PRINCIPAL EIGENVALUE FOR THE RANDOM WALK AMONG RANDOM TRAPS ON $\mathbb{Z}^d$

JEAN-CHRISTOPHE MOURRAT

Abstract. Let $(\tau_x)_{x\in\mathbb{Z}^d}$ be i.i.d. random variables with heavy (polynomial) tails. Given $a \in [0,1]$, we consider the Markov process defined by the jump rates $\omega_{x\to y} = {\tau_x}^{-(1-a)}{\tau_y}^{a}$ between two neighbours x and y in $\mathbb{Z}^d$. We give the asymptotic behaviour of the principal eigenvalue of the generator of this process, with Dirichlet boundary condition. The prominent feature is a phase transition that occurs at some threshold depending on the dimension.

1. Introduction

For each site $x \in \mathbb{Z}^d$, let $\tau_x > 0$ be a random variable, so that $(\tau_x)_{x\in\mathbb{Z}^d}$ are independent and identically distributed. We call $\tau = (\tau_x)_{x\in\mathbb{Z}^d}$ the *environment*, and write its law $\mathbb{P}$ (and the corresponding expectation $\mathbb{E}$). Fixing $a \in [0,1]$ and an environment τ, we define the Markov process $(X_t)_{t\geqslant 0}$ by the following jump rates :

$$\omega_{x\to y} = \left| \begin{array}{ll} {\tau_x}^{-(1-a)}{\tau_y}^{a} & \text{if } \|x-y\| = 1, \\ 0 & \text{otherwise.} \end{array} \right.$$

We write $\mathbf{P}^\tau_x$ for the law of the process starting from site x, and $\mathbf{E}^\tau_x$ for the corresponding expectation. The associated infinitesimal generator is :

$$\mathcal{L}f(x) = \sum_{y:\|x-y\|=1} \omega_{x\to y}(f(y) - f(x)).$$

The aim of this note is to investigate the behaviour of the principal eigenvalue of $\mathcal{L}$ restricted to a large box. Define the box of size n by $B_n = \{-n,\ldots,n\}^d$, and $\mathcal{L}_n$ the operator $\mathcal{L}$ restricted to this box, with Dirichlet boundary conditions. That is to say $\mathcal{L}_n f = \mathbf{1}_{B_n}\mathcal{L}f$, defined for any function $f : \mathbb{Z}^d \to \mathbb{R}$ that vanishes outside the box. Let λ_n be the smallest eigenvalue of $-\mathcal{L}_n$. We write λ_n° for the eigenvalue obtained in the particular case when $a = 0$.

We are particularly interested in the study of heavy tailed laws for the environment. A natural assumption (see the remark just after Theorem 1.2) is that the tail probability $\mathbb{P}[\tau_0 > y]$, that we will write $F(y)$, decays like a power of y as y goes to infinity. We say that a function f varies regularly with index ρ at infinity, and write $f \in RV_\rho$, if for all $\kappa > 0$, $f(\kappa x)/f(x) \to \kappa^\rho$ as $x \to +\infty$ (see [BGT] for a monograph on regular variation).

Assumption 1. There exists $\alpha > 0$ such that $F \in RV_{-\alpha}$.

Roughly speaking, this assumption can be reformulated as

$$\mathbb{P}[\tau_0 > y] \simeq \frac{1}{y^\alpha} \qquad (y \to +\infty), \tag{1.1}$$

although it is in fact more general than just assuming the equality (or equivalence) in equation (1.1). Note that, for $0 < \alpha < 2$, τ_0 belongs to the domain of attraction of an α-stable law if and only if $F \in RV_{-\alpha}$ (see [Fe2, Corollary XVII.5.2]).

Assumption 2. We will always assume that $\tau_0 \geqslant 1$, concentrating on "bad behaviours" at infinity.

We need to introduce the generalized inverse of $1/F$, defined by :

$$h(x) = \inf\{y : 1/F(y) \geqslant x\}.$$

As F belongs to $RV_{-\alpha}$, one can see that $h \in RV_{1/\alpha}$ (see for instance [Re, Proposition 0.8 (v)]). Loosely speaking, $h(y) \simeq y^{1/\alpha}$. We will recall later how h is related to the asymptotic behaviour of maxima and sums of (τ_x) (see Proposition 2.1), but let us first state and comment our main results. We stress that they hold for any $a \in [0, 1]$.

Theorem 1.1. *For almost every environment, we have :*

$$\lim_{n\to\infty} -\frac{\ln(\lambda_n)}{\ln(n)} = \left| \begin{array}{ll} \max\left(2, 1+\dfrac{1}{\alpha}\right) & \text{if } d = 1, \\ \max\left(2, \dfrac{d}{\alpha}\right) & \text{if } d \geqslant 2. \end{array} \right.$$

For certain values of the parameters α and d, we are able to describe more precisely the behaviour of λ_n.

Theorem 1.2. (1) *If $d \geqslant 2$ and $\alpha > d/2$, or if $d = 1$ and $\alpha > 1$, then there exist $k_1, k_2 > 0$ such that for almost every environment and n large enough :*

$$\frac{k_1}{n^2} \leqslant \lambda_n \leqslant \frac{k_2}{n^2}.$$

(2) *If $\alpha < 1$ and $d \neq 2$, then for any $\varepsilon > 0$, there exist $\eta, M > 0$ such that for all n large enough :*

$$\mathbb{P}[\eta \leqslant a_n\lambda_n \leqslant M] \geqslant 1 - \varepsilon,$$

where

$$a_n = \left| \begin{array}{ll} nh(n) & \text{if } d = 1, \\ h(n^d) & \text{if } d \geqslant 3. \end{array} \right.$$

(3) *Let $a_n = \ln(n)h(n^2)$. If $d = 2$ and $\alpha < 1$, then for any $\varepsilon > 0$, there exist $\eta, M > 0$ such that for all n large enough :*

$$\mathbb{P}[\eta \leqslant a_n\lambda_n^\circ \leqslant M] \geqslant 1 - \varepsilon,$$

$$\mathbb{P}[\eta \leqslant a_n\lambda_n \leqslant \ln(n)M] \geqslant 1 - \varepsilon.$$

Let us now give some heuristics about the behaviour of (X_t). If $a = 0$, the walk is in fact a time-change of the simple random walk : arriving at some site x, it waits an exponential time of mean τ_x before jumping to a neighbouring site chosen uniformly. When $a \neq 0$, things get more complicated. Suppose that the walk arrives at some deep trap, that is a site x where τ_x is very large. Compared with the $a = 0$ case, the walk will leave site x faster. On the other hand, once on a neighbouring site, it will come back to x with very high probability. These two competing effects can compensate remarkably in the limit, and indeed our main results are independent of a (as they also are in [BČ05]).

We propose to call $(X_t)_{t\geqslant 0}$ a *random walk among random traps.* It seems to us that for its relative simplicity, it should be considered one of the basic types of random walks in random environments to study, just as is the random walk among random conductances. Although one could have the feeling that these two types are basically the same, one attaching randomness to edges of the graph and the other to sites, they exhibit very different behaviours. For instance, the reversible measure is not the uniform one in the case of random traps (it gives weight τ_x to site x). Also, if $d \geqslant 2$, the random walk in random conductances tends to avoid visiting regions where conductance is very low (and where time spent to "get out" may be high). On the other hand, when walking among random traps, say for $a = 0$, the path is the same as for the simple random walk, and the walk is not inclined to avoid regions from which it takes a long time to get out. See [Al81] for a nice discussion about this issue.

This type of walk gained interest when J.P. Bouchaud [Bo92] proposed it as a phenomenological model to explain aging of glassy systems, and as a consequence, what we call "random walk among random traps" is also known as *Bouchaud's trap model*. Later on, [RMB00] introduced the full model as presented here (including the $a \in [0,1]$), which allows them to get more diverse aging behaviours.

When $\mathbb{E}[\tau_0]$ is finite (in particular when $\alpha > 1$), one can apply results of [DFGW89] to prove that, under the averaged law, (X_t) is diffusive and converges to Brownian motion after rescaling.

For $a = 0$, $\alpha < 1$ and in dimension 1, [FIN02] proved that the process was subdiffusive, and obtained convergence of the rescaled process to a singular diffusion, as well as aging. The results have been extended to general a in [BČ05]. Another (also subdiffusive) scaling limit, called the *fractional kinetics process*, was identified when $a = 0$, $\alpha < 1$ and $d \geqslant 2$ in [BČ07]. We refer to [BČ06] for a review on the subject.

To our knowledge, these were the only results available when this note was made public. More recently, [BČ09] have shown that, for $d \geqslant 3$, the convergence towards the fractional kinetics process holds for any $a \in [0,1]$ (see also [Mo09] for a different proof of this result when $d \geqslant 5$). For $a = 0$, $\alpha < 1$ and in dimension 1, [Fa09] have now obtained a detailed description of the spectrum. Finally, for $a = 0$, $\alpha < 1$ and $d \geqslant 2$, [JLT09] have shown that, in the time scale of $(\lambda_n)^{-1}$, the random walk on B_n with periodic boundary rescales towards the K-process introduced in [FM08].

This note comes as a partial answer to a question of [BČ06], asking for the "nature of the spectrum of the Markov chain close to its edge. Naturally, the long time behaviour of X_t can be understood from the edge of the spectrum of the generator $\mathcal{L}$. This question deserves further study (see [BF05], [BF08] and also [MB97])."

Upper bounds on λ_n are obtained rather easily, using its variational characterisation (see equation (1.2)), and then choosing appropriate test functions. An exception should however be pointed out for the upper bound on λ_n in part (3) of Theorem 1.2. While the upper bound can probably be improved to match the lower bound if $\mathbb{E}[(\tau_0)^a]$ is finite, a complete answer remains unclear to us.

As far as lower bounds are concerned, a simple argument shows that it suffices to consider the case when $a = 0$ (see inequality (1.3)). When the random variables (τ_x) are not integrable, the matching lower bounds can be obtained using the fact that the sum and the maximum of $(\tau_x)_{x \in B_n}$ are of the same order of magnitude. Finding the missing lower bounds when $\mathbb{E}[\tau_0]$ is finite is however more difficult. Remarkably, the classical techniques exposed for instance in the review [SC97], although giving the appropriate bounds in certain cases, did not enable us to conclude in general. We show in section 6 that the distinguished path method (see e.g. [SC97, Theorem 3.2.3]), that proved efficient for instance in [FM06, Section 3] for random walks among random conductances, is bound to give an extra 1 in the exponent when $d \geqslant 2$ (for the one-dimensional case, [Ch, Section 3.7] proves that the method is sharp, as can be checked directly in our context). In order to solve the problem, we use the fact that $(\lambda_n)^{-1}$ is comparable to

$$\sup_{x \in B_n} \mathbf{E}_x^\tau[T_n],$$

where T_n is the exit time from B_n (Proposition 4.1). Using the properties of the Green function of the embedded discrete time random walk, one can see that $\mathbf{E}_0^\tau[T_n]$ is typically of the order of n^2. Loosely speaking, we show by a computation of moments that for any $\varepsilon > 0$, the probability that the fluctuations of $\mathbf{E}_0^\tau[T_n]$

exceed $n^{d/\alpha+\varepsilon}$ is $o(n^{-d})$. This ensures that, for any $\varepsilon > 0$,

$$\sup_{x \in B_n} \mathbf{E}_x^\tau[T_n] \leqslant C(n^2 + n^{d/\alpha+\varepsilon}),$$

which gives us sufficient information to derive the almost sure lower bounds of Theorem 1.1 and part 1 of Theorem 1.2. We point out however that, as concerns the precise asymptotics of Theorem 1.2, it leaves a gap for $\alpha \in [1, d/2]$.

We would also like to draw the reader's attention to the fact that this method gives little indication on how to extend the results to a conservative dynamics (for instance, with periodic boundary conditions instead of Dirichlet).

Remark. A natural choice of (τ_x) from the statistical physics' point of view is the following : first choose independently for each site a random variable $-E_x$ with law exponential of parameter 1, and define τ_x to be $\exp(-\beta E_x)$, where β represents the inverse of the temperature. Then one can check that $F \in RV_{-1/\beta}$, and the irregularity that appears at $\beta = 1$ for $d \leqslant 2$ and at $\beta = 2/d$ for larger d can be regarded as a phase transition (the anomalous behaviour occurring for β large, that is for small temperature, or in our context, small α).

It may seem surprising that this new phase transition does not appear at the same threshold than the diffusive/subdiffusive transition, which, as far as one knows, occurs when $\alpha (= 1/\beta) = 1$ in any dimension. The reason for this is the following : although the principal eigenvalue will "feel" the very deepest traps of the box (of order $n^{d/\alpha}$), the process started at the origin will exit the box after visiting only some n^2 sites, thus having seen only traps of order at most $n^{2/\alpha}$.

Lastly, we would like to mention that on the complete graph and for $a = 0$, [BF05] got explicit formulas for the whole spectrum and managed to link them with aging properties.

Apart from this introduction, the paper is divided into five sections. In section 2, we recall some classical consequences of Assumption 1 concerning the asymptotic behaviour of sums and maxima of (τ_x). We begin the analysis of the problem in section 3 using the variational characterisation of the principal eigenvalue, which gives bounds on λ_n° and λ_n that are sharp when $\alpha \leqslant 1$ or $d = 1$. In order to find a good lower bound on λ_n° (easily extended to a lower bound on λ_n) when $d \geqslant 2$ and $\alpha > 1$, we introduce in section 4 the embedded discrete time random walk. When $a = 0$, it is the simple random walk, and the explicit knowledge of its Green function enables us to conclude. In section 5, upper bounds for λ_n are computed. Finally, we analyse the limitation of the distinguished path method in section 6.

Let us see how to deduce Theorem 1.1 from the rest of the paper. Regarding lower bounds on λ_n, an elementary observation is that $\lambda_n \geqslant \lambda_n^\circ$ (see (1.3)). As a consequence, part (2) of Proposition 3.3 gives an upper bound on the exponent of the principal eigenvalue, that needs to be improved when $d \geqslant 3$ and $\alpha > 1$. This is done by Proposition 4.6. Now for the associated lower bounds on the exponent of the principal eigenvalue, they come from Proposition 5.1 and part (2) of Proposition 2.1 if $d = 1$; from part (2) of Proposition 5.2 and Proposition 5.5 if $d \geqslant 2$.

Concerning part (1) of Theorem 1.2, if $d = 1$ and $\alpha > 1$, the lower bound on λ_n comes from part (3) of Proposition 3.3. If $d \geqslant 2$ and $\alpha > d/2$, the lower bound is given by part (2) of Proposition 4.6. In any case, Proposition 5.5 gives the desired upper bound on λ_n.

Finally, for parts (2) and (3) of Theorem 1.2, part (1) of Proposition 3.3 gives the desired result for λ_n° as well as a lower bound on λ_n. In dimension one, the upper estimate on λ_n is given by Proposition 5.1 and part (4) of Proposition 2.1, while if $d \geqslant 2$, it comes from part (1) of Proposition 5.2 together with part (3) of Proposition 2.1.

Notations. We write $(\cdot,\cdot)$ for the scalar product defined by :

$$(f,g) = \sum_{x\in\mathbb{Z}^d} f(x)g(x)\tau_x,$$

and $L^2(B_n)$ for the set of functions that vanish outside B_n (equipped with the above scalar product). The operator $\mathcal{L}_n$ is self-adjoint in $L^2(B_n)$.

For two points $x, y \in \mathbb{Z}^d$, we write $x \sim y$ when they are neighbours (that is, when $\|x-y\| = 1$). We define the Dirichlet form associated to $\mathcal{L}$:

$$\mathcal{E}(f,g) = (-\mathcal{L}f, g) = \frac{1}{2}\sum_{\substack{x,y\in\mathbb{Z}^d\\ x\sim y}} \tau_x^a \tau_y^a (f(y)-f(x))(g(y)-g(x)),$$

and $\mathcal{E}^\circ$ the Dirichlet form obtained when $a = 0$. We have :

$$\lambda_n = \inf_{\substack{f\in L^2(B_n)\\ f\neq 0}} \frac{\mathcal{E}(f,f)}{(f,f)}. \tag{1.2}$$

Assumption 2 gives that $\mathcal{E}(f,f) \geqslant \mathcal{E}^\circ(f,f)$, so it is clear that

$$\lambda_n \geqslant \lambda_n^\circ. \tag{1.3}$$

We further need to define the boundary of B_n, as $\partial B_n = B_{n+1} \setminus B_n$. If K is some set, $|K|$ stands for its cardinal.

The real number $C > 0$ represents a generic constant that need not be the same from one occurrence to another.

2. Asymptotic behaviour of sums and maxima

In this section, we briefly recall some classical consequences of Assumption 1. First of all, it implies that for any $\varepsilon > 0$:

$$F(y)y^{\alpha+\varepsilon} \xrightarrow[y\to+\infty]{} +\infty \quad \text{and} \quad F(y)y^{\alpha-\varepsilon} \xrightarrow[y\to+\infty]{} 0, \tag{2.1}$$

and as a consequence, $\mathbb{E}[\tau_0^\beta]$ is finite for all $\beta < \alpha$, infinite for all $\beta > \alpha$ (and may be finite or infinite when $\beta = \alpha$).

The following proposition describes the asymptotic behaviour of the sum and the maximum of (τ_x) over the box B_n.

Proposition 2.1. (1) *For any $\varepsilon > 0$ and almost every environment :*

$$n^{-(\max(d,d/\alpha)+\varepsilon)} \sum_{x\in B_n} \tau_x \to 0 \qquad (n \to +\infty).$$

(2) *For any $\varepsilon > 0$ and almost every environment :*

$$n^{-(\max(d,d/\alpha)-\varepsilon)} \sum_{x\in B_n} \tau_x \to +\infty \qquad (n \to +\infty).$$

(3) *There exists a random variable M_∞ with values in $(0,+\infty)$ such that the rescaled maxima converge in law to M_∞ :*

$$\frac{1}{h(n^d)} \max_{x\in B_n} \tau_x \to M_\infty \qquad (n \to +\infty).$$

(4) *If $\alpha < 1$, then there exists a random variable S_∞ with values in $(0,+\infty)$ such that the rescaled partial sums converge in law to S_∞ :*

$$\frac{1}{h(n^d)} \sum_{x\in B_n} \tau_x \to S_\infty \qquad (n \to +\infty).$$

Proof. For the first statement, it is a consequence of the law of large numbers if $\alpha > 1$, otherwise it is an application of [Pe, Theorem 6.9]. For the second one, it comes again from the law of large numbers if $\alpha > 1$. Otherwise, observe that the sum is larger than the maximum of its terms, and

$$\mathbb{P}\left[\max_{x \in B_n} \tau_x \leqslant M n^{d/\alpha - \varepsilon}\right] = (1 - F(M n^{d/\alpha-\varepsilon}))^{(2n+1)^d}.$$

Using the properties of F (see (2.1)), we see that the latter is the general term of a convergent series, and we can apply the Borel-Cantelli lemma. Now the convergence of the rescaled maxima is given in [Fe2, Section VIII.8] or [Re, Proposition 1.11]. For the convergence of the partial sums, see [Fe2, Section XVII.5]. □

3. The variational formula

We will use here the variational characterisation of λ_n° :

$$\lambda_n^\circ = \inf_{\substack{f \in L^2(B_n) \\ f \neq 0}} \frac{\mathcal{E}^\circ(f,f)}{(f,f)}. \tag{3.1}$$

We define the conductance between the origin and ∂B_n as

$$C_n = \inf\left\{\mathcal{E}^\circ(f,f) \mid f \in L^2(B_n), f(0) = 1\right\}.$$

Noting that B_n is a finite set, one can see by a compactness argument that the infimum is reached for some function V_n. The behaviours of C_n and λ_n° are related in the following way.

Proposition 3.1. *For any n and any environment, we have :*

$$\frac{C_{2n}}{\sum_{x \in B_n} \tau_x} \leqslant \lambda_n^\circ,$$

$$\lambda_{2n+1}^\circ \leqslant \lambda_{2n}^\circ \leqslant \frac{C_n}{\max_{B_n} \tau}.$$

Proof. Considering the homogeneity of the quotient in (3.1), we can restrict the infimum to be taken over all f with $\|f\|_\infty = 1$. Let f be such a function, and $x_0 \in B_n$ such that $|f(x_0)| = 1$. Possibly changing f to $-f$, we can assume $f(x_0) = 1$. Noting that the function $g = f(\cdot + x_0)$ is in $L^2(B_{2n})$ and satisfies $g(0) = 1$, we have :

$$\mathcal{E}^\circ(f,f) = \mathcal{E}^\circ(g,g) \geqslant C_{2n}.$$

On the other hand, as $\|f\|_\infty = 1$, we have :

$$(f,f) \leqslant \sum_{x \in B_n} \tau_x,$$

and these lead to the first desired inequality.

The fact that $\lambda_{2n+1}^\circ \leqslant \lambda_{2n}^\circ$ is clear from (3.1). Now let $x_1 \in B_n$ be such that $\max_{B_n} \tau = \tau_{x_1}$, and consider the function $h = V_n(\cdot - x_1) \in L^2(B_{2n})$. We get :

$$\mathcal{E}^\circ(h,h) = \mathcal{E}^\circ(V_n, V_n) = C_n.$$

But note that $h(x_1) = 1$, therefore :

$$(h,h) \geqslant \tau_{x_1} = \max_{B_n} \tau,$$

and we get the second inequality. □

We now describe the asymptotic behaviour of C_n.

Proposition 3.2. *If $d = 1$, then :*

$$C_n = \frac{2}{n+1}.$$

If $d = 2$, then there exist k_1, k_2 such that for all n :

$$\frac{k_1}{\ln(n)} \leqslant C_n \leqslant \frac{k_2}{\ln(n)}.$$

If $d \geqslant 3$, then C_n converges to a strictly positive number.

Proof. We can regard B_{n+1} as an electrical network (see [LP, Chapter 2]), with each edge representing a resistance of value 1. One can see that V_n is harmonic on every point that is not 0 nor a point of ∂B_n. Thus it coincides with the potential on the electrical network, with the constraints that $V_n(0) = 1$ and $V_{n|\partial B_n} = 0$. The number C_n is the effective conductance between 0 and ∂B_n. In dimension 1, a direct computation gives the result. If $d = 2$, then we can use [LP, Proposition 2.14]. In larger dimensions, the simple random walk is transient, and therefore (see [LP, Theorem 2.3]) C_n converges to a strictly positive number. □

From this, we can deduce the following.

Proposition 3.3. (1) *If $\alpha < 1$, then for any $\varepsilon > 0$, there exist $\eta, M > 0$ such that for all n large enough :*

$$\mathbb{P}\left[\eta \leqslant \frac{h(n^d)}{C_n}\lambda_n^\circ \leqslant M\right] \geqslant 1 - \varepsilon,$$

(2) *For almost every environment, we have :*

$$\limsup_{n\to\infty} -\frac{\ln(\lambda_n^\circ)}{\ln(n)} \leqslant \left| \begin{array}{ll} \max\left(2, 1 + \dfrac{1}{\alpha}\right) & \text{if } d = 1, \\ \max\left(d, \dfrac{d}{\alpha}\right) & \text{if } d \geqslant 2. \end{array} \right.$$

(3) *If $\mathbb{E}[\tau_0]$ is finite, then for almost every environment and all n large enough :*

$$\lambda_n^\circ \geqslant \frac{C_{2n}}{(2n+1)^d(\mathbb{E}[\tau_0] + 1)}.$$

Proof. The first part of the proposition is a consequence of Propositions 3.1, 3.2 and parts (3) and (4) of Proposition 2.1. For the second part, use part (1) of Proposition 2.1 instead. The last part is an application of the law of large numbers. □

We recall from inequality (1.3) that $\lambda_n \geqslant \lambda_n^\circ$. Hence, as far as lower bounds are concerned, parts (2) and (3) of Theorem 1.2 are now obtained. However, part (1) is proved only for $d = 1$, and Theorem 1.1 only for $d \leqslant 2$ or $\alpha \leqslant 1$. The following section provides the missing lower bounds.

4. Exit time upper bounds when $a = 0$

This section aims at finding good lower bounds for λ_n when $d \geqslant 2$ and $\alpha > 1$. To do so, we will use the exit times T_n from B_n :

$$T_n = \inf\{t \geqslant 0 : X_t \notin B_n\}.$$

The principal eigenvalue and the exit time from B_n are indeed related by the following (general) result :

Proposition 4.1. *For any environment τ, any $n \in \mathbb{N}$ and $t \geqslant 0$, we have*

$$e^{-t\lambda_n} \leqslant \sup_{x\in B_n} \mathbf{P}_x^\tau[T_n > t] \leqslant \frac{\sup_{x\in B_n} \mathbf{E}_x^\tau[T_n]}{t}.$$

Proof. Let ψ_n be the eigenfunction associated with the principal eigenvalue λ_n such that $\sup \psi_n = 1$.

$$\mathbf{E}^\tau_x[\psi_n(X_t)\mathbf{1}_{\{T_n>t\}}] = e^{-t\lambda_n}\psi_n(x).$$

Choosing $x \in B_n$ such that $\psi_n(x) = 1$, we have :

$$\mathbf{P}^\tau_x[T_n > t] \geqslant \mathbf{E}^\tau_x[\psi_n(X_t)\mathbf{1}_{\{T_n>t\}}] = e^{-t\lambda_n}.$$

The second inequality is Markov's inequality. □

Our objective is to find a sharp upper bound for $\sup_{x\in B_n} \mathbf{E}^\tau_x[T_n]$. As noted in inequality (1.3), finding a lower bound for λ°_n is sufficient. Therefore, we assume in this section that $\mathbf{a} = \mathbf{0}$, and also that $\mathbf{d} \geqslant \mathbf{2}$.

We introduce the embedded discrete time random walk $(Y_n)_{n\in\mathbb{N}}$, and the jump instants $(J_n)_{n\in\mathbb{N}}$, so that

$$J_n \leqslant t < J_{n+1} \quad \Rightarrow \quad X_t = Y_n.$$

As we assumed here that $a = 0$, it is clear that conditionally on $Y_n = x$, the time $J_{n+1} - J_n$ spent by the walk at site x is an exponential variable of mean τ_x. Let $G_n(x, y)$ be the number of visits before exiting B_n at site y for the walk Y starting at x :

$$\hat{T}_n = \inf\{k : Y_k \notin B_n\} \quad \text{and} \quad G_n(x, y) = \mathbf{E}^\tau_x\left[\sum_{k=0}^{\hat{T}_n-1} \mathbf{1}_{\{Y_k=y\}}\right].$$

Note that $G_n(x, y)$, as the expectation of a functional of Y, is non-random. As a consequence of the above remark, the expected total time spent by the walk X at site x before exiting B_n is τ_x times the number of visits of Y at site x. In other words :

$$\mathbf{E}^\tau_x[T_n] = \sum_{y\in B_n} G_n(x, y)\tau_y. \tag{4.1}$$

Roughly speaking, we will see that the expectation of this sum behaves like n^2 (assuming $\alpha > 1$), and that the probability to be far from the expectation by $n^{d/\alpha}$ is of order n^{-d}. To estimate these fluctuations, our method will be to compute moments after truncation and centring of the τ_x. To do so, the first thing we need is to find convenient upper bounds for $G_n(\cdot, \cdot)$.

Proposition 4.2. (1) *There exists $C_1 > 0$ such that for any integer n :*

$$\sum_{y\in B_n} G_n(0, y) \leqslant C_1 n^2.$$

(2) *If $d \geqslant 3$, then there exists $C_2 > 0$ such that for any integer n and any $x \in \mathbb{Z}^d$:*

$$G_n(0, x) \leqslant \frac{C_2}{(1 + \|x\|)^{d-2}}.$$

(3) *If $d = 2$, then there exists $C_3 > 0$ such that for any integer n and any $x \in \mathbb{Z}^d$:*

$$G_n(0, x) \leqslant C_3 \ln(n).$$

Proof. For the first part, note that

$$\sum_{y\in B_n} G_n(0, y) = \mathbf{E}^\tau_0\left[\sum_{k=0}^{\hat{T}_n-1} \mathbf{1}_{\{Y_k\in B_n\}}\right] = \mathbf{E}^\tau_0[\hat{T}_n].$$

As given for instance by [Fe1, Section XIV.3]), the expectation of the exit time of the first coordinate of Y from $\{-n, \ldots, n\}$ is bounded by a constant times n^2. It is clear that this quantity is an upper bound for $\mathbf{E}^\tau_0[\hat{T}_n]$. The second inequality is a consequence of [La, Theorem 1.5.4], while the last comes from [La, Theorem 1.6.6]. □

We begin by truncating and centring the random variables (τ_x). Let $\alpha' < \alpha$ (remember that $\mathbb{E}[\tau_0^{\alpha'}]$ is finite). For convenience, we impose on α' the additional condition

$$\alpha' \leqslant 2 \text{ if } d \leqslant 3. \tag{4.2}$$

As we will see in the proof of Proposition 4.6, this restriction is of no consequence for our purpose. We define the following truncation of τ_x :

$$\tilde{\tau}_{x,n} = \left| \begin{array}{ll} \tau_x & \text{if } \tau_x \leqslant n^{d/\alpha'}, \\ 0 & \text{otherwise} \end{array} \right.$$

(observe that with high probability, we have $\tau_x = \tilde{\tau}_{x,n}$ for every $x \in B_n$), and let $\overline{\tau}_{x,n} = \tilde{\tau}_{x,n} - \mathbb{E}[\tilde{\tau}_{x,n}]$.

We proceed to show the following proposition, that roughly speaking states that fluctuations of order $n^{d/\alpha'}$ of the exit time from 0 occur with probability smaller than n^{-d}.

Proposition 4.3. *For any $\beta > d/\alpha'$, there exist $\delta, C > 0$ such that for all n :*

$$\mathbb{P}\left[\left| \sum_{x \in B_n} G_n(0,x) \overline{\tau}_{x,n} \right| > n^\beta \right] \leqslant \frac{C}{n^{d+\delta}}.$$

Proof. Let m be an integer. We have :

$$\begin{aligned} &\mathbb{E}\left[\left(\sum_{x \in B_n} G_n(0,x) \overline{\tau}_{x,n} \right)^{2m} \right] \\ &\quad = \sum_{x_1, \ldots, x_{2m}} G_n(0,x_1) \cdots G_n(0,x_{2m}) \mathbb{E}[\overline{\tau}_{x_1,n} \cdots \overline{\tau}_{x_{2m},n}] \\ &\quad = \sum_{k=1}^{m} \sum_{\substack{e_1 + \cdots + e_k = 2m \\ e_i \geqslant 2}} C_{e_1,\ldots,e_k} \sum_{\substack{y_1,\ldots,y_k \\ y_i \neq y_j}} \prod_{i=1}^{k} G_n(0,y_i)^{e_i} \mathbb{E}[\overline{\tau}_{y_i,n}^{e_i}] \qquad (4.3) \\ &\quad \leqslant C(m) \sum_{k=1}^{m} \sum_{\substack{e_1 + \cdots + e_k = 2m \\ e_i \geqslant 2}} \underbrace{\prod_{i=1}^{k} \sum_{x \in B_n} G_n(0,x)^{e_i} |\mathbb{E}[\overline{\tau}_{0,n}^{e_i}]|}_{=:\Pi^n_{e_1,\ldots,e_k}}, \end{aligned}$$

where, to get the second equality, we chose to decompose $x_1, \ldots, x_{2m}$ the following way : let k be the cardinal of $\{x_1, \ldots, x_{2m}\}$. We have $\{x_1, \ldots, x_{2m}\} = \{y_1, \ldots, y_k\}$. Then e_i represents then number of occurrences of y_i in $x_1, \ldots, x_{2m}$. We then use the fact that the random variables $(\overline{\tau}_{x,n})_{x \in \mathbb{Z}^d}$ are independent to split the expectation in product form. Note that as $\overline{\tau}_{x,n}$ is a centred random variable, the cases when $e_i = 1$ for some i do not contribute to the sum, so it is enough to consider cases when $e_i \geqslant 2$ (and this implies $k \leqslant m$). It is a nice combinatorics exercise to check that $C_{e_1,\ldots,e_k}$ is the multinomial coefficient associated with $(e_1, \ldots, e_k)$ divided by $k!$, but the important fact is that this term does not depend on n.

We will now determine the asymptotic behaviour of the $\Pi^n_{e_1,\ldots,e_k}$. If $d \geqslant 3$, using part (2) of Proposition 4.2, one knows that

$$\sum_{x \in B_n} G_n(0,x)^{e_i} \leqslant C \sum_{x \in B_n} (1 + \|x\|)^{-e_i(d-2)},$$

which, by comparison with an integral, is bounded by :

$$\left| \begin{array}{ll} C\ln(n) & \text{if } d \geqslant 4 \text{ or } e_i \geqslant 3, \\ Cn & \text{if } d \geqslant 3. \end{array} \right.$$

On the other hand, $|\mathbb{E}[\overline{\tau}_{0,n}^{e_i}]|$ is bounded when n goes to infinity if $e_i \leqslant \alpha'$, and otherwise

$$|\mathbb{E}[\overline{\tau}_{0,n}^{e_i}]| \leqslant \mathbb{E}[|\overline{\tau}_{0,n}|^{(e_i-\alpha')+\alpha'}] \leqslant (n^{d/\alpha'})^{e_i-\alpha'}\mathbb{E}[|\overline{\tau}_{0,n}|^{\alpha'}] \leqslant Cn^{e_i d/\alpha' - d}. \tag{4.4}$$

We first treat the case $d \geqslant 4$. We choose m as the smallest integer larger than (or equal to) $\alpha'/2$. All the $\Pi^n_{e_1,\ldots,e_k}$ are bounded by $C\ln(n)^m$ when n goes to infinity except :

$$\Pi^n_{2m} \leqslant C\ln(n)n^{2md/\alpha'-d}.$$

It comes, using Markov's inequality, that there exists C such that for any n :

$$\mathbb{P}\left[\left|\sum_{x\in B_n} G_n(0,x)\overline{\tau}_{x,n}\right| > n^\beta\right] \leqslant Cn^{-d}\ln(n)^m n^{2m(d/\alpha'-\beta)},$$

which proves the desired result.

When $d = 3$, remember from (4.2) that $\alpha' \leqslant 2$. We choose $m = 2$ in (4.3) and get :

$$\Pi^n_{2,2} \leqslant Cn^2 n^{12/\alpha'-6} \quad \text{and} \quad \Pi^n_4 \leqslant C\ln(n)n^{12/\alpha'-3},$$

and it comes that :

$$\mathbb{P}\left[\left|\sum_{x\in B_n} G_n(0,x)\overline{\tau}_{x}\right| > n^\beta\right] \leqslant Cn^{-3}\ln(n)n^{4(3/\alpha'-\beta)},$$

which proves the proposition, and we are left with the two-dimensional case. From the estimates of Proposition 4.2, we know that

$$\sum_{x\in B_n} G_n(0,x)^{e_i} \leqslant (C_3\ln(n))^{e_i-1}\sum_{x\in B_n} G_n(0,x) \leqslant C\ln(n)^{e_i}n^2,$$

from which we obtain that, provided $e_1+\cdots+e_k = 2m$:

$$\Pi^n_{e_1,\ldots,e_k} \leqslant C\ln(n)^{2m}n^{2k}\prod_{i=1}^k |\mathbb{E}[\overline{\tau}_{0,n}^{e_i}]|.$$

Recalling that (from equation (4.4) and the fact that $\alpha' \leqslant 2$),

$$|\mathbb{E}[\overline{\tau}_{0,n}^{e_i}]| \leqslant Cn^{2e_i/\alpha'-2},$$

we obtain, for any sequence $e_1,\ldots,e_k$ such that $e_1+\cdots+e_k = 2m$:

$$\Pi^n_{e_1,\ldots,e_k} \leqslant C\ln(n)^{2m}n^{4m/\alpha'}.$$

Now we choose m large enough so that :

$$\left(\frac{4}{\alpha'} - 2\beta\right)m < -2$$

and apply Markov's inequality. □

The next step is to lift this estimate to the sum of $G_n(0,x)\tilde{\tau}_{x,n}$.

Proposition 4.4. *Assuming that $\mathbb{E}[\tau_0]$ is finite, there exists M such that for any $\beta > d/\alpha'$, there exist $\delta, C > 0$ such that for all n :*

$$\mathbb{P}\left[\sum_{x\in B_n} G_n(0,x)\tilde{\tau}_{x,n} > Mn^2 + n^\beta\right] \leqslant \frac{C}{n^{d+\delta}}.$$

Proof. Note that as $\mathbb{E}[\tilde{\tau}_{x,n}] \leqslant \mathbb{E}[\tau_0]$, and using part (1) of Proposition 4.2 :

$$\sum_{x \in B_n} G_n(0,x)\mathbb{E}[\tilde{\tau}_{x,n}] \leqslant C_1 \mathbb{E}[\tau_0] n^2.$$

It comes that

$$\mathbb{P}\left[\sum_{x \in B_n} G_n(0,x)\tilde{\tau}_{x,n} > C_1\mathbb{E}[\tau_0]n^2 + n^\beta\right] \leqslant \mathbb{P}\left[\sum_{x \in B_n} G_n(0,x)\overline{\tau}_{x,n} > n^\beta\right],$$

on which we apply Proposition 4.3. □

We can now carry this result back to $\sup_{x \in B_n} \mathbf{E}_x^\tau[T_n]$.

Proposition 4.5. *Assuming that $\mathbb{E}[\tau_0]$ is finite, there exists M' such that for any $\beta > d/\alpha'$, almost every environment and n large enough :*

$$\sup_{x \in B_n} \mathbf{E}_x^\tau[T_n] \leqslant n^\beta + M'n^2.$$

Proof. We first need to relate $\mathbf{E}_x^\tau[T_n]$ with the estimates proved before (which concern only $\mathbf{E}_0^\tau[T_n]$). Let T_n^x be the exit time from $x + B_n$. Since for any $x \in B_n$, we have $B_n \subseteq x + B_{2n}$, it comes that almost surely $T_n \leqslant T_{2n}^x$, so $\mathbf{E}_x^\tau[T_n] \leqslant \mathbf{E}_x^\tau[T_{2n}^x]$, the latter having same law as $\mathbf{E}_0^\tau[T_{2n}]$ under $\mathbb{P}$.

Let $M' > 0$ and let i be an integer. We consider :

$$\mathbb{P}\left[\sup_{n \geqslant 2^i} \frac{\sup_{x\in B_n} \mathbf{E}_x^\tau[T_n]}{n^\beta + M'n^2} > 1\right] \leqslant \sum_{j=i}^{\infty} \mathbb{P}\left[\sup_{2^j \leqslant n < 2^{j+1}} \frac{\sup_{x\in B_n} \mathbf{E}_x^\tau[T_{2n}^x]}{n^\beta + M'n^2} > 1\right]. \tag{4.5}$$

We bound the general term of this series by

$$\mathbb{P}\left[\sup_{x \in B_{2^{j+1}}} \mathbf{E}_x^\tau[T_{2^{j+2}}^x] > 2^{j\beta} + M'2^{2j}\right],$$

which we bound by $A_j + |B_{2^{j+1}}|A'_j$, where :

$$A_j = \mathbb{P}\left[\exists x \in B_{2^{j+2}} : \tau_x > 2^{(j+2)d/\alpha'}\right], \tag{4.6}$$

$$A'_j = \mathbb{P}\left[\sum_{x \in B_{2^{j+2}}} G_n(0,x)\tilde{\tau}_{x,2^{j+2}} > 2^{j\beta} + M'2^{2j}\right].$$

We first estimate A_j. Take α'' such that $\alpha' < \alpha'' < \alpha$. It comes from assumption 1 (see (2.1)) that for all y large enough :

$$\mathbb{P}[\tau_0 > y] \leqslant y^{-\alpha''}.$$

One gets that for j large enough :

$$A_j \leqslant 1 - \left(1 - 2^{-jd\alpha''/\alpha'}\right)^{|B_{2^{j+2}}|} = 1 - \exp\left(|B_{2^{j+2}}|2^{-jd\alpha''/\alpha'}(1+o(1))\right),$$

which is the general term of a convergent series.

Now for A'_j, using Proposition 4.4, we see that choosing $M' = 16M$, the term $|B_{2^{j+1}}|A'_j$ is bounded by $C2^{-j\delta}$ for some $\delta > 0$. Therefore, the series in the right-hand side of 4.5 converges (and tends to 0 when i goes to infinity), which proves the proposition. □

We can now conclude :

Proposition 4.6. (1) *If $\alpha > 1$ and $d/\alpha \geqslant 2$, then for almost every environment :*

$$\limsup_{n\to\infty} -\frac{\ln(\lambda_n^\circ)}{\ln(n)} \leqslant \frac{d}{\alpha}.$$

(2) *If $d/\alpha < 2$, then there exists C such that for almost every environment and all n large enough :*

$$\lambda_n^\circ \geqslant \frac{C}{n^2}.$$

Proof. If $d \geqslant 4$, we can make α' tend to α in Proposition 4.5, which, together with Proposition 4.1, gives the desired result. When $d \in \{2, 3\}$, one needs to take care of the additional restriction (4.2). If $\alpha \leqslant 2$, then one can make α' tend to α, and obtain the results. Otherwise, we are in the case when $d/\alpha < 2$. As a consequence, we can choose $\alpha' = 2$, and part (2) of the proposition still holds. □

5. Upper bounds on λ_n

We now give upper bounds on λ_n. Our method is clear from equation (1.2), that we recall here :

$$\lambda_n = \inf_{\substack{f \in L^2(B_n) \\ f \neq 0}} \frac{\mathcal{E}(f, f)}{(f, f)}.$$

Picking a function in $L^2(B_n)$ gives an upper bound, and the problem is to choose the function well enough (i.e. looking more or less like the eigenfunction) to get a sharp bound.

5.1. The one-dimensional case.

Proposition 5.1. *We assume $d = 1$. There exists $C > 0$ such that for almost every environment and all n large enough :*

$$\lambda_n \leqslant \frac{C}{n \sum_{x \in B_{n/4}} \tau_x}.$$

Proof. For $a = 0$, a "triangle function" that takes the value 0 on $-(n+1)$ and $(n+1)$, the value 1 on 0 and is piecewise linear would do well. But for general a, this function is not appropriate, and we will construct instead a function that looks like it, but is constant around deep traps.

Let $M > 0$ be such that $\mathbb{P}[\tau_0 > M] \leqslant 1/8$. Because of the law of large numbers, one gets :

$$\frac{1}{n} \ |\{k \in \{-n-1, \ldots, 0\} \ : \ \tau_k > M\}| \xrightarrow[n \to \infty]{a.s.} \frac{1}{8}.$$

Almost surely, for n large enough, the two following conditions are satisfied :

$$|\{k \in \{-n-1, \ldots, 0\} \ : \ \tau_k > M\}| \leqslant \frac{n}{4}, \tag{5.1}$$

$$|\{k \in \{0, \ldots, n+1\} \ : \ \tau_k > M\}| \leqslant \frac{n}{4}. \tag{5.2}$$

Let us first construct the left part of our function : let $l : -\mathbb{N} \to \mathbb{R}$ be such that $l(k) = 0$ for all $k < -n$, and for all $k \in \{-n, \ldots, 0\}$:

$$l(k) - l(k-1) = \left| \begin{array}{ll} 0 & \text{if } \tau_{k-1} > M \text{ or } \tau_k > M, \\ 1/n & \text{otherwise.} \end{array} \right.$$

The function l is made in such a way that for all k for which it makes sense :

$$\tau_k^a \tau_{k+1}^a (l(k+1) - l(k))^2 \leqslant \frac{M^{2a}}{n^2}. \tag{5.3}$$

Moreover, when (5.1) is satisfied, there are at most half of the edges on which the function is constant, so $l(0) \geqslant 1/2$. In this case, and as for any k we have $l(k) - l(k-1) \leqslant 1/n$, it comes that $l(k) \geqslant 1/4$ when $k \geqslant -n/4$.

We define in the same way a right part $r : \mathbb{N} \to \mathbb{R}$ such that $r(k) = 0$ for all $k > n$, and for all $k \in \{n, \dots, 0\}$:

$$r(k) - r(k+1) = \left| \begin{array}{ll} 0 & \text{if } \tau_k > M \text{ or } \tau_{k+1} > M, \\ 1/n & \text{otherwise.} \end{array} \right.$$

The function r satisfies the same small variation property as in (5.3). Similarly, when (5.2) is satisfied, we have that $r(0) \geqslant 1/2$ and $r(k) \geqslant 1/4$ for all $k \leqslant n/4$.

Now we connect the two parts l and r preserving this small variation property. Let $m = \min(l(0), r(0))$. We define $f : \mathbb{Z} \to \mathbb{R}$ by

$$f(x) = \left| \begin{array}{ll} \min(l(x), m) & \text{if } x < 0, \\ \min(r(x), m) & \text{otherwise.} \end{array} \right.$$

We have therefore :

$$\mathcal{E}(f,f) \leqslant \frac{2M^{2a}}{n}.$$

On the other hand, for n large enough, (5.1) and (5.2) are satisfied, and in this case $m \geqslant 1/2$ and $f(k) \geqslant 1/4$ for all k such that $-n/4 \leqslant k \leqslant n/4$. Thus :

$$(f,f) \geqslant \frac{1}{16} \sum_{-n/4 \leqslant k \leqslant n/4} \tau_k,$$

and we finally obtain, for all n large enough :

$$\lambda_n \leqslant \frac{\mathcal{E}(f,f)}{(f,f)} \leqslant \frac{32M^{2a}}{n \ \sum_{x \in B_{n/4}} \tau_x}.$$

□

5.2. **Large dimension, anomalous behaviour.** The results proved in this part are in fact valid in any dimension and for any $\alpha > 0$, but they are sharp only in the regime given in the title, that is for $d \geqslant 2$ and $2\alpha \leqslant d$.

Proposition 5.2. (1) *For any $\varepsilon > 0$, there exists $M > 0$ such that for all n large enough :*

$$\mathbb{P}\left[\lambda_n \max_{B_{n-1}} \tau \leqslant M\right] \geqslant 1 - \varepsilon.$$

(2) *For any $\varepsilon > 0$ and almost every environment :*

$$n^{d/\alpha - \varepsilon} \lambda_n \xrightarrow[n\to\infty]{} 0.$$

Proof. Let K be the set of first and second neighbours of 0, namely $K = \{x \in \mathbb{Z}^d : 1 \leqslant \|x\| \leqslant 2\}$, and c the number of edges from a point of $\{x : \|x\| = 1\}$ to a point of $\{x : \|x\| = 2\}$. Write $M_x = \max_{x+K} \tau$. If we choose the function that takes value 1 on site $x \in B_{n-1}$ and its neighbours, and 0 elsewhere, namely :

$$f(z) = \left| \begin{array}{ll} 1 & \text{if } \|z - x\| \leqslant 1, \\ 0 & \text{otherwise,} \end{array} \right.$$

then we see that for any $x \in B_{n-1}$:

$$\lambda_n \leqslant \frac{c(M_x)^{2a}}{\tau_x}. \tag{5.4}$$

Let $x_n \in B_{n-1}$ be such that $\tau_{x_n} = \max_{B_{n-1}} \tau$. We have :

$$\lambda_n \leqslant \frac{c(M_{x_n})^{2a}}{\max_{B_{n-1}} \tau}.$$

So we get :

$$\mathbb{P}\left[\lambda_n \max_{B_{n-1}} \tau \geqslant M\right] \leqslant \mathbb{P}\left[c(M_{x_n})^{2a} \geqslant M\right].$$

Now recall that M_{x_n} is the maximum over all neighbours and second neighbours of x_n, so it should look like taking the maximum over all neighbours and second neighbours of, say, 0. More precisely, conditionally on $\max_{B_{n-1}} \tau = \tau_z$ for some fixed z, the law of $(\tau_x)_{x \in B_{n-1} \setminus \{z\}}$ is invariant under permutation. Therefore, provided $z \in B_{n-2} \setminus K$ and conditionally on $\max_{B_{n-1}} \tau = \tau_z$, the random variables M_z and M_0 have the same law. Summing over all $z \in B_{n-2} \setminus K$, we get that conditionally on the event E_n that $x_n \in B_{n-2} \setminus K$, the random variables M_0 and M_{x_n} have the same law. We obtain :

$$\mathbb{P}\left[c(M_{x_n})^{2a} \geqslant M\right] \leqslant \mathbb{P}\left[c(M_0)^{2a} \geqslant M\right] + \mathbb{P}\left[E_n^c\right].$$

The law of x_n being uniform in B_{n-1}, we have that $\mathbb{P}\left[E_n^c\right]$ goes to 0 when n goes to infinity. First part of the theorem comes choosing M large enough.

We now turn to the second assertion of the proposition. Defining :

$$\overline{M}_n = \max_{x \in B_{n-1}} \frac{\tau_x}{(M_x)^{2a}},$$

we will show that for any $\varepsilon > 0$:

$$\frac{\overline{M}_n}{n^{d/\alpha - \varepsilon}} \xrightarrow[n \to \infty]{\text{a.s.}} +\infty, \tag{5.5}$$

which will prove the result via equation (5.4). There exists $k > 0$ such that $\mathbb{P}[(M_x)^{2a} > k] < 1/2$. Thus (note that M_x and τ_x are independent) :

$$\mathbb{P}\left[\frac{\tau_x}{(M_x)^{2a}} \geqslant y\right] \geqslant \frac{\mathbb{P}[\tau_x \geqslant ky]}{2} = \frac{F(ky)}{2}.$$

Hence, for all $K > 0$:

$$\mathbb{P}[\overline{M}_n \leqslant n^{d/\alpha - \varepsilon} K] \leqslant \left(1 - \frac{F(kKn^{d/\alpha - \varepsilon})}{2}\right)^{(2n-1)^d},$$

and recalling that, as a consequence of assumption 1 (see (2.1)), for all $\beta < \alpha$, $F(y) \leqslant y^{-\beta}$ for all y large enough, one can see that the term on the right-hand side of the former equality is the general term of a convergent series, and thus apply the Borel-Cantelli lemma. □

5.3. **Regular behaviour.** In what follows our assumption will be that $\mathbb{E}[\tau_0^a]$ is finite. In particular, all results will be valid under the condition that $\mathbb{E}[\tau_0]$ is finite (or if $a = 0$).

We write $(e_i)_{1 \leqslant i \leqslant d}$ for the canonical base of $\mathbb{R}^d$.

Proposition 5.3. *Let $f : [-1,1]^d \to \mathbb{R}$ be a continuous function. If $\mathbb{E}[\tau_0^a]$ is finite, then for all $i \in \{1, \ldots, d\}$:*

$$\frac{1}{(2n+1)^d} \sum_{x \in B_n} \tau_x^a \tau_{x+e_i}^a \ f(x/n) \xrightarrow[n \to \infty]{a.s.} \mathbb{E}[\tau_0^a]^2 \int_{[-1,1]^d} f(x) \mathrm{d}x. \tag{5.6}$$

Proof. If f is piecewise constant, then the limit (5.6) is proved by separating the sum over B_n into two parts B_n' and B_n'' so that $(\tau_x^a \tau_{x+e_i}^a)_{x \in B_n'}$ and $(\tau_x^a \tau_{x+e_i}^a)_{x \in B_n''}$ are two families of independent random variables, and then applying the law of large numbers. For a continuous f, one can approximate uniformly f by piecewise constant functions from above and below, and the result follows. □

For all $f : [-1,1]^d \to \mathbb{R}$ and all integer n, we define the function $f_n : \mathbb{Z}^d \to \mathbb{R}$ by $f_n(x) = f(x/n)$ if $x \in B_n$, and $f_n(x) = 0$ otherwise. Note that $f_n \in L^2(B_n)$.

Proposition 5.4. *Let $f : [-1,1]^d \to \mathbb{R}$ be a twice continuously differentiable function that takes value 0 on the boundary of $[-1,1]^d$. If $\mathbb{E}[\tau_0^a]$ is finite, then :*

$$\frac{n^2}{(2n)^d}\mathcal{E}(f_n, f_n) \xrightarrow[n\to\infty]{a.s.} \mathbb{E}[\tau_0^a]^2 \int_{[-1,1]^d} \|\nabla f(x)\|_2^2 \mathrm{d}x.$$

Recall the following equality :

$$\mathcal{E}(f_n, f_n) = \sum_{i=1}^{d} \sum_{x\in B_n} \tau_x^a \tau_{x+e_i}^a \left(f\left(\frac{x}{n}\right) - f\left(\frac{x+e_i}{n}\right)\right)^2.$$

As we assumed f to be twice continuously differentiable, it comes that for all $\varepsilon > 0$ and n large enough :

$$\forall x \in B_n : x + e_i \in B_n \Rightarrow \left| \left(f\left(\frac{x}{n}\right) - f\left(\frac{x+e_i}{n}\right)\right)^2 - \frac{1}{n^2}\frac{\partial f}{\partial x_i}\left(\frac{x}{n}\right)^2 \right| \leqslant \frac{\varepsilon}{n^2},$$

and note that if $x \in B_n$ and $x + e_i \notin B_n$, then $f(x/n) = f((x + e_i)/n) = 0$, so this case does not contribute to the sum. The result follows using the previous proposition.

Proposition 5.5. *If $\mathbb{E}[\tau_0^a]$ is finite, then there exists C such that almost surely, for all n large enough :*

$$\lambda_n \leqslant \frac{C}{n^2} \frac{n^d}{\sum_{x\in B_{n/2}} \tau_x}.$$

Proof. Taking $f(x) = \prod_{i=1}^d \sin\left(\frac{\pi x_i}{2}\right)$ in Proposition 5.4, we get that for almost every environment :

$$\mathcal{E}(f_n, f_n) \sim \frac{d\pi^2}{4}\frac{(2n)^d}{n^2}\mathbb{E}[\tau_0^a]^2 \qquad (n \to +\infty).$$

On the other hand, if $x \in B_{n/2}$, then $f(x) \geqslant 2^{-d/2}$, thus :

$$(f_n, f_n) \geqslant 2^{-d/2} \sum_{x\in B_{n/2}} \tau_x,$$

therefore the proposition holds for any $C > 2^{3d/2-2} d\pi^2 \mathbb{E}[\tau_0^a]^2$. □

6. The distinguished path method

We present here a more direct method to get a lower bound on λ_n (close to the one presented e.g. in [SC97, Theorem 3.2.3], but adapted to treat the case of Dirichlet boundary condition), and show that it does not provide a sharp estimate when $d \geqslant 2$. Note that in dimension one, [Ch, Section 3.7] proves that this technique is always sharp, and one can verify that it gives indeed the expected lower bound. This method also proved efficient in larger dimension in [FM06, Section 3] in the context of random walks among random conductances.

For all $x \in B_n$, we give ourselves a path $\gamma_n(x)$ from some point of ∂B_n to x (that apart from the starting point, visits only points in B_n). Let $\gamma_n(x) = (x^0, \dots, x^l)$. For an edge e, we note $e \in \gamma_n(x)$ if $e = (x^i, x^{i+1})$ for some i, and in this case, we write $\mathrm{d}f(e) = f(x^{i+1}) - f(x^i)$, and $\mathfrak{Q}(e) = \tau_{x^i}^a \tau_{x^{i+1}}^a$. Let E_n be the set of edges that go from a point of B_n to a point of $B_n \cup \partial B_n$. We give ourselves a weight function $W_n : E_n \to (0, +\infty)$. We define the W_n-length of a path γ as :

$$l_n(\gamma) = \sum_{e\in\gamma} \frac{1}{W_n(e)}.$$

Note that, as we assumed that $\tau \geqslant 1$, we have that $\mathfrak{Q}(e) \geqslant 1$ (and there is equality when $a = 0$). Using Cauchy-Schwarz inequality, we get :

$$\begin{aligned} f(x)^2 &= \left(\sum_{e \in \gamma_n(x)} \mathrm{d}f(e) \right)^2 \\ &\leqslant \sum_{e \in \gamma_n(x)} \frac{1}{W_n(e)\mathfrak{Q}(e)} \sum_{e \in \gamma_n(x)} \mathrm{d}f(e)^2 W_n(e) \mathfrak{Q}(e) \\ &\leqslant l_n(\gamma_n(x)) \sum_{e \in \gamma_n(x)} \mathrm{d}f(e)^2 W_n(e) \mathfrak{Q}(e) \end{aligned}$$

$$\begin{aligned} \sum_{x \in B_n} f(x)^2 \tau_x &\leqslant \sum_{x \in B_n} l_n(\gamma_n(x)) \tau_x \sum_{e \in \gamma_n(x)} \mathrm{d}f(e)^2 W_n(e) \mathfrak{Q}(e) \\ &\leqslant \sum_{e \in E_n} \mathrm{d}f(e)^2 \mathfrak{Q}(e) W_n(e) \sum_{x : e \in \gamma_n(x)} l_n(\gamma_n(x)) \tau_x. \end{aligned}$$

Note that

$$\mathcal{E}(f,f) = \sum_{e \in E_n} \mathrm{d}f(e)^2 \mathfrak{Q}(e),$$

so letting

$$\mathcal{M}_n := \max_{e \in E_n} W_n(e) \sum_{x : e \in \gamma_n(x)} l_n(\gamma_n(x)) \tau_x,$$

we obtain the following lower bound on λ_n (similar to [SC97, Theorem 3.2.3]) :

$$\lambda_n \geqslant \frac{1}{\mathcal{M}_n}.$$

Let us see that, however W_n and $\gamma_n(x)$ are chosen, it cannot lead to a sharp bound if $d \geqslant 2$ and $\alpha < d$. Let $z \in B_{n/2}$ be such that τ_z is maximal. The site z is such that $\tau_z \simeq n^{d/\alpha}$ and $|\gamma_n(z)| \geqslant n/2$. Now choose $e \in \gamma_n(z)$ so that $W_n(e)$ is maximal. We have :

$$\mathcal{M}_n \geqslant \sum_{e' \in \gamma_n(z)} \frac{W_n(e)}{W_n(e')} \tau_z \geqslant |\gamma_n(z)| \tau_z \gtrsim n^{1+d/\alpha},$$

where we would have hoped to find $n^{\max(2,d/\alpha)}$. So this method cannot give the appropriate exponent if $\alpha < d$.

Still, note that if one chooses W_n constant equal to 1, and the shortest paths for $(\gamma_n(x))_{x \in B_n}$, one can show using results of [BK65] that $\mathcal{M}_n$ is indeed of order $n^{\max(2,1+d/\alpha)}$, which gives an alternative proof of a lower bound for the principal eigenvalue when $\alpha \geqslant d$.

Acknowledgments. The author would like to thank his Ph.D. advisors, Pierre Mathieu and Alejandro Ramírez, for many insightful discussions about this work as well as detailed comments on earlier drafts, and Gérard Ben Arous for suggesting this problem.

References

[Al81] S. Alexander. Anomalous transport properties for random-hopping and random-trapping models. *Phys. Rev. B* **23** (6), 2951-2955 (1981)

[BČ09] M. Barlow, J. Černý. Convergence to fractional kinetics for random walks associated with unbounded conductances. Preprint (2009).

[BK65] L.E. Baum, M. Katz. Convergence rates in the law of large numbers. *Trans. Amer. Math. Soc.* **120** (1) 108-123 (1965).

[BČ05] G. Ben Arous, J. Černý. Bouchaud's model exhibits two different aging regimes in dimension one. *Ann. Appl. Probab.* **15** (2), 1161-1192 (2005).

[BČ06] G. Ben Arous, J. Černý. Dynamics of trap models. *Les Houches summer school lecture notes*, Elsevier (2006).

[BČ07] G. Ben Arous, J. Černý. Scaling limit for trap models on $\mathbb{Z}^d$. *Ann. Probab.* **35** (6) 2356-2384 (2007).

[BGT] N.H. Bingham, C.M. Goldie, J.L. Teugels. *Regular variation.* Cambridge University Press (1989).

[BF05] A. Bovier, A. Faggionato. Spectral characterization of aging: the REM-like trap model. *Ann. Appl. Probab.* **15** (3), 1997-2037 (2005).

[BF08] A. Bovier, A. Faggionato. Spectral analysis of Sinai's walk for small eigenvalues. *Ann. Probab.* **36** (1), 198-254 (2008).

[Bo92] J.-P. Bouchaud. Weak ergodicity breaking and aging in disordered systems. *J. Phys. I* (France) **2**, 1705-1713 (1992).

[Ch] M.-F. Chen. *Eigenvalues, inequalities, and ergodic theory.* Springer (2005).

[DFGW89] A. De Masi, P.A. Ferrari, S. Goldstein, W.D. Wick. An invariance principle for reversible Markov processes. Applications to random motions in random environments. *J. Statistical Physics* **55** (3-4), 787-855 (1989).

[Fa09] A. Faggionato. Spectral analysis of 1D nearest-neighbor random walks with applications to subdiffusive random trap and barrier models. ArXiv:0905.2900v1 (2009).

[Fe1] W. Feller. *An introduction to probability theory and its applications*, vol. I, third edition. John Wiley & Sons, Inc. (1968).

[Fe2] W. Feller. *An introduction to probability theory and its applications*, vol. II, second edition. John Wiley & Sons, Inc. (1971).

[FIN02] L.R.G. Fontes, M. Isopi, C.M. Newman. Random walks with strongly inhomogeneous rates and singular diffusions: convergence, localization and aging in one dimension. *Ann. Probab.* **30** (2), 579-604 (2002).

[FM06] L.R.G. Fontes, P. Mathieu. On symmetric random walks with random conductances on $\mathbb{Z}^d$. *Probab. Theory Relat. Fields* **134**, 565-602 (2006).

[FM08] L.R.G. Fontes, P. Mathieu. K-processes, scaling limit and aging for the trap model in the complete graph. *Ann. Probab.* **36** (4), 1322-1358 (2008).

[JLT09] M. Jara, C. Landim, A. Teixeira. Quenched scaling limits of trap models. ArXiv:0902.3334v1 (2009).

[La] G.F. Lawler. *Intersections of random walks.* Probability and its applications, Birkhäuser (1991).

[LP] R. Lyons, with Y. Peres. *Probability on trees and networks.* Cambridge University Press, in preparation. Current version available at `http://mypage.iu.edu/~rdlyons/`.

[MB97] R. Mélin, P. Butaud. Glauber dynamics and ageing. *J. Physique I* **7** (5), 691-710 (1997).

[Mo09] J.-C. Mourrat. Scaling limit of the random walk among random traps on $\mathbb{Z}^d$. In preparation.

[Pe] V.V. Petrov. *Limit Theorems of Probability Theory - Sequences of Independent Random Variables.* Oxford studies in probability (1995).

[Re] S.I. Resnick. *Extreme values, regular variation, and point processes.* Springer-Verlag (1987).

[RMB00] B. Rinn, P. Maass, J.-P. Bouchaud. Multiple scaling regimes in simple aging models. *Phys. Rev. Lett.* **84** (23), 5403-5406 (2000).

[SC97] L. Saloff-Coste. Lectures on finite Markov chains. *Lectures on probability theory and statistics (Saint-Flour 1996)*, Lecture Notes in Math. **1665**, Springer, 301-413 (1997).

Université de Provence, CMI, 39 rue Joliot Curie, 13013 Marseille, France ; PUC de Chile, Facultad de Matemáticas, Vicuña Mackenna 4860, Macul, Santiago, Chile.