ON THE DELOCALIZED PHASE OF THE RANDOM PINNING MODEL

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ABSTRACT. We consider the model of a directed polymer pinned to a line of i.i.d. random charges, and focus on the interior of the delocalized phase. We first show that in this region, the partition function remains bounded. We then prove that for almost every environment of charges, the probability that the number of contact points in $[0, n]$ exceeds $c \log n$ tends to 0 as n tends to infinity. The proofs rely on recent results of [BGdH10, CdH10].

1. INTRODUCTION

Let $\tau = (\tau_i)_{i \in \mathbb{N}}$ be a sequence such that $\tau_0 = 0$ and $(\tau_{i+1} - \tau_i)_{i \geqslant 0}$ are independent and identically distributed random variables with values in $\mathbb{N}^* = \{1, 2, \ldots\}$. Let $\mathbf{P}$ be the distribution of τ, $\mathbf{E}$ the associated expectation, and $K(n) = \mathbf{P}[\tau_1 = n]$. We assume that there exists $\alpha \geqslant 0$ such that

$$(1.1) \qquad \frac{\log K(n)}{\log n} \xrightarrow[n \to \infty]{} -(1 + \alpha).$$

As an example, one can think about the sequence τ as the sequence of arrival times at 0 of a one-dimensional simple random walk (and in this case, $\alpha = 1/2$). In a slight abuse of notation, we will look also at the sequence τ as a set, and write for instance $n \in \tau$ instead of $\exists i : n = \tau_i$.

Let $\omega = (\omega_k)_{k \in \mathbb{N}}$ be independent and identically distributed random variables. We write $\mathbb{P}$ for the law of ω, and $\mathbb{E}$ for the associated expectation. We will refer to ω as the *environment*. We assume that the ω_k are centred random variables, and that they have exponential moments of all order. Let $\beta \geqslant 0, h \geqslant 0$, and $n \in \mathbb{N}^*$. We consider the probability measure $\mathbf{P}_n^{\beta,h,\omega}$ (expectation $\mathbf{E}_n^{\beta,h,\omega}$) which is defined as the following Gibbs transformation of the measure $\mathbf{P}$:

$$\frac{\mathrm{d}\mathbf{P}_n^{\beta,h,\omega}}{\mathrm{d}\mathbf{P}}(\tau) = \frac{1}{Z_n^{\beta,h,\omega}} \, \exp \left(\sum_{k=0}^{n-1} (\beta \omega_k - h) \mathbf{1}_{\{k \in \tau\}} \right) \mathbf{1}_{\{n \in \tau\}}.$$

In the above definition, β can be thought of as the inverse temperature, h as the disorder bias, and $Z_n^{\beta,h,\omega}$ is a normalization constant called the *partition function*,

$$Z_n^{\beta,h,\omega} = \mathbf{E} \left[\exp \left(\sum_{k=0}^{n-1} (\beta \omega_k - h) \mathbf{1}_{\{k \in \tau\}} \right) \mathbf{1}_{\{n \in \tau\}} \right].$$

At the exponential scale, the asymptotic behaviour of the partition function is captured by the *free energy* $\mathrm{F}(\beta, h)$ defined as

$$\mathrm{F}(\beta, h) = \lim_{n \to +\infty} \frac{1}{n} \, \log Z_n^{\beta,h,\omega}.$$

Superadditivity of the partition function implies that this limit is well defined almost surely, and that it is deterministic (see for instance [G, Theorem 4.1]). Assumption (1.1) implies that $\mathrm{F}(\beta, h) \geqslant 0$. It is intuitively clear that the free energy can become strictly positive only if the set $\tau \cap [0, n]$ is likely to contain many points under the measure $\mathbf{P}_n^{\beta,h,\omega}$. We thus say that we are in the *localized phase*

if $\mathrm{F}(\beta, h) > 0$, and in the *delocalized phase* otherwise. One can show [dH, Theorem 11.3] that for every $\beta \geqslant 0$, there exists $h_c(\beta) \geqslant 0$ such that

$$
\begin{aligned}
h < h_c(\beta) &\quad\Rightarrow\quad \text{localized phase, i.e. } \mathrm{F}(\beta, h) > 0, \\
h \geqslant h_c(\beta) &\quad\Rightarrow\quad \text{delocalized phase, i.e. } \mathrm{F}(\beta, h) = 0,
\end{aligned}
$$

and moreover, the function $\beta \mapsto h_c(\beta)$ is strictly increasing.

2. Statement of the main results

We focus here on the interior of the delocalized phase, that is to say when $h > h_c(\beta)$. Note that, due to the strict monotonicity of the function $h_c(\cdot)$, one sits indeed in the interior of the delocalized phase if one fixes $h = h_c(\beta_0)$ and considers any inverse temperature $\beta < \beta_0$.

By definition, the partition function is known to grow subexponentially in this region. In [BS10, Remark p. 417], the authors ask whether the partition function remains bounded there. We answer positively to this question, and can in fact be slightly more precise.

Theorem 2.1. *Let $\beta \geqslant 0$ and $h > h_c(\beta)$. For almost every environment, one has*

$$
\sum_{n=1}^{+\infty} Z_n^{\beta, h, \omega} < +\infty.
$$

Remark. This result implies that, in the interior of the delocalized phase, the *unconstrained* (or *free*) partition function $Z_{n,f}^{\beta, h, \omega}$ is also almost surely bounded (in fact, tends to 0) as n tends to infinity. Indeed, $Z_{n,f}^{\beta, h, \omega}$ is defined by

$$
Z_{n,f}^{\beta, h, \omega} = \mathbf{E}\left[\exp\left(\sum_{k=0}^{n-1} (\beta\omega_k - h)\mathbf{1}_{\{k \in \tau\}}\right)\right],
$$

which is equal to

$$
\sum_{n'=n}^{+\infty} \mathbf{E}\left[\exp\left(\sum_{k=0}^{n-1} (\beta\omega_k - h)\mathbf{1}_{\{k \in \tau\}}\right) ; \tau \cap [n, n'] = \{n'\}\right] \leqslant \sum_{n'=n}^{+\infty} Z_{n'}^{\beta, h, \omega} \xrightarrow[n \to \infty]{\text{a.s.}} 0.
$$

Our second result concerns the size of the set $\tau \cap [0, n]$, that we may call the set of *contact points*, under the measure $\mathbf{P}_n^{\beta, h, \omega}$. Let us write $E_{n,N}$ for the event that $|\tau \cap [0, n]| > N$ (where we write $|A|$ for the cardinal of a set A).

Theorem 2.2. *Let $\beta \geqslant 0$ and $h > h_c(\beta)$. For every $\varepsilon > 0$ and for almost every environment, there exists $N_\varepsilon, C_\varepsilon > 0$ such that for any $N \geqslant N_\varepsilon$ and any n :*

$$
\mathbf{P}_n^{\beta, h, \omega}(E_{n,N}) \leqslant \frac{C_\varepsilon}{K(n)} \, e^{-N(h - h_c(\beta) - \varepsilon)}.
$$

In particular, for every constant c such that

$$
c > \frac{1 + \alpha}{h - h_c(\beta)}
$$

and for almost every environment, one has

$$
\mathbf{P}_n^{\beta, h, \omega}(E_{n,c \log n}) \xrightarrow[n \to \infty]{} 0.
$$

To my knowledge, results of this kind were known only under the averaged measure $\mathbb{P}\mathbf{P}_n^{\beta, h, \omega}$, and with some restrictions on the distribution of ω due to the use of concentration arguments (see [GT05] or [G, Section 8.2]). In particular, in the interior of the delocalized phase and for almost every environment, the polymer intersects the pinning line less that the simple random walk does.

It is worth comparing this result with the case when randomness of the medium is absent, that is, when $\beta = 0$. In this context, the distribution of the number of contact points of the polymer forms a tight sequence as n varies (see for instance [dH, Theorem 7.3]). It is only natural to expect that a similar result holds true in the disordered case as well.

Interestingly, boundedness of the number of contact points in the delocalized phase was recently obtained for a specific model of pinning on a random interface with long-range correlations in [BL11], even at criticality ($h = h_c(\beta)$). In this work, the specific structure of the environment enables the authors to identify the critical point explicitly, a feature which makes the subsequent analysis more tractable.

3. PROOFS

In this section, we present the proofs of Theorems 2.1 and 2.2. Although one might think at first that such an approach cannot be of much help as far as the delocalized phase is concerned, we will rely on recent results obtained in [BGdH10, CdH10], where the authors develop a large deviations point of view of the problem. Let us define

$$(3.2) \qquad F_N^{\beta,h,\omega} = \sum_{0=l_0<l_1<\cdots<l_N} \prod_{i=0}^{N-1} K(l_{i+1} - l_i)e^{(\beta\omega_{l_i}-h)}.$$

Our results are based on the following fact, due to [CdH10], that holds both in the delocalized and in the localized phases.

Lemma 3.1. *For almost every environment, one has*

$$\limsup_{N\to+\infty} \frac{1}{N} \log F_N^{\beta,h,\omega} = h_c(\beta) - h.$$

Proof of Lemma 3.1. Although this result is not stated as a proposition in [CdH10], the authors give all the necessary elements to prove it. Indeed, we can start from [CdH10, (3.11)], which reads

$$\limsup_{N\to+\infty} \frac{1}{N} \log F_N^{\beta,h,\omega} = -h + S^{\text{que}}(\beta; 1),$$

where $S^{\text{que}}(\beta; z)$ is defined in [CdH10, (3.10)]. We then learn from [CdH10, (3.13)] that

$$h_c(\beta) = S^{\text{que}}(\beta; 1^-),$$

so what remains to see is that

$$S^{\text{que}}(\beta; 1^-) = S^{\text{que}}(\beta; 1).$$

The proof of this fact is first obtained assuming further that the support of the distribution of ω_0 is finite, see [CdH10, Lemma 3.2]. The general case, where one assumes only finiteness of all exponential moments, is considered in [CdH10, Section 3.3]. One can start by observing that, since $z \mapsto S^{\text{que}}(\beta, z)$ is increasing (in the wide sense), one has $S^{\text{que}}(\beta; 1^-) \leqslant S^{\text{que}}(\beta; 1)$. On the other hand, it is shown in step 1 of the proof of [CdH10, Lemma 3.3] that $S^{\text{que}}(\beta; 1) \leqslant A(\beta)$, where $A(\beta)$ is defined in [CdH10, (3.21)]. Moreover, steps 2 to 4 of the proof of [CdH10, Lemma 3.3] are devoted to the justification of the fact that $A(\beta) \leqslant S^{\text{que}}(\beta; 1^-)$. It thus follows that $S^{\text{que}}(\beta; 1) \leqslant S^{\text{que}}(\beta; 1^-)$, which finishes the proof. $\square$

Proof of Theorem 2.1. The proof is close to [CdH10, Section 3.2]. We can decompose Z_n the following way :

$$Z_n^{\beta,h,\omega} = \sum_{N=1}^{+\infty} \sum_{0=l_0<l_1<\cdots<l_N=n} \prod_{i=0}^{N-1} K(l_{i+1} - l_i)e^{(\beta\omega_{l_i}-h)}.$$

An interversion of sums then leads to

$$\sum_{n=1}^{+\infty} Z_n^{\beta,h,\omega} = \sum_{N=1}^{+\infty} F_N^{\beta,h,\omega},$$

and Lemma 3.1 ensures the almost sure convergence of the second series when $h > h_c(\beta)$. □

For an event A, let us write $Z_n^{\beta,h,\omega}(A)$ for the quantity

$$\mathbf{E}\left[\exp\left(\sum_{k=0}^{n-1}(\beta\omega_k - h)\mathbf{1}_{\{k\in\tau\}}\right)\mathbf{1}_{\{n\in\tau\}} \ ; \ A\right].$$

In words, $Z_n^{\beta,h,\omega}(A)$ is a partition function in which one integrates with respect to $\mathbf{P}$ only on the event A. In order to prove Theorem 2.2, we first give a refined version of Theorem 2.1, which goes as follows.

Proposition 3.2. *Let $\beta \geqslant 0$ and $h > h_c(\beta)$. For every $\varepsilon > 0$ and for almost every environment, there exist $N_\varepsilon, C_\varepsilon$ such that for any $N \geqslant N_\varepsilon$:*

$$\sum_{n=1}^{+\infty} Z_n^{\beta,h,\omega}(E_{n,N}) \leqslant C_\varepsilon e^{-N(h-h_c(\beta)-\varepsilon)}.$$

Proof. We can assume that $\varepsilon < h - h_c(\beta)$. Note that, for any n and N_0,

$$Z_n^{\beta,h,\omega}(E_{n,N_0}) = \sum_{N=N_0}^{+\infty} \sum_{0=l_0<l_1<\cdots<l_N=n} \prod_{i=0}^{N-1} K(l_{i+1}-l_i)e^{(\beta\omega_{l_i}-h)}.$$

By an interversion of sums, we obtain that

$$\sum_{n=1}^{+\infty} Z_n^{\beta,h,\omega}(E_{n,N_0}) = \sum_{N=N_0}^{+\infty} F_N^{\beta,h,\omega}.$$

By Lemma 3.1, there exists N_ε such that for every $N \geqslant N_\varepsilon$,

$$F_N^{\beta,h,\omega} \leqslant e^{-N(h-h_c(\beta)-\varepsilon/2)},$$

and as a consequence, for every $N_0 \geqslant N_\varepsilon$, one has

$$\sum_{n=1}^{+\infty} Z_n^{\beta,h,\omega}(E_{n,N_0}) \leqslant \sum_{N=N_0}^{+\infty} e^{-N(h-h_c(\beta)-\varepsilon/2)},$$

which implies the announced claim. □

Proof of Theorem 2.2. Note that

$$\mathbf{P}^{\beta,h,\omega}(E_{n,N}) = \frac{Z_n^{\beta,h,\omega}(E_{n,N})}{Z_n^{\beta,h,\omega}}.$$

The numerator can be bounded from above using Proposition 3.2. For the denominator, one can use the bound

$$Z_n^{\beta,h,\omega} \geqslant K(n)e^{\beta\omega_0-h},$$

which proves the desired result. □

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