,j}$).

Lemma 6.9. *There exists a constant C_φ such that for all $\epsilon, \epsilon' \in (0,1]$,*

$$|\mathbb{E}[Z^\epsilon_W - Z_W]| \leq C_\varphi \epsilon.$$

Proof. Let F be the vector field such that $\operatorname{curl} F = \delta_{0,1}$ and $\operatorname{div} F = 0, \Delta F = 0$. Then, $F \in W^{1,\infty} \cap W^{1,1}$. In particular,

$$\begin{aligned}\|\varphi^\epsilon \star F - F\|_\infty &\leq \sup_{z\in\mathbb{R}^2} \int_{B_{\epsilon K_\varphi}} \epsilon^{-2}\varphi(\epsilon^{-1}v)|F(z+v) - F(z)|\,\mathrm{d}v \\ &\leq \int_{B_{\epsilon K_\varphi}} \epsilon^{-2}\varphi(\epsilon^{-1}v)|v|\|DF\|_\infty\,\mathrm{d}v \\ &\leq \epsilon K_\varphi \|F\|_{W^{1,\infty}},\end{aligned}$$

hence

$$\|\varphi^\epsilon \star F - \varphi^{\epsilon'} \star F\|_\infty \leq (\epsilon + \epsilon')K_\varphi \|F\|_{W^{1,\infty}}$$

Using the stochastic Fubini's theorem, we can write Z^ϵ as

$$Z^\epsilon = \int_{\frac{1}{2}}^1 \varphi^\epsilon \star F(W_s) \circ \mathrm{d}W_s = \int_{\frac{1}{2}}^1 \varphi^\epsilon \star F(W_s)\,\mathrm{d}W_s.$$

Remark that F is random but measurable with respect to $(W_{1/2}, W_1)$. Let $B : [0, \frac{1}{2}] \to \mathbb{R}^2$ be such that $B_0 = 0$ and for $s \in (0, 1/2)$

$$\mathrm{d}W_s = \mathrm{d}B_s + \frac{W_{1/2} - W_s}{1/2 - s}\,\mathrm{d}s.$$

Conditionally on $(W_{1/2}, W_1)$, the restriction of W to $[0, \frac{1}{2}]$ is a Brownian bridge, and it follows that conditionally on $(W_{1/2}, W_1)$, B is a Brownian motion. In particular, we deduce that for any $\epsilon, \epsilon' \in (0,1]$,

$$\begin{aligned}\left|\mathbb{E}\left[Z^{1,\epsilon}_W - Z^{1,\epsilon'}_W \middle| W_{1/2}, W_1\right]\right| &= \left|\int_0^{\frac{1}{2}} \frac{\mathbb{E}\left[\left(\varphi^\epsilon \star F(W_s) - \varphi^{\epsilon'} \star F(W_s)\ \right)(W_{1/2} - W_s)|W_{1/2}, W_1\right]}{1/2 - s}\,\mathrm{d}s\right| \\ &\leq \|\varphi^\epsilon \star F - \varphi^{\epsilon'} \star F\|_\infty \int_0^{\frac{1}{2}} \frac{\mathbb{E}\left[|W_{1/2} - W_s| \middle| W_{1/2}, W_1\right]}{1/2 - s}\,\mathrm{d}s\end{aligned}$$

Conditionnally on $(W_{1/2}, W_1)$, $W_{1/2} - W_s$ is distributed as a Gaussian random variable centered at $(1-2s)W_{1/2}$ and with covariance $2s(1-2s)$. Thus,

$$\mathbb{E}\left[|W_{1/2} - W_s| \middle| W_{1/2}, W_1\right] \leq (1-2s)|W_{1/2}| + \sqrt{2s(1-2s)},$$

and we deduce

$$\begin{aligned}\left|\mathbb{E}\left[Z^{1,\epsilon}_W - Z^{1,\epsilon'}_W \middle| W_{1/2}, W_1\right]\right| &\leq (\epsilon + \epsilon')K_\varphi \|F\|_{W^{1,\infty}} \left(|W_{1/2}| + 2\sqrt{2}\int_0^{\frac{1}{2}} \sqrt{\tfrac{s}{1-2s}}\,\mathrm{d}s\right) \\ &= (\epsilon + \epsilon')K_\varphi \|F\|_{W^{1,\infty}}(|W_{1/2}| + \tfrac{\pi}{2}),\end{aligned}$$

thus

$$|\mathbb{E}[Z^{1,\epsilon}_W - Z^{1,\epsilon'}_W]| \leq (\epsilon + \epsilon')K_\varphi \|F\|_{W^{1,\infty}}(1 + \tfrac{\pi}{2}).$$

Letting $\epsilon' \to 0$, we deduce

$$|\mathbb{E}[Z_W^{1,\epsilon} - Z_W^1]| \leq \epsilon K_\varphi \|F\|_{W^{1,\infty}}(1 + \tfrac{\pi}{2}).$$

By symmetries of the Brownian motion, the couples $(Z_W^{1,\epsilon}, Z_W^1)$ and $(Z_W^{2,\epsilon}, Z_W^2)$ are equal in distribution, and we deduce by summation

$$|\mathbb{E}[Z_W^\epsilon - Z_W]| \leq \epsilon K_\varphi \|F\|_{W^{1,\infty}}(2 + \pi),$$

which concludes the proof. □

Lemma 6.10. *There exists constants C, C' such that, for $\epsilon = 2^{-n/2}, n \in \mathbb{N}$, as $\epsilon \to 0$,*

$$\mathbb{E}[X_W^\epsilon] = C\log(\epsilon^{-1}) + C' + O(\epsilon).$$

Proof. By scaling arguments,

$$\mathbb{E}[X_{n,j}^{2^{-n/2}}] = 2^{-n}\mathbb{E}[X_W^1],\ \mathbb{E}[T_{k,j}] = 2^{-k}\mathbb{E}[T_{0,1}],\ \mathbb{E}[Z_{k,j}] = 2^{-k}\mathbb{E}[Z_{0,1}],\ \mathbb{E}[Y_{k,j}^{2^{-n/2}}] = 0$$

$$\mathbb{E}[Z_{k,j}^{2^{-n/2}} - Z_{k,j}] = 2^{-k}\mathbb{E}[Z_{0,1}^{2^{-(n-k)/2}} - Z_{0,1}].$$

By Lemma 6.9,

$$R_n := \sum_{l=n}^{\infty} \mathbb{E}[Z_{0,1}^{2^{-l/2}} - Z_{0,1}] \leq C_\varphi \sum_{l=n}^{\infty} 2^{-l/2} < C'_\varphi 2^{-n/2},$$

Hence,

$$\begin{aligned}
\mathbb{E}[X^{2^{-n/2}}] &= \mathbb{E}[X_W^1] + \sum_{k=0}^{n-1}(\mathbb{E}[T_{0,1}] + \mathbb{E}[Z_{0,1}] + \mathbb{E}[Z_{0,1}^{2^{-(n-k)/2}} - Z_{0,1}]) \\
&= \mathbb{E}[X_W^1] + n(\mathbb{E}[T_{0,1}] + \mathbb{E}[Z_{0,1}]) + R_1 - R_{n+1} \\
&= Cn + C' + O(2^{-n/2}),
\end{aligned}$$

which concludes the proof. □

Remark that when we consider a Brownian motion with duration $t \neq 1$, we can easily see how these constants C, C' depends on t by a scaling argument:

$$\mathbb{E}_t[X_W^\epsilon] = t\mathbb{E}_1[X_W^{\epsilon/\sqrt{t}}] = tC\log(\epsilon^{-1}\sqrt{t}) + C' + O(\epsilon) = tC\log(\epsilon^{-1}) + (tC' + \frac{t\log(t)C}{2}) + O(\epsilon).$$

Remark also that the constant C' is not universal: it necessarily depends on the initial mollifier φ, as it can be seen by using the rescaled φ^λ instead of φ as initial mollifier.

The constant C, however, is universal and we will now prove $C = \frac{1}{2\pi}$. Although this result is, as far as I know, new, it is not *per se* novel, and this value should be expected from [10, Theorem 1]. The regularisation method used in [10] is not the same as our regularisation by mollification, and I don't think there is any simple argument that would allow to rigorously prove the constants arising from both regularisation methods should be the same. However, at an intuitive level, it is very clear that they should be. The argument goes as follows: for points whose distance to Range(W) is much greater than ϵ, both regularisation can be neglected, and they should thus give similar contribution to X_W^ϵ. For points whose distance to Range(W) is of order ϵ or less, their contribution to X_W^ϵ is of order 1, in both normalisation. Thus, the difference between the two normalisation should be of order 1. In other words, for both normalisation, the cut-off happens at scales of the same order $\approx \epsilon$.

6.4. **Computation of the Constant C.** In order to compute C explicitly, it would be practical to use $\mathbf{n}_W(z)$ instead of either $\mathbf{n}_{\bar{W}}(z)$ or $\hat{\mathbf{n}}_W(z)$. Yet, as we have see the planar integral of $\mathbf{n}_W^\epsilon(z)^2$ is divergent because of the infrared issues. We fix a large radius C_ϵ, which is such that there is only a small probability for W to exit B_{C_ϵ}. Inside this ball, we accept to make an error by replacing $\mathbf{n}_{\bar{W}}^\epsilon(z)$ with $\mathbf{n}_W^\epsilon(z)$. Of course this produce an error term that grows large when C_ϵ grows. Outside the ball, we do not make this replacement. It is more difficult to deal with $\mathbf{n}_{\bar{W}}^\epsilon(z)$ directly, but we can do it has we only need a rough upper bound, rather than a precise estimation, since we treat this an error term. The choice of C_ϵ is then made to balance the error term inside the ball with the one outside the ball.

Lemma 6.11. *Let C_ϵ be a function of ϵ such that $C_\epsilon \geq 4\sqrt{2}$. As $\epsilon \to 0$,*

$$\mathbb{E}_0\Big[\int_{B_{C_\epsilon}} \mathbf{n}^\epsilon_{\bar{W}}(z)^2\,\mathrm{d}z\Big] = \frac{1}{2\pi}\log(\epsilon^{-1}) + O(C_\epsilon^2 + e^{-\frac{C_\epsilon^2}{8}}\log(\epsilon^{-1}) + C_\epsilon\sqrt{\log(\epsilon^{-1})}),$$

Proof. Let $\bar{I}$ be the left-hand side, and let

$$I = \mathbb{E}_0\Big[\int_{B_{C_\epsilon}} \mathbf{n}^\epsilon_{W}(z)^2\,\mathrm{d}z\Big].$$

Using

$$\mathbb{E}[\mathbf{n}^\epsilon_W(z)^2] = \epsilon^{-2}\int_0^1\int_{\mathbb{R}^2} p_t(z,y)|\psi(\epsilon^{-1}y)|^2\,\mathrm{d}y\,\mathrm{d}t \tag{21}$$

and splitting the integral over $y \in \mathbb{R}^2$ into three parts, we get $I = J + R_1 + R_2$ with

$$J := \epsilon^{-2}\int_{B_{C_\epsilon}}\int_{B_{\frac{C_\epsilon}{2}}}\int_0^1 p_t(z,y)|\psi(\epsilon^{-1}y)|^2\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}z,$$

$$R_1 := \epsilon^{-2}\int_{B_{C_\epsilon}}\int_{B_{2C_\epsilon}\setminus B_{\frac{C_\epsilon}{2}}}\int_0^1 p_t(z,y)|\psi(\epsilon^{-1}y)|^2\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}z,$$

and

$$R_2 := \epsilon^{-2}\int_{B_{C_\epsilon}}\int_{\mathbb{R}^2\setminus B_{2C_\epsilon}}\int_0^1 p_t(z,y)|\psi(\epsilon^{-1}y)|^2\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}z.$$

To estimate J, we swap the two integrals in its definition, then we use the change of variable $z' = y - z$, and then the inclusions

$$\forall y \in B_{\frac{C_\epsilon}{2}}, \qquad B_{\frac{C_\epsilon}{2}} \subset B_{C_\epsilon}(y) \subset B_{\frac{3}{2}C_\epsilon}. \tag{22}$$

We so obtain the inequalities

$$J \geq \epsilon^{-2}\int_{B_{\frac{C_\epsilon}{2}}}\int_{B_{\frac{C_\epsilon}{2}}}\int_0^1 p_t(0,z')|\psi(\epsilon^{-1}y)|^2\,\mathrm{d}t\,\mathrm{d}z'\,\mathrm{d}y,$$

$$J \leq \epsilon^{-2}\int_{B_{\frac{C_\epsilon}{2}}}\int_{B_{\frac{3C_\epsilon}{2}}}\int_0^1 p_t(0,z')|\psi(\epsilon^{-1}y)|^2\,\mathrm{d}t\,\mathrm{d}z'\,\mathrm{d}y.$$

In both right-hand sides, the spatial integrals can now be performed independently from each other. With the change of variable $v = \epsilon^{-1}y$, and since $\int_{\mathbb{R}^2} p_t(0,z)\,\mathrm{d}z = 1$, we deduce

$$J \leq \int_{B_{\frac{\epsilon^{-1}C_\epsilon}{2}}} |\psi|^2(v)\,\mathrm{d}v = \frac{1}{2\pi}\log(\epsilon^{-1}C_\epsilon) + O(1), \tag{23}$$

and we similarly deduce that

$$J \geq \frac{\int_{B_{\frac{C_\epsilon}{2}}}\int_0^1 p_t(0,z)\,\mathrm{d}t\,\mathrm{d}z}{2\pi}\log(\epsilon^{-1}C_\epsilon) + O(1).$$

For $x \geq \sqrt{2t}$, $\partial_t p_t(0,z) \geq 0$, and it follows from the condition $C_\epsilon \geq 4\sqrt{2}$ that for all $z \in \mathbb{R}^2\setminus B_{C_\epsilon/2}$ and all $t \in [0,1]$, $p_t(0,z) \leq p_1(0,z)$. Thus,

$$\int_{\mathbb{R}^2\setminus B_{\frac{C_\epsilon}{2}}}\int_0^1 p_t(0,z)\,\mathrm{d}t\,\mathrm{d}z \leq \int_{\mathbb{R}^2\setminus B_{\frac{C_\epsilon}{2}}} p_1(0,z)\,\mathrm{d}z = e^{-\frac{C_\epsilon^2}{8}},$$

thus

$$J \geq \frac{1}{2\pi}\log(\epsilon^{-1}C_\epsilon) - e^{-\frac{C_\epsilon^2}{8}}\log(\epsilon^{-1}C_\epsilon) + O(1).$$

Together with (23), we obtain

$$J = \frac{1}{2\pi}\log(\epsilon^{-1}C_\epsilon) + O(e^{-\frac{C_\epsilon^2}{8}}\log(\epsilon^{-1}C_\epsilon)) + O(1).$$

We now focus on the estimation of the residual terms R_1 and R_2. We know by Lemma 3.1, that there exists C such that $|\psi|^2(y) \leq C|y|^{-2}$ for all $y \in \mathbb{R}^2$, so that

$$R_1 \leq \epsilon^{-2}\int_{B_{2C_\epsilon}\setminus B_{C_\epsilon/2}} \frac{C}{\epsilon^{-2}|y|^2}\,\mathrm{d}y \leq \pi(2C_\epsilon)^2\frac{C}{(C_\epsilon/2)^2} = 16\pi C.$$

As for R_2, we notice that for all $y \in \mathbb{R}^2\setminus B_{2C_\epsilon}$ and $z \in B_{C_\epsilon}$, $|y-z| \geq \frac{|y|}{2}$, thus $p_t(y,z) \leq p_t(0,\frac{y}{2})$. Hence,

$$R_2 \leq \int_{B_{C_\epsilon}}\int_{\mathbb{R}^2\setminus B_{2C_\epsilon}}\int_0^1 p_t(0,\frac{y}{2})\,\mathrm{d}t\frac{C}{|y|^2}\,\mathrm{d}y\,\mathrm{d}z = (\pi C_\epsilon^2)2\pi\int_0^1\int_{2C_\epsilon}^\infty p_t(0,\frac{r}{2})\frac{C}{r^2}r\,\mathrm{d}r\,\mathrm{d}t = o(1).$$

Altogether, we obtain

$$\begin{aligned} I = J + R_1 + R_2 &= \frac{1}{2\pi}\log(\epsilon^{-1}C_\epsilon) + O(e^{-\frac{C_\epsilon^2}{8}}\log(\epsilon^{-1}C_\epsilon)) + O(1) \\ &= \frac{1}{2\pi}\log(\epsilon^{-1}) + O(\log(C_\epsilon) + e^{-\frac{C_\epsilon^2}{8}}\log(\epsilon^{-1})). \end{aligned} \tag{24}$$

Now we need to go from I to $\bar{I}$, for which we apply the generic identity

$$\big|\|a\|^2 - \|b\|^2\big| = \big|\|a-b\|^2 + 2\langle a-b,b\rangle\big| \leq \|a-b\|^2 + 2\|a-b\|\|b\| \tag{25}$$

with $a = \mathbf{n}^\epsilon_{\bar{W}}$, $b = \mathbf{n}^\epsilon_W$, and with the Hilbert norm $\|f\|^2 = \int \mathbb{E}_0[f(z)^2]\,\mathrm{d}z$ on $L^2(\Omega\times B_{C_\epsilon})$. Since $|\mathbf{n}^\epsilon_{\bar{W}}(z) - \mathbf{n}^\epsilon_W(z)| \leq 1$ for all z, we have $\|a-b\| \leq \sqrt{\pi}C_\epsilon$. By (24), $\|b\|^2 = O(\log(\epsilon^{-1}C_\epsilon))$, and using (24) a second time we obtain

$$\begin{aligned} \int_{B_{C_\epsilon}} \mathbb{E}_0[\mathbf{n}^\epsilon_{\bar{W}}(z)^2]\,\mathrm{d}z &= \|b\|^2 + O(\|a-b\|^2 + \|a-b\|\|b\|) \\ &= \frac{1}{2\pi}\log(\epsilon^{-1}) + O(C_\epsilon^2 + e^{-\frac{C_\epsilon^2}{8}}\log(\epsilon^{-1}) + C_\epsilon\sqrt{\log(\epsilon^{-1})}), \end{aligned} \tag{26}$$

where we simplified some smaller order terms such as $O(C_\epsilon\sqrt{\log(C_\epsilon)})$ for the last inequality. □

Lemma 6.12. *Let C_ϵ be a function of ϵ such that $C_\epsilon \geq 4\sqrt{2}$. Assume φ is compactly supported. Then, as $\epsilon t \to 0$,*

$$\mathbb{E}_0[\int_{\mathbb{R}^2\setminus B_{C_\epsilon}} \mathbf{n}^\epsilon_{\bar{W}}(z)^2]\,\mathrm{d}z = O\Big(\frac{e^{-\frac{17}{512}C_\epsilon^2}}{C_\epsilon}\log(\epsilon^{-1})\Big) + o(1).$$

Proof. As we explained several times already, one must be very careful here with replacing $\bar{W}$ with W. For $|z| \geq C_\epsilon$, let τ_z be the first time when $|W| = \frac{|z|}{2}$, and notice that, for topological reasons,

$$\tau_z \geq 1 \implies \forall y \in \mathbb{R}^2\setminus B_{|z|/2},\ \mathbf{n}_{\bar{W}}(y) = 0.$$

For ϵ sufficiently small (which we now assume), $C_\epsilon \geq 2\epsilon\sup\{|y|, y \in \mathrm{Supp}(\varphi)\}$. Then, for all $z \in \mathbb{R}^2\setminus B_{C_\epsilon}$,

$$\tau_z \geq 1 \implies \mathbf{n}^\epsilon_{\bar{W}}(z) = 0.$$

This can be proved rigorously using the fact that $\psi^{z,\epsilon}$, seen as complex function, is analytical on $B_{\|W\|_\infty}$, hence its Stratonovich integral along the closed curve $\bar{W}$ is zero. Alternatively, it follows trivially from the stochastic Green's formula from [8].

On the even $\tau_z < 1$, let $\mathcal{W} : t \in [0, 1-\tau_z] \mapsto W_{\tau_z+t}$. Conditional on $(\tau_z, (W_t)_{t\leq\tau_z})$, $\mathcal{W}$ is a Brownian motion of duration τ_z started from W_{τ_z}. The integer $\mathbf{n}^\epsilon_{\bar{W}}(z) - \mathbf{n}^\epsilon_{\bar{\mathcal{W}}}(z)$ is equal (up to sign) to the winding around z of the triangle with vertices 0, W_{τ_z}, and W_1, which is at most 1. Thus, $\mathbf{n}^\epsilon_{\bar{W}}(0)$ and $\mathbf{n}^\epsilon_{\mathcal{W}}(0)$ differs from each other from at most 2. Denoting by $\mathbb{E}_{t,y}$ the expectation under which Y is a planar Brownian motion started from y and with duration t, and using $(a+b)^2 \leq 2(a^2+b^2)$, we deduce

$$\begin{aligned}
\mathbb{E}_0[\mathbf{n}^\epsilon_{\bar{\mathcal{W}}}(z)^2] &\leq \mathbb{P}_0(\tau_z \leq 1) \sup_{\substack{y\in\partial B_{|z|/2}\\ t\in[0,1]}} \mathbb{E}_{t,y}[(\mathbf{n}^\epsilon_{\mathcal{W}}(z)+R^\epsilon)^2], \qquad |R^\epsilon| \leq 2 \\
&\leq 2\mathbb{P}_0(\tau_z \leq 1)(\sup_{\substack{y\in\partial B_{|z|/2}\\ t\in[0,1]}} \mathbb{E}_{t,y}[(\mathbf{n}^\epsilon_{\mathcal{W}}(z))^2]+4) \\
&= 2\mathbb{P}_0(\tau_z \leq 1)(\sup_{\substack{y\in\partial B_{|z|/2}\\ t\in[0,1]}} \mathbb{E}_{t,0}[(\mathbf{n}^\epsilon_{\mathcal{W}s}(z-y))^2]+4) \qquad (27)
\end{aligned}$$

For $z \in \mathbb{R}^2 \setminus B_{4\sqrt{2}}$, $y \in \partial B_{|z|/2}$, and $w \in B_1$, we have $|z-y|-|w| \geq |z|/2-1 \geq |z|/4 \geq \sqrt{2}$, hence for all and $s \in [0,1]$, $p_s(z-y,w) \leq p_s(0,|z|/4) \leq p_1(0,|z|/4)$. From Equation (21) and Lemma 3.1, we thus have, uniformly over $z \in [4\sqrt{2},\infty)$ and $\epsilon \in (0,1]$

$$\begin{aligned}
\mathbb{E}_{t,0}[(\mathbf{n}^\epsilon_{W}(z-y))^2] &\leq \int_0^t \int_{\mathbb{R}^2} p_s(z-y,w)\frac{C}{\epsilon^2+|w|^2}\,\mathrm{d}w\,\mathrm{d}s \\
&\leq p_1(0,\tfrac{z}{4})\,\mathrm{d}s \int_{B_1} \frac{C}{\epsilon^2+|w|^2}\,\mathrm{d}w + \int_0^t \int_{\mathbb{R}^2\setminus B_1} p_s(z-y,w)C\,\mathrm{d}w\,\mathrm{d}s \\
&\leq Cp_1(0,\tfrac{z}{4})(\log(\epsilon^{-1})+\frac{\log(2)}{2})+Ct \\
&= O(e^{-\frac{|z|^2}{32}}\log(\epsilon^{-1}))+O(1). \qquad (28)
\end{aligned}$$

Besides, since $\{(x^1,x^2): x^1 \leq c, x^2 \leq c\} \subset B_{\sqrt{2}c}$ for all c, using the reflection principle for the 1-dimensional Brownian motion, we have

$$\begin{aligned}
\mathbb{P}_0(\tau_z \leq 1) &\leq 2\mathbb{P}_0(\exists s \in [0,1] : |W^1_s| \geq \frac{|z|}{4}) \\
&\leq 4\mathbb{P}_0(\exists s \in [0,1] : W^1_s \geq \frac{|z|}{4}) \\
&= 4\mathbb{P}_0(|W^1_1| \geq \frac{|z|}{4}) = 8\mathbb{P}_0(W^1_1 \geq \frac{|z|}{4}) = \frac{8}{\sqrt{2\pi}}\int_{\frac{|z|}{4}}^\infty e^{-\frac{r^2}{32}}\,\mathrm{d}r \underset{|z|\to\infty}{=} O\Big(\frac{e^{-\frac{|z|^2}{512}}}{|z|}\Big). \qquad (29)
\end{aligned}$$

Together with (27) and (28), we get

$$\mathbb{E}[(\mathbf{n}^\epsilon_{\bar{W}}(z))^2] = O\Big(\frac{e^{-\frac{17|z|^2}{512}}}{|z|}\log(\epsilon^{-1})+\frac{e^{-\frac{|z|^2}{512}}}{|z|}\Big).$$

By integrating over z, we deduce that

$$\int_{\mathbb{R}^2\setminus B_{C_\epsilon}} \mathbb{E}[(\mathbf{n}^\epsilon_{\bar{W}}(z))^2]\,\mathrm{d}z = O\Big(\frac{e^{-\frac{17}{512}C_\epsilon^2}}{C_\epsilon}\log(\epsilon^{-1})\Big)+o(1).$$

□

Corollary 6.13. *Assume φ is compactly supported. For all $x \in \mathbb{R}^2$, as $\epsilon \to 0$,*

$$\mathbb{E}_x\Big[\int_{\mathbb{R}^2} \mathbf{n}^\epsilon_{\bar{W}}(z)^2\,\mathrm{d}z\Big] = \frac{\log(\epsilon^{-1})}{2\pi} + O\big(\log(\log(\epsilon^{-1}))\sqrt{\log(\epsilon^{-1})}\big).$$

Proof. From translation invariance, we can assume $x=0$ without loss of generality. Then we conclude by applying Lemma 6.11 and Lemma 6.12 with $C_\epsilon = 8\sqrt{\log(\log(\epsilon^{-1}))}$. Notice this is the (up to a multiplicative constant) the minimal scale for which the residual term coming from integration over $\mathbb{R}^2 \setminus B_{C_\epsilon}$ is not prevalent over the other ones. □

Appendix A. Formal relation between Higgs–Yang–Mills and the Amperean area

Recall the Higgs–Yang–Mills measure (without self-interaction term) is the formally defined probability measure $\mathbb{P}$ on couples (A, Φ), where $\Phi : \mathbb{R}^2 \to \mathbb{C}$ and $A : \mathbb{R}^2 \to \mathbb{R}^2$. One can as well consider a proper domains of $\mathbb{R}^2$, in which case $f = 1$ must be replaced with $f = \mathbb{1}_D$ in some expressions.

One can also consider a surface Σ endowed with a Riemannian metric instead of $\mathbb{R}^2$. Geometrically speaking, A is then not a vector field but rather a connection over some complex line bundle, whilst Φ is a section of this bundle.

The probability measure $\mathbb{P}$ is given formally by

$$\mathrm{d}\mathbb{P}(\Phi, A) := \frac{1}{Z} \exp(-\frac{\| \operatorname{grad} \Phi + i\alpha A\Phi\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^2)} + \| \operatorname{curl} A\|^2_{L^2(\mathbb{R}^2,\mathbb{R})}}{2})\mathcal{D}\Phi\mathcal{D}A.$$

Here the discussion is too rough to consider counter-terms. For a given A, Z_A is the formally defined normalising constant such that the measure $\mathbb{P}_A$ given by

$$\mathrm{d}\mathbb{P}_A(\Phi) := \frac{1}{Z_A} \exp(-\frac{\| \operatorname{grad} \Phi + i\alpha A\Phi\|^2}{2})\mathcal{D}\Phi$$

is a probability measure, that γ is the straight line segment from y to x, and that

$$S_{x,y} := \big\langle \Phi(x), \exp\big(i\alpha \int_\gamma A\big)\Phi(y)\big\rangle.$$

Let $G_A = (-\Delta_A)^{-1}$ be the Green function associated with the Dirichlet Magnetic Laplacian

$$\Delta_A = (\nabla + i\alpha A)^*(\nabla + i\alpha A)$$

acting on complex-valued functions. For a genuine and smooth enough A, the objects $\mathbb{P}_A$, Δ_A, and G_A are rigorously defined (although the given expression for $\mathbb{P}_A$ is not: it has to be interpreted as a Gaussian measure in an appropriate distributional space), and furthermore the partition function Z_A which is formally the square root of the determinant of the Laplacian can be rigorously defined, as a positive real number, through ζ-regularisation. The relation we now write formally would hold rigorously for such a smooth A, but the reader should understand that the typical A under the measure $\mathbb{P}$ is too irregular for these rigorous definitions to apply.

We write $\mathbb{P}_{YM}$ for the formally defined probability measure on A given by

$$\mathrm{d}\mathbb{P}_{YM}(A) := \frac{1}{Z'} \exp(-\frac{\| \operatorname{curl} A\|^2_{L^2(\mathbb{R}^2,\mathbb{R})}}{2})\mathcal{D}A.$$

Expanding the Green function as a time integral of the heat kernel and using Feynman-Kac's formula, one gets

$$G_A(x,y) = \int_0^\infty p_t^A(x,y)\,\mathrm{d}t = \int_0^\infty p_t(x,y)\mathbb{E}_{t,x,y}[e^{i\alpha\int_W A}]\,\mathrm{d}t,$$

where W is a Brownian bridge from x to y with duration t. Hence,

$$\mathbb{E}[(\frac{Z_A}{ZZ'})^{-1}S_{x,y}] = \int_0^\infty p_t(x,y)\mathbb{E}_{t,x,y}\otimes\mathbb{E}_{YM}[\exp\big(-i\alpha\int_{W\cdot\gamma^{-1}} A\big)]\,\mathrm{d}t.$$

Yet, $\xi := \operatorname{curl} A$ is a Gaussian white noise under $\mathbb{P}_{YM}$, and Green's formula gives

$$\mathbb{E}_{YM}[\exp\big(-i\alpha\int_{W\cdot\gamma^{-1}} A\big)] = \mathbb{E}_{YM}[\exp\big(-i\alpha\int_{\mathbb{R}^2} \mathbf{n}_{W\cdot\gamma^{-1}}(z)\xi(\,\mathrm{d}z)\big)] = \exp\big(-\frac{\alpha^2}{2}\int_{\mathbb{R}^2}\mathbf{n}_{W\cdot\gamma^{-1}}(z)^2\,\mathrm{d}z\big),$$

which finally gives

$$\mathbb{E}[(\frac{Z_A}{ZZ'})^{-1}S_{x,y}] = \int_0^\infty p_t(x,y)\mathbb{E}_{t,x,y}[\exp\big(-\frac{\alpha^2}{2}\int_{\mathbb{R}^2}\mathbf{n}_{W\cdot\gamma^{-1}}(z)^2\,\mathrm{d}z\big)]\,\mathrm{d}t.$$

We recognize the Amperean area in the right-hand side.

When considering higher order moments of the $S_{x,y}$'s, say monomial of order $2k$, we must first use Isserlis' theorem (also known as Wick's formula). Then, we end up with k independant Brownian bridges. When using the Gaussianity of A, appears the expression $\int_{\mathbb{R}^2}\Big(\sum_i \mathbf{n}_{W\cdot\gamma^{-1}}(z)\Big)^2\,\mathrm{d}z$. Expanding the square explains with not only the variables $:X_W(1):$ but also the $Y_{W,W'}(1)$ matter, as crossed terms. When adding some Wilson loops, we get extra diagonal term that corresponds to the usual Wilson loop expectations in Abelian Yang–Mills theory (without Higgs field), and extra crossed terms given by the classical stochastic integrals $I_W(f)$.

As for the partition function Z_A, it can be dealt with in a similar manner thanks to its representation as an expected product over a Brownian loop soup $\mathcal{L}$,[6]

$$Z_A = Z''\mathbb{E}[\prod_{\ell\in\mathcal{L}}\exp(i\int_\ell A)]^{\frac{1}{2}}.$$

Remark that the addition of a mass term $\langle\varphi, m\varphi\rangle$ in the definition of $\mathbb{P}_{YM}$ would introduce an extra factor e^{-mt} inside the time integral if m is constant, or more generally an extra factor $\int_0^t m(W_s)\,\mathrm{d}s$ inside the expectation if m varies in space (i.e. "is a potential"). In particular, the renormalisation we used to define X_W do translate into a renormalisation of $\mathbb{P}_{YM}$ by addition of a negative diverging mass term.

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[6] Even for smooth A, the product is not absolutely convergent due to ultraviolet issues, but it is defined almost surely as a limit as $\epsilon \to 0$ of the product over loops with size at least ϵ. Infrared issues only appear for very specific choices of A (such as $A = 0$), but not in the typical case we are interested in.