LYAPUNOV EXPONENTS OF RANDOM WALKS IN SMALL RANDOM POTENTIAL: THE UPPER BOUND

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Abstract. We consider the simple random walk on $\mathbb{Z}^d$ evolving in a random i.i.d. potential taking values in $[0, +\infty)$. The potential is not assumed integrable, and can be rescaled by a multiplicative factor $\lambda > 0$. Completing the work started in a companion paper, we give the asymptotic behaviour of the Lyapunov exponents for $d \geqslant 3$, both annealed and quenched, as the scale parameter λ tends to zero.

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1. Introduction

Let $(X_n)_{n\in\mathbb{N}}$ be the simple random walk in $\mathbb{Z}^d$, whose law starting from x we write $\mathbf{P}_x$ (with expectation $\mathbf{E}_x$). Independently of the random walk, we give ourselves a random potential $V = (V(x))_{x\in\mathbb{Z}^d}$, which is a family of i.i.d. random variables taking values in $[0, +\infty)$. We write $\mathbb{P} = \mu^{\otimes\mathbb{Z}^d}$ for the law of this family, with associated expectation $\mathbb{E}$. Let $\ell \in \mathbb{R}^d$ be a vector of unit Euclidean norm, and

$$\mathsf{T}_n(\ell) = \inf\{k : X_k \cdot \ell \geqslant n\}$$

be the first time at which the random walk crosses the hyperplane orthogonal to ℓ lying at distance n from the origin. For every $\lambda > 0$, we define the *quenched* and *annealed* point-to-hyperplane Lyapunov exponents associated with the direction ℓ and the potential λV by, respectively,

$$\alpha_\lambda(\ell) = \lim_{n\to+\infty} -\frac{1}{n} \log \mathbf{E}_0 \left[\exp\left(-\sum_{k=0}^{\mathsf{T}_n(\ell)-1} \lambda V(X_k) \right) \right], \tag{1.1}$$

$$\overline{\alpha}_\lambda(\ell) = \lim_{n\to+\infty} -\frac{1}{n} \log \mathbb{E}\mathbf{E}_0 \left[\exp\left(-\sum_{k=0}^{\mathsf{T}_n(\ell)-1} \lambda V(X_k) \right) \right]. \tag{1.2}$$

The first limit holds almost surely, and is deterministic (see [Mo12] for the existence of the first limit, and [Fl07] for the second one).

Our goal is to complete the proof of the following result.

Theorem 1.1. *If $d \geqslant 3$, then as λ tends to 0,*

$$\alpha_\lambda(\ell) \sim \overline{\alpha}_\lambda(\ell) \sim \left(2d \int q_d \, \frac{1 - e^{-\lambda z}}{1 - (1-q_d)e^{-\lambda z}} \, \mathrm{d}\mu(z) \right)^{1/2}, \tag{1.3}$$

where q_d is the probability that the simple random walk never returns to its starting point, and $a_\lambda \sim b_\lambda$ stands for $a_\lambda / b_\lambda \to 1$.

We refer to [MM13] for a detailed review of previous results and motivations. The heuristic picture (explained in more details in [MM13]) behind Theorem 1.1 is as follows. The sites of $\mathbb{Z}^d$ can be classified according to the value of λ times their associated potential. The walk does not optimize its behaviour on sites whose potential is much smaller than λ^{-1}: their contribution to the sum in (1.1) or (1.2) is that of the law of large numbers. At the opposite end of the range, the sites whose potential is much larger than λ^{-1} are avoided by the walk. The behaviour of the walk with respect to sites whose potential is about $\lambda^{-1} z$ is described by a more balanced strategy, consisting of (1) reducing the proportion of such visited sites by a factor $(1-e^{-z})/z$; and (2) once on such a site, to return to it with probability only $(1-q_d)e^{-z}$ instead of $(1-q_d)$.

Theorem 1.1 can be extended in several directions. In particular, one can also consider a probability law μ that depends on λ as well; this enables in particular to obtain the behaviour of the Green function of the operator $-\Delta + \lambda V$, but also describe some sparse potentials in the spirit of [Ko13]. We refer to [MM13, Section 6] for more on this. We believe that our techniques can be modified to provide for a two-dimensional version of Theorem 1.1, but the details have not been worked out.

We define $f : \mathbb{R}_+ \to \mathbb{R}$ by

$$f(z) = q_d \, \frac{1-e^{-z}}{1-(1-q_d)e^{-z}}, \tag{1.4}$$

so that the integral appearing in the right-hand side of (1.3) can be rewritten as

$$\mathfrak{I}_\lambda = \int f(\lambda z) \, \mathrm{d}\mu(z). \tag{1.5}$$

It was shown in [KMZ11] that if $\mathbb{E}[V] < +\infty$ (we use $\mathbb{E}[V]$ as shorthand for $\mathbb{E}[V(0)]$), then as λ tends to 0,

$$\alpha_\lambda(\ell) \sim \overline{\alpha}_\lambda(\ell) \sim (2d\,\lambda\,\mathbb{E}[V])^{1/2}.$$

Since the function f is concave and $f(z) \sim z$ as z tends to 0, this is consistent with Theorem 1.1. For $\mathbb{E}[V] = +\infty$ and $d \geqslant 3$, it was shown in [MM13] that

$$\liminf_{\lambda \to 0} \frac{\overline{\alpha}_\lambda(\ell)}{\sqrt{2d\,\mathfrak{I}_\lambda}} \geqslant 1. \tag{1.6}$$

A standard convexity argument ensures that $\overline{\alpha}_\lambda(\ell) \leqslant \alpha_\lambda(\ell)$. As a consequence, in order to prove Theorem 1.1, it suffices to show the following.

Theorem 1.2. *If $d \geqslant 3$ and $\mathbb{E}[V] = +\infty$, then*

$$\limsup_{\lambda \to 0} \frac{\alpha_\lambda(\ell)}{\sqrt{2d\,\mathfrak{I}_\lambda}} \leqslant 1.$$

Our task here is thus to show Theorem 1.2. Its proof is simpler than that of the converse bound in (1.6). Indeed, instead of having to consider all possible combinations of paths and environments, we must simply, given a typical environment, construct a scenario whose probability is appropriately bounded from below and for which the walk travels to the distant hyperplane. As a first step, we use the following observation, due to [BK93, Ze98].

Lemma 1.3. *For every $\varepsilon > 0$, let $\tilde{V}_\varepsilon = (\tilde{V}_\varepsilon(x))_{x \in \mathbb{Z}^d}$ be the potential defined by*

$$\tilde{V}_\varepsilon(x) = \left| \begin{array}{ll} V(x) & \text{if } \ V(x) \geqslant \frac{\varepsilon}{\lambda}, \\ \mathbb{E}\left[V(0) \mid V(0) < \frac{\varepsilon}{\lambda}\right] & \text{if } \ V(x) < \frac{\varepsilon}{\lambda}, \end{array} \right.$$

and $\alpha_{\varepsilon,\lambda}(\ell)$ be the quenched Lyapunov exponent associated with the potential $\tilde{V}_\varepsilon$. We have $\alpha_{\varepsilon,\lambda}(\ell) \geqslant \alpha_\lambda(\ell)$.

This follows as in the proof of the last statement of [Ze98, Proposition 4], using the fact that the convergence in [Mo12, (6.12)] holds in L^1. We define

$$\mathfrak{I}_{\varepsilon,\lambda} = \int f(\lambda z) \, \mathrm{d}\mu_\varepsilon(z), \tag{1.7}$$

where μ_ε is the law of $\tilde{V}_\varepsilon(0)$. Elementary bounds yield that $\mathfrak{I}_{\varepsilon,\lambda} \leqslant (1 + C\varepsilon)\mathfrak{I}_\lambda$ for some universal constant C, and thus Theorem 1.2 will be a consequence of

Proposition 1.4. *There exists $K < \infty$ (independent of ε) such that for $\alpha_{\varepsilon,\lambda}(\ell)$ as above,*

$$\limsup_{\lambda \to 0} \frac{\alpha_{\varepsilon,\lambda}(\ell)}{\sqrt{2d\, \mathfrak{I}_{\varepsilon,\lambda}}} \leqslant 1 + K\varepsilon.$$

The advantage of this reduction is that it permits us to deal with a simpler environment than the original one. For λ small the great majority of points $x \in \mathbb{Z}^d$ will have $\tilde{V}_\varepsilon(x)$ equal to the constant value $\mathbb{E}[V(0) \mid V(0) < \frac{\varepsilon}{\lambda}]$. We call the remaining points (that is, the sites x such that $V(x) \geqslant \varepsilon/\lambda$) the *important points*. Those sites will (when suitably renormalized) resemble a Poisson cloud, and can be analysed by coarse-graining techniques.

In this paper, the main work is done in Section 3. This is preceded by Section 2 which gives some simple technical results for random walks, chiefly based on the invariance principle. Section 3 exploits the law of large numbers for the environment as λ becomes small. It culminates in Proposition 3.13 which states that within a "good" environment, the random walk can move forward in the ℓ-direction while incurring appropriate costs. This "building block" is transformed into a result about the Lyapunov exponent in Section 4. In order to do so, we make sure by an oriented-percolation argument that one can find a path of consecutive good boxes from the starting point to a target hyperplane with probability close to one. Section 5 concerns a special case that had to be left apart in the previous arguments.

For notational reasons, we will treat explicitly the case $\ell = (1, 0, \cdots 0)$, but this entails no loss of generality, since the main tool is the invariance principle, and there are no subtle lattice effects to take account of.

2. Technical estimates

In this section, we wish to estimate the probability for our random walk to advance by n in the ℓ direction, with appropriate speed. We first analyse the limiting object for rescaled random walks, that is, Brownian motion with a drift. We are interested in Brownian motion with covariance matrix $\frac{1}{d}$ Id, since it is the scaling limit of our discrete-time random walk.

Let $B = (B_t)_{t \geqslant 0}$ denote the canonical process on the space $C(\mathbb{R}_+, \mathbb{R}^d)$ of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}^d$, and let $\mathcal{F}_t = \sigma(B_s, s \leqslant t)$. For $x \in \mathbb{R}^d$ and $h \in \mathbb{R}$, we denote by Q_x^h the law of the Brownian motion with covariance matrix $\frac{1}{d}$ Id, drift $h\ell$ and starting at x, where $\ell = (1, 0, \ldots, 0) \in \mathbb{R}^d$. We simply write Q_x for Q_x^0. As is well-known, the measure Q_x^h has Radon-Nikodym derivative

$$e^{dh\ell \cdot (B_t - B_0) - dh^2 t/2} \tag{2.1}$$

with respect to the measure Q_x on $\mathcal{F}_t$ (see [RY]).

For every $M \geqslant 1$, consider the event A_M^h defined by

(i) the path hits $\{M\} \times \left(\frac{\sqrt{M}}{2}, \frac{3\sqrt{M}}{2}\right) \times \left(\frac{-\sqrt{M}}{2}, \frac{\sqrt{M}}{2}\right)^{d-2}$ in time M/h or less,

(ii) before this time, the path B does not leave $(B_0 - 2, \infty) \times \left(-\sqrt{M}, 2\sqrt{M}\right) \times \left(-\sqrt{M}, \sqrt{M}\right)^{d-2}$.

Remark 2.1. We have introduced an asymmetry in the second coordinate in order to make room for the oriented-percolation argument of Section 4. Indeed, we cannot hope to find a tower of consecutive "good" boxes on top of one another, but we will be able to find such a sequence of consecutive boxes by allowing to move sideways in the second coordinate.

Since under Q^h_x, the first component essentially moves linearly with speed h while the other components vary diffusively, one has

Lemma 2.2. *For every $h \neq 0$, there exists a constant $c = c(d,h) > 0$ such that for every $M \geqslant 1$ and $x \in \left(-\sqrt{M}/2, \sqrt{M}/2\right)^d$,*

$$Q^h_x \left[A^h_M\right] \geqslant c.$$

This and the Radon-Nikodym derivative given in (2.1) yield

Corollary 2.3. *For every $h \neq 0$, there exists a constant $c_1 = c_1(d,h) > 0$ such that for every $M \geqslant 1$ and $x \in (-1/2, 1/2) \times \left(-\sqrt{M}/2, \sqrt{M}/2\right)^{d-1}$,*

$$Q_x \left[A^h_M\right] \geqslant c_1 e^{-M\frac{dh}{2}}.$$

This result can now be applied to the original object of interest, the random walk. On the space of random walk trajectories X, we define the event $A^{h,L}_M$ by

(i) the random walk hits $\{ML\} \times \left(\frac{\sqrt{M}L}{2}, \frac{3\sqrt{M}L}{2}\right) \times \left(\frac{-\sqrt{M}L}{2}, \frac{\sqrt{M}L}{2}\right)^{d-2}$ in time ML^2/h or less,

(ii) before this time, the walk X does not leave $(X_0 - 2L, \infty) \times \left(-\sqrt{M}L, 2\sqrt{M}L\right) \times \left(-\sqrt{M}L, \sqrt{M}L\right)^{d-2}$.

From the invariance principle, we thus get the following.

Corollary 2.4. *Let $h \neq 0$ and $c_1 > 0$ be given by Corollary 2.3. For every $M \geqslant 1$, every L sufficiently large, and every $x \in (-L/2, L/2) \times \left(-\sqrt{M}L/2, \sqrt{M}L/2\right)^{d-1}$,*

$$\mathbf{P}_x \left[A^{h,L}_M\right] \geqslant \frac{2c_1}{3} e^{-M\frac{dh}{2}}.$$

We conclude this section with a technical variation of this result that will be more adapted to our needs. Given $0 < \varepsilon_0 \leqslant \varepsilon$ and L, we define the stopping times $(\sigma_i)_{i \geqslant 0}$ recursively by $\sigma_0 = 0$ and, for $i \geqslant 1$,

$$\sigma_i = \inf\{n > \sigma_{i-1} : |X_n - X_{\sigma_{i-1}}| \geqslant \varepsilon_0 L\}. \tag{2.2}$$

These stopping times are introduced as a substitute for fixed times. The point is that, given M, we can choose the parameter ε_0 to be sufficiently small that the σ_i exhibit good behaviour even on the "extreme" event $A^{h,L}_M$. Let $\tilde{A}^{h,L}_M$ be the event that

(i) the random walk hits $\{ML\} \times \left(\frac{\sqrt{M}L}{2}, \frac{3\sqrt{M}L}{2}\right) \times \left(\frac{-\sqrt{M}}{2}, \frac{\sqrt{M}L}{2}\right)^{d-2}$ before time $\sigma_{\frac{M}{h}\frac{(1+\varepsilon)}{\varepsilon_0^2}}$,

(ii) before this time, the walk X does not leave $(X_0 - 2L, \infty) \times \left(-\sqrt{M}L, 2\sqrt{M}L\right) \times \left(-\sqrt{M}L, \sqrt{M}L\right)^{d-2}$.

Let us denote by F the (random) smallest index k such that σ_k is at least as large as the hitting time of the hyperplane $\{x : x_1 = ML\}$. So on the event $\tilde{A}^{h,L}_M$, we have $F \leqslant \frac{M}{h}\frac{(1+\varepsilon)}{\varepsilon_0^2}$.

Lemma 2.5. *Let $h \neq 0$ and $c_1 > 0$ be given by Corollary 2.3. For every $M \geqslant 1$, every L large enough, every $\varepsilon_0 > 0$ small enough and every $x \in (-L/2, L/2) \times \left(-\sqrt{M}L/2, \sqrt{M}L/2\right)^{d-1}$,*

$$\mathbf{P}_x\left[\tilde{A}_M^{h,L}\right] \geqslant \frac{c_1}{2} e^{-M\frac{dh}{2}}.$$

Proof. This follows from the fact that

$$\begin{aligned}\mathbf{P}_x\left[A_M^{h,L} \setminus \tilde{A}_M^{h,L}\right] &\leqslant \mathbf{P}_x\left[\sigma_{\frac{M}{h}\frac{(1+\varepsilon)}{\varepsilon_0^2}} < \frac{ML^2}{h}\right] \\ &\leqslant e^{-\frac{hM}{2}\frac{C\varepsilon}{\varepsilon_0^2}},\end{aligned}$$

which is arbitrarily small compared with $e^{-M\frac{dh}{2}}$ provided that we choose ε_0 sufficiently small. □

Remark 2.6. From now on we will take $h = \frac{1}{\sqrt{d}}$, and the superscript h will be dropped in any notation for events or variables.

Remark 2.7. From now on we suppose ε_0 to be small enough and L large enough to ensure that the conclusion of Lemma 2.5 is true. We also introduce δ and δ_1 so that $0 < \delta_1 \ll \delta \ll \varepsilon_0$. They will need to be small enough to satisfy a finite number of conditions given below, but are otherwise kept fixed.

3. Coarse graining and the environment

In this section we begin to consider the environment. Our first task is to show that on the scale L_λ, chosen as below, the environment is highly regular for points of high $V(.)$ value and, given only "reasonable" law of large numbers behaviour, these points will be such that they are struck in a "Poisson" manner.

We now choose $L = L_\lambda$ as a function of λ as

$$L_\lambda^{-1} = \sqrt{2\mathfrak{I}_{\varepsilon,\lambda}}, \tag{3.1}$$

where we recall that $\mathfrak{I}_{\varepsilon,\lambda}$ was defined in (1.7). We aim to show that the essential features of the problem become visible at this scale, and that the useful random walk paths will behave at this scale as random with a bias. At lower scales they will just be unbiased random walks, at higher they become deterministic motion.

We first suppose that

$$L_\lambda^{-1} = \sqrt{2\mathfrak{I}_{\varepsilon,\lambda}} \leqslant \varepsilon^{-2}\sqrt{\mathbb{P}\left[V \geqslant \frac{\varepsilon}{\lambda}\right]}. \tag{3.2}$$

We will later sketch the (easier) case when this assumption does not hold, i.e. when $\{x : V(x) > \frac{\varepsilon}{\lambda}\}$ makes little contribution to $\mathfrak{I}_\lambda$. An immediate consequence of this assumption is that

$$\int_{\frac{\varepsilon}{\lambda}}^{+\infty} f(\lambda z)\, \mathrm{d}\mu(z) \geqslant f(\varepsilon)\mathbb{P}\left[V \geqslant \frac{\varepsilon}{\lambda}\right] \geqslant \frac{\varepsilon^5}{L_\lambda^2}. \tag{3.3}$$

We divide up the values in $\left[\frac{\varepsilon}{\lambda}, \frac{1}{\varepsilon\lambda}\right)$ into intervals of length $\frac{\varepsilon^2}{\lambda}$ (except for the last) $I_0 = [a_0, b_0), I_1 = [a_1, b_1), \ldots I_R = [a_R, b_R)$, and we let $I_{R+1} = \left[\frac{1}{\varepsilon\lambda}, \infty\right)$, the interval of values best avoided. Note that the number R depends only on ε (which will be chosen sufficiently small but otherwise kept fixed), and not on λ (which we will let tend to 0).

We divide up the intervals into two classes, as in [MM13]. We say that the interval I_j is *relevant* if

$$\mathbb{P}[V(0) \in I_j] \geqslant \frac{\varepsilon^9}{L_\lambda^2}. \tag{3.4}$$

We say that it is *irrelevant* otherwise.

As the name indicates, the key is that points with values in irrelevant intervals are not relevant to scale L_λ, while the number of relevant important points in a "good cube" (to be specified later) of side length L_λ should be of order L_λ^{d-2}, and so there should be a reasonable chance that one of these points will be hit by the random walk (or by a lightly conditioned random walk) before exiting the cube.

For $\delta_1 > 0$ (to be chosen much smaller than ε, and otherwise kept fixed) and every $\underline{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$, we define the mesoscopic box

$$C_{\underline{n},\delta_1} = \big[\delta_1 L_\lambda \underline{n}, \delta_1 L_\lambda(\underline{n} + \underline{1})\big) = \prod_{i=1}^{d} \big[\delta_1 L_\lambda n_i, \delta_1 L_\lambda(n_i + 1)\big). \tag{3.5}$$

We say that an environment $(V(x))_{x\in\mathbb{Z}^d}$ is (L_λ, δ_1)*-good on* $A \subseteq \mathbb{Z}^d$ (or that A *is* (L_λ, δ_1)*-good*) if for every mesoscopic box $C_{\underline{n},\delta_1}$ (with $\underline{n} \in \mathbb{Z}^d$) intersecting A, the following two properties hold:

- for every j such that I_j is relevant,

$$\sum_{x\in C_{\underline{n},\delta_1}} \mathbf{1}_{V(x)\in I_j} \leqslant (1+\varepsilon)(\delta_1 L_\lambda)^d \ \mathbb{P}[V(0) \in I_j]; \tag{3.6}$$

- for every j such that I_j is irrelevant,

$$\sum_{x\in C_{\underline{n},\delta_1}} \mathbf{1}_{V(x)\in I_j} \leqslant 2(\delta_1 L_\lambda)^d \ \frac{\varepsilon^9}{L_\lambda^2}. \tag{3.7}$$

We start by showing that sets with size of order L_λ are good with high probability.

Lemma 3.1. *For every fixed M, ε and δ_1, the probability that $[-ML_\lambda, ML_\lambda]^d$ is (L_λ, δ_1)-good tends to one as λ tends to 0.*

Proof. For M, ε and δ_1 fixed, the number $(R+2)$ of relevant and irrelevant intervals to be considered remains bounded as λ tends to 0. Similarly, the number of cubes of the form $C_{\underline{n},\delta_1}$ with $\underline{n} \in \mathbb{Z}^d$ that intersect $[-ML_\lambda, ML_\lambda]^d$ remains bounded as λ tends to 0.

We consider a relevant interval I_j and any cube $C_{\underline{n},\delta_1}$. The random quantity

$$\sum_{x\in C_{\underline{n},\delta_1}} I_{V(x)\in I_j}$$

is a binomial random variable with parameters $|C_{\underline{n},\delta_1}|$ and $\mathbb{P}[V(0) \in I_j]$. This is stochastically dominated by a Binomial random variable with parameters $(L_\lambda\delta_1 + 1)^d$ and $\mathbb{P}[V(0) \in I_j]$. By elementary bounds on Binomial tails (see for instance [MM13, (2.16)]), we have for λ small enough,

$$\begin{aligned} \mathbb{P}\left[\sum_{x\in C_{\underline{n},\delta_1}} I_{V(x)\in I_j} \geqslant (1+\varepsilon)\ (\delta_1 L_\lambda)^d \ \mathbb{P}[V(0) \in I_j]\right] \\ \leqslant \mathbb{P}\left[\sum_{x\in C_{\underline{n},\delta_1}} I_{V(x)\in I_j} \geqslant (1+\varepsilon/2)\ |C_{\underline{n},\delta_1}| \ \mathbb{P}[V(0) \in I_j]\right] \leqslant\ e^{-c(\varepsilon)|C_{\underline{n},\delta_1}|} \end{aligned} \tag{3.8}$$

for some strictly positive $c(\varepsilon)$. A similar reasoning applies to irrelevant intervals. □

The next result shows that in a good environment, the walk will typically not visit important points close to the boundary of $B(X_0, L_\lambda\varepsilon_0)$.

Lemma 3.2. *There exists K (depending on ε) such that for every ε_0 and δ satisfying $0 < \delta < \varepsilon_0/2$ and every δ_1 small enough, the following holds. If the environment in $B(0, L_\lambda(\varepsilon_0 + 3\delta))$ is (L_λ, δ_1)-good, then the probability that a random walk beginning at the origin hits a site of value $\frac{\varepsilon}{\lambda}$ or more in $B(0, L_\lambda(\varepsilon_0 + 2\delta)) \setminus B(0, L_\lambda(\varepsilon_0 - \delta))$ is bounded by $K\delta\varepsilon_0$.*

Proof. Recalling (1.4) and (1.7), we note that

$$\mathfrak{I}_{\varepsilon,\lambda} = \int f(\lambda z) \, \mathrm{d}\mu_\varepsilon(z) \geqslant f(\varepsilon)\mathbb{P}[V(0) \geqslant \varepsilon/\lambda],$$

and in particular, by the definition of L_λ, see (3.1),

$$\mathbb{P}[V(0) \geqslant \varepsilon/\lambda] \leqslant \frac{L_\lambda^{-2}}{2f(\varepsilon)}.$$

If δ_1 is sufficiently small and the environment is (L_λ, δ_1)-good, then it follows that the number of important points in $B(0, L_\lambda(\varepsilon_0 + 2\delta)) \setminus B(0, L_\lambda(\varepsilon_0 - \delta))$ is less than $(1 + \varepsilon + 2\varepsilon^2)L_\lambda^{-2}/f(\varepsilon)$ times the total number of points that are in cubes $C_{\underline{n},\delta_1}$ intersecting this region. Clearly for δ_1 small this number is bounded by a universal constant times

$$\frac{L_\lambda^{-2}}{f(\varepsilon)} \varepsilon_0^{d-1} \delta L_\lambda^d.$$

Recall that, as given by [La, Theorem 1.5.4], there exists $K_1 > 0$ such that

$$\mathbf{P}_0[X \text{ visits } y] \leqslant K_1 \; |y|^{2-d}. \tag{3.9}$$

Hence, the probability that a given point in $B(0, L_\lambda(\varepsilon_0 + 2\delta)) \setminus B(0, L_\lambda(\varepsilon_0 - \delta))$ is touched by the random walk is thus bounded by

$$\frac{K_1}{(L_\lambda(\varepsilon_0 - \delta))^{d-2}} \leqslant \frac{K_1 2^{d-2}}{(L_\lambda \varepsilon_0)^{d-2}}.$$

The probability described in the lemma is thus bounded by a constant times

$$f(\varepsilon)^{-1} L_\lambda^{-2} \varepsilon_0^{d-1} \delta L_\lambda^d \; (L_\lambda \varepsilon_0)^{2-d} = f(\varepsilon)^{-1} \delta \varepsilon_0,$$

which is the desired result. □

For the random walk $(X_n)_{n \geqslant 0}$, let σ be the stopping time defined by

$$\sigma = \inf\{n \geqslant 0 : |X_n - X_0| \geqslant \varepsilon_0 L_\lambda\}. \tag{3.10}$$

We say that the pair $(x, y) \in (\mathbb{Z}^d)^2$ is *generic* if

$$\begin{aligned} \mathbf{P}_x\big[&X \text{ hits an important point in} \\ &B(X_0, L_\lambda(\varepsilon_0 + 2\delta)) \setminus B(X_0, L_\lambda(\varepsilon_0 - \delta)) \mid X_\sigma = y\big] < \varepsilon_0^2 \delta^{1/3}. \end{aligned} \tag{3.11}$$

Although this is not explicit in the terminology, we stress that the notion of being generic depends on ε_0 and δ. Recalling the definition of the stopping times (σ_j) in (2.2), we let

$$F = \inf\left\{j : X_{\sigma_j} \in \big((M - \varepsilon_0)L_\lambda, \infty\big) \times \mathbb{Z}^{d-1}\right\}$$

and let $A(M, \varepsilon_0, \delta)$ be the event that

(i) the random walk $(X_{\sigma_j})_{j \geqslant 0}$ hits $((M - \varepsilon_0)L_\lambda, ML_\lambda) \times \left(\frac{\sqrt{M}L_\lambda}{2}, \frac{3\sqrt{M}L_\lambda}{2}\right) \times \left(-\frac{\sqrt{M}L_\lambda}{2}, \frac{\sqrt{M}L_\lambda}{2}\right)^{d-2}$ before leaving $\big(-3L_\lambda, ML_\lambda\big) \times \big(-\sqrt{M}L_\lambda, 2\sqrt{M}L_\lambda\big) \times \big(-\sqrt{M}L_\lambda, \sqrt{M}L_\lambda\big)^{d-2}$;

(ii) $F \leqslant (1 + \varepsilon)M\varepsilon_0^{-2}\sqrt{d}$;

(iii) for every $j \leqslant F$, the pair $(X_{\sigma_j}, \; X_{\sigma_{j+1}})$ is generic.

As a consequence of Lemmas 2.5 and 3.2, we obtain the following result.

Proposition 3.3. *Let M, ε_0 be given with ε_0 small. There exists $\overline{\delta} \in (0,1)$ such that for every $\delta < \overline{\delta}$, there exists $\overline{\delta}_1 \in (0, \overline{\delta})$ such that for every $\delta_1 < \overline{\delta}_1$, if the environment in $(-4L_\lambda, (M+1)L_\lambda) \times (-3\sqrt{M}L_\lambda, 3\sqrt{M}L_\lambda)^{d-1}$ is (L_λ, δ_1)-good, then for every $x \in (-L_\lambda/2, L_\lambda/2) \times (-\sqrt{M}L_\lambda/2, \sqrt{M}L_\lambda/2)^{d-1}$, we have*

$$\mathbf{P}_x\left[A(M, \varepsilon_0, \delta)\right] \geqslant \frac{c_1}{3} e^{-M\sqrt{d}/2}.$$

Proof. Let K $(= K(\varepsilon))$ be given by Lemma 3.2, and let $0 < \delta < \varepsilon_0/2$. We write $\mathcal{E}(\varepsilon_0, \delta)$ for the event that the random walk hits an important point in $B(X_0, L_\lambda(\varepsilon_0 + 2\delta)) \backslash B(X_0, L_\lambda(\varepsilon_0 - \delta))$. By Lemma 3.2, if the environment in $(-4L_\lambda, (M+1)L_\lambda) \times (-3\sqrt{M}L_\lambda, 3\sqrt{M}L_\lambda)^{d-1}$ is (L_λ, δ_1)-good and δ_1 is small enough, then for any x in $\big(-3L_\lambda, ML_\lambda\big) \times \big(-\sqrt{M}L_\lambda, 2\sqrt{M}L_\lambda\big) \times \big(-\sqrt{M}L_\lambda, \sqrt{M}L_\lambda\big)^{d-2}$,

$$\mathbf{P}_x\left[\mathcal{E}(\varepsilon_0, \delta)\right] \leqslant K\delta\varepsilon_0.$$

As a consequence, by Chebyshev's inequality, the probability that

$$\mathbf{P}_x[\mathcal{E}(\varepsilon_0, \delta) \mid X_\sigma] \geqslant \varepsilon_0^2 \delta^{1/3}$$

is bounded by

$$\frac{K\delta\varepsilon_0}{\varepsilon_0^2\delta^{1/3}} = \frac{K\delta^{2/3}}{\varepsilon_0} < \varepsilon_0^2\delta^{1/3},$$

provided that $\delta < \overline{\delta} \leqslant \varepsilon_0^9/K^3$. Thus, we have for x as above,

$$\mathbf{P}_x[(x, X_\sigma) \text{ is not generic}] < \varepsilon_0^2\delta^{1/3}.$$

Let $\mathcal{E}^i(\varepsilon_0, \delta)$ be the event that $(X_{\sigma_i}, X_{\sigma_{i+1}})$ is not generic. By the strong Markov property, the probability that there exists $i \leqslant (1+\varepsilon)M\varepsilon_0^{-2}\sqrt{d}$ such that

$$\begin{cases} X_{\sigma_i} \in \big(-3L_\lambda, ML_\lambda\big) \times \big(-\sqrt{M}L_\lambda, 2\sqrt{M}L_\lambda\big) \times \big(-\sqrt{M}L_\lambda, \sqrt{M}L_\lambda\big)^{d-2} \quad \text{and} \\ \mathcal{E}^i(\varepsilon_0, \delta) \text{ occurs} \end{cases}$$

is bounded by

$$(1+\varepsilon)M\sqrt{d}\delta^{1/3}.$$

The result then follows provided $\overline{\delta}$ is less than $\left(\frac{c_1}{6M\sqrt{d}(1+\varepsilon)} e^{-M\sqrt{d}/2}\right)^3$, using Lemma 2.5 with $h = d^{-1/2}$. □

Remark 3.4. We wish to emphasize that the event $A(M, \varepsilon_0, \delta)$ does not (explicitly) depend on the actual hitting time of the hyperplane $\{ML_\lambda\} \times \mathbb{Z}^{d-1}$ other than through the rough clock provided by the σ_is.

Remark 3.5. The part (iii) in the definition of $A(M, \varepsilon_0, \delta)$ enables us to conclude that with high probability on the event $A(M, \varepsilon_0, \delta)$ the important points within δL_λ of the points X_{σ_i} may be ignored.

The next major point is to examine the killing probabilities as the random walk passes from X_{σ_i} to $X_{\sigma_{i+1}}$. We have (see [La, Theorem 1.5.4]) that for $T_x = \inf\{n : X_n = x\}$,

$$|x|^{d-2}\mathbf{P}_0(T_x < \infty) \to c$$

for some $c \in (0, \infty)$. From this it follows that

Lemma 3.6. *Let $\tau_r = \inf\{n : |X_n| > r\}$. There exists a continuous $\varphi : [0,1] \to \mathbb{R}$, strictly positive on $(0,1)$, such that for every $\eta > 0$, uniformly on $\frac{|x|}{r} \in (\eta, 1-\eta)$, we have*

$$|x|^{d-2}\mathbf{P}_0[T_x < \tau_r] - \varphi\left(\frac{|x|}{r}\right) \to 0$$

as r tends to infinity.

Indeed, the lemma can be obtained (with an explicit expression for φ) by noting that the law of the random walk when hitting the sphere is asymptotically uniformly distributed, and decomposing the event of touching x according to whether it occurs before or after hitting the sphere. We refer to [BČ07, Lemma A.2] for details. The only important point for us is that since $\mathbf{E}_0(\tau_r)/r^2 \xrightarrow[r\to\infty]{} 1$, we have

$$\int_{D(0,1)} \frac{\varphi(|v|)}{q_d\,|v|^{d-2}}\,\mathrm{d}v = 1, \tag{3.12}$$

where $D(0,1)$ is the unit ball in $\mathbb{R}^d$. The following lemma follows from this observation.

Lemma 3.7. *If δ_1 is fixed sufficiently small (in terms of δ, ε and ε_0) and if $B(0,\varepsilon_0 L_\lambda)$ is an (L_λ,δ_1)-good environment, then for every j such that I_j is relevant,*

$$\sum_{\delta L_\lambda \leqslant |x| \leqslant (\varepsilon_0-\delta)L_\lambda} \mathbf{P}_0[X \textit{ hits } x \textit{ before time } \sigma]\ \mathbf{1}_{V(x)\in I_j} \leqslant (1+3\varepsilon)\varepsilon_0^2 L_\lambda^2 q_d\ \mathbb{P}[V(0)\in I_j],$$

while

$$\sum_{j:I_j \textit{ is irrel.}}\ \sum_{\delta L_\lambda \leqslant |x| \leqslant (\varepsilon_0-\delta)L_\lambda} \mathbf{P}_0[X \textit{ hits } x \textit{ before time } \sigma]\ \mathbf{1}_{V(x)\in I_j} \leqslant 5\varepsilon_0^2\varepsilon^6.$$

Proof. We begin with the proof concerning relevant intervals. Note first that

$$\begin{aligned}\sum_{\delta L_\lambda \leqslant |x| \leqslant (\varepsilon_0-\delta)L_\lambda} &\mathbf{P}_0[T_x<\sigma]\ \mathbf{1}_{V(x)\in I_j} \\ &\leqslant \sum_{\substack{\underline{n}\in\mathbb{Z}^d: \\ C_{\underline{n},\delta_1}\cap(B(0,(\varepsilon_0-\delta)L_\lambda)\setminus B(0,\delta L_\lambda))\neq\varnothing}}\ \sum_{x\in C_{\underline{n},\delta_1}:V(x)\in I_j} \mathbf{P}_0[T_x<\sigma].\end{aligned} \tag{3.13}$$

For $\delta_1 < \delta/3d$, if $C_{\underline{n},\delta_1}$ intersects $B(0,(\varepsilon_0-\delta)L_\lambda)\setminus B(0,\delta L_\lambda)$, then necessarily $C_{\underline{n},\delta_1} \subseteq B(0,(\varepsilon_0-\delta/2)L_\lambda)\setminus B(0,\delta L_\lambda/2)$. By Lemma 3.6,

$$\left| |x|^{d-2}\mathbf{P}_0[T_x<\sigma] - \varphi\left(\frac{|x|}{\varepsilon_0 L_\lambda}\right)\right|$$

tends to 0 as λ tends to 0, uniformly over all x in $B(0,(\varepsilon_0-\delta/2)L_\lambda)\setminus B(0,\delta L_\lambda/2)$. These observations imply that the right-hand side of (3.13) is asymptotically equivalent to

$$\sum_{\substack{\underline{n}\in\mathbb{Z}^d: \\ C_{\underline{n},\delta_1}\cap(B(0,(\varepsilon_0-\delta)L_\lambda)\setminus B(0,\delta L_\lambda))\neq\varnothing}}\ \sum_{x\in C_{\underline{n},\delta_1}:V(x)\in I_j} \varphi\left(\frac{|x|}{\varepsilon_0 L_\lambda}\right)|x|^{2-d}$$

as λ tends to 0. For δ_1 sufficiently small (and since φ is continuous), this is smaller than

$$\sum_{\substack{\underline{n}\in\mathbb{Z}^d: \\ C_{\underline{n},\delta_1}\cap(B(0,(\varepsilon_0-\delta)L_\lambda)\setminus B(0,\delta L_\lambda))\neq\varnothing}} (1+\varepsilon)\frac{|\{x\in C_{\underline{n},\delta_1} : V(x)\in I_j\}|}{(\varepsilon_0 L_\lambda)^{d-2}}\ \frac{1}{|D_{\underline{n},\delta_1}|}\int_{D_{\underline{n},\delta_1}} \frac{\varphi(|v|)}{|v|^{d-2}}\,\mathrm{d}v,$$

where $D_{\underline{n},\delta_1}$ is the (continuous) rectangle corresponding to $\frac{C_{\underline{n},\delta_1}}{\varepsilon_0 L_\lambda}$, and $|D_{\underline{n},\delta_1}|$ is its Lebesgue measure, that is, $(\delta_1/\varepsilon_0)^d$. Using the fact that our environment is (L_λ,δ_1)-good (see (3.6)) and (3.12), we obtain that

$$\sum_{\substack{\delta L_\lambda \leqslant |x| \leqslant (\varepsilon_0-\delta)L_\lambda \\ V(x)\in I_j}} \mathbf{P}_0[T_x<\sigma] \leqslant (1+3\varepsilon)\varepsilon_0^2 L_\lambda^2 q_d\ \mathbb{P}[V(0)\in I_j],$$

for all λ sufficiently small, as announced.

The proof for irrelevant intervals is the same, except that one uses (3.7) and the fact that there are no more than ε^{-3} irrelevant intervals to conclude. □

We can strengthen Lemma 3.7 as follows.

Lemma 3.8. *Let δ, ε_0, δ_1 be as in the previous lemma. If $B(0, \varepsilon_0 L_\lambda)$ is an (L_λ, δ_1)-good environment, then for every λ sufficiently small (i.e. L_λ sufficiently large), every $y \in \partial B(0, L_\lambda \varepsilon_1)$ and every j such that I_j is relevant,*

$$\sum_{\delta L_\lambda \leqslant |x| \leqslant (\varepsilon_0 - \delta) L_\lambda} \mathbf{P}_0[X \textit{ hits } x \textit{ before time } \sigma \mid X_\sigma = y] \, \mathbf{1}_{V(x) \in I_j} \leqslant (1 + 3\varepsilon)\varepsilon_0^2 L_\lambda^2 q_d \, \mathbb{P}[V(0) \in I_j]$$

while

$$\sum_{j : I_j \textit{ is irrel.}} \sum_{\varepsilon_0 \delta L_\lambda \leqslant |x| \leqslant (\varepsilon_0(1-\delta)) L_\lambda} \mathbf{P}_0[X \textit{ hits } x \textit{ before time } \sigma \mid X_\sigma = y] \, \mathbf{1}_{V(x) \in I_j} \leqslant 5\varepsilon_0^2 \varepsilon^6.$$

Proof. As before we consider only the case of relevant intervals. Associated with the random walk starting at 0 and conditioned to exit the ball $\mathcal{B}(0, L_\lambda \varepsilon_0)$ at $y \in \partial \mathcal{B}(0, L_\lambda \varepsilon_0)$ is the harmonic function

$$h(z) = \frac{\mathbf{P}_z[X_\sigma = y]}{\mathbf{P}_0[X_\sigma = y]}.$$

The conditioned random walk is the h-process with this harmonic function. The Harnack principle for the positive harmonic function h (see [La, Section 1.7]) asserts that for every $\delta > 0$, there exists $c = c(\delta) \in (0, \infty)$ such that

$$z, w \in \mathcal{B}(0, L_\lambda(\varepsilon_0 - \delta/2)) \Rightarrow h(z) \leqslant c \, h(w).$$

As a consequence, there exists $C \in (0, \infty)$ such that

$$z, w \in \mathcal{B}(0, L_\lambda(\varepsilon - \delta/2)) \Rightarrow |h(z) - h(w)| \leqslant C|z - w|. \tag{3.14}$$

For any $z \in \mathcal{B}(0, L_\lambda(\varepsilon - \delta/2))$, we have

$$\mathbf{P}_0[X \text{ hits } z \text{ before } \sigma | X_\sigma = y] = \frac{h(z)}{h(0)} \mathbf{P}_0[X \text{ hits } z \text{ before } \sigma].$$

Given ε_0 and $\varepsilon' > 0$, for L_λ sufficiently large (or λ sufficiently small),

$$\left| \frac{\mathbf{P}_0[X \text{ hits } z \text{ before } \sigma | X_\sigma = y]}{\varphi\left(\frac{z}{\varepsilon_0 L_\lambda}\right) h(z) \frac{1}{|z|^{d-2}}} - 1 \right| < \frac{\varepsilon'}{100},$$

uniformly over $z \in \mathcal{B}(0, L_\lambda(\varepsilon_0 - \delta/2)) \backslash \mathcal{B}(0, L_\lambda \delta/2)$. From this, we see as in the proof of the previous lemma that for j relevant,

$$\sum_{\delta L_\lambda \leqslant |x| \leqslant (\varepsilon_0 - \delta) L_\lambda} \mathbf{P}_0[T_x < \sigma | X_\sigma = y] \, \mathbf{1}_{V(x) \in I_j} \leqslant (1 + \frac{\varepsilon'}{50}) \sum_{\underline{n}} \sum_{x \in C_{\underline{n}, \delta_1} : V(x) \in I_j} h(x) \varphi\left(\frac{|x|}{\varepsilon_0 L_\lambda}\right) |x|^{2-d},$$

for λ small enough, where the outer sum on the right side is over every $\underline{n} \in \mathbb{Z}^d$ such that $C_{\underline{n}, \delta_1} \cap (B(0, (\varepsilon_0 - \delta) L_\lambda) \backslash B(0, \delta L_\lambda)) \neq \varnothing$. Using the continuity of the function

φ as well as the Harnack bound (3.14) for h, we can upper bound this last quantity by

$$\sum_{\underline{n}}(1+\varepsilon)\frac{|\{x \in C_{\underline{n},\delta_1} : V(x) \in I_j\}|}{(\varepsilon_0 L_\lambda)^{d-2}}(\varepsilon_0 L_\lambda)^{d-2} \inf_{v \in C_{\underline{n},\delta_1}} h(v)\varphi\left(\frac{|v|}{\varepsilon_0 L_\lambda}\right)|v|^{2-d},$$

provided δ_1 is fixed sufficiently small. In order to conclude, it remains to adapt (3.12) to deduce that

$$\sum_{\delta L_\lambda \leqslant |x| \leqslant (\varepsilon_0 - \delta)L_\lambda} h(x)\varphi\left(\frac{|x|}{\varepsilon_0 L_\lambda}\right)|x|^{2-d} \leqslant (1+\varepsilon')(\varepsilon_0 L_\lambda)^{-(d-2)} q_d$$

for λ small. This relies on the following three facts:
a) the isotropy of φ and $|v|^{d-2}$;
b) the fact that $1 = h(0) = \mathbf{E}_0[h(X_{\sigma_R})]$ $\forall$ $\delta L_\lambda \leqslant R \leqslant (\varepsilon_0 - \delta)L_\lambda$ and that uniformly over such R, the distribution of $\frac{X_{\sigma_R}}{R}$ tends to the uniform distribution on the sphere as L_λ tends to infinity.
c) The Harnack bound (3.14) for the function h.

In order to justify this, we give ourselves a small $\varepsilon'' > 0$ and divide up the $d-1$ dimensional unit sphere into sets $S_1, S_2, \ldots, S_M$, so that each S_i has diameter less than ε'' and has the property that the Haar measure of its interior is equal to that of its closure. The interval $[\delta L_\lambda, (\varepsilon - \delta)L_\lambda]$ is itself divided up into intervals $L_1, L_2, \cdots L_N$ each of length less than $L_\lambda \varepsilon''$. Altogether, this gives us a partition of the set $\{x : \delta L_\lambda \leqslant |x| \leqslant (\varepsilon - \delta)L_\lambda\}$ into the subsets $G_{ij} := \{x : (|x|, \frac{x}{|x|}) \in L_i \times S_j\}$. Properties b) and c) ensure that

$$\sum_j \sum_{z \in G_{ij}} h(z)\varphi(\frac{z}{\varepsilon_0 L_\lambda})\frac{1}{|z|^{d-2}}$$

is upper-bounded by $(1+\varepsilon')|\{z : |z| \in L_i\}| \sup_{z:|z| \in L_i} \varphi(\frac{z}{\varepsilon_0 L_\lambda})\frac{1}{|z|^{d-2}}$ for ε'' sufficiently small and then λ sufficiently small. The continuity of φ and property a) then complete the bound by a Riemann-sum argument. □

As is well known, for a random walk in dimensions three and higher, given that a point is visited, the number of visits is geometric of parameter q_d. From this, it is almost immediate that if a point is well distanced from the boundary of $B(0, \varepsilon_0 L_\lambda)$ and is visited by a random walk before time σ, then the number of visits ought to be approximately geometric with parameter q_d. The following is our formulation of this.

Lemma 3.9. *For fixed ε_0, $\delta > 0$, there exists $c(L_\lambda, \varepsilon_0, \delta)$ tending to one as L_λ tends to infinity such that for all $x \in B(0, (\varepsilon_0 - \delta)L_\lambda) \setminus B(0, \delta L_\lambda)$,*

$$(1 - q_d c(L_\lambda, \varepsilon_0, \delta))^{r-1} \leqslant \mathbf{P}_0\left[N(x) \geqslant r \mid N(x) \geqslant 1\right] \leqslant (1-q_d)^{r-1},$$

where $N(x)$ is the number of visits to site x before time σ.

Proof. For the right-hand side we simply note that by the strong Markov property, given that the point x is visited by a random walk, the number of visits to site x is geometric with parameter q_d. For the left-hand side inequality, we need a lower bound on the probability that the random walk starting at x returns there before time σ. We can decompose this probability as

$$1 - q_d - \mathbf{P}_x[X \text{ hits } x \text{ after time } \sigma] \geqslant 1 - q_d - \sup_{y \in \partial B(0, \varepsilon_0 L_\lambda)} \mathbf{P}_y[T_x < +\infty],$$

and this finishes the proof. □

Lemma 3.9 and convexity yield

Proposition 3.10. *Given $\varepsilon_0, \delta, \delta_1$ (and ε fixed small enough), for all λ small enough, if $B(0, \varepsilon_0 L_\lambda)$ is (L_λ, δ_1)-good, then*

$$\mathbf{E}_0\left[\exp\left(-\sum_{s<\sigma} \lambda V(X_s) \mathbf{1}_{V(X_s)\geqslant \frac{\varepsilon}{\lambda}} \mathbf{1}_{X_s \in B(0,(\varepsilon_0-\delta)L_\lambda)\setminus B(0,\delta L_\lambda)}\right) \,\Big|\, X_\sigma = y\right] \geqslant \exp\left(-(1+4\varepsilon) L_\lambda^2 \varepsilon_0^2 \int_{\frac{\varepsilon}{\lambda}}^{\infty} f(\lambda z) \, \mathrm{d}\mu(z)\right). \tag{3.15}$$

Proof. Since for all u and v positive, $1 - e^{-u+v} \leqslant 1 - e^{-u} + 1 - e^{-v}$, we have

$$1 - \mathbf{E}_0\left[\exp\left(-\sum_{s<\sigma} \lambda V(X_s) \mathbf{1}_{V(X_s)\geqslant \frac{\varepsilon}{\lambda}} \mathbf{1}_{X_s \in B(0,(\varepsilon_0-\delta)L_\lambda)\setminus B(0,\delta L_\lambda)}\right) \,\Big|\, X_\sigma = y\right] \leqslant \sum_{\substack{x \in B(0,(\varepsilon_0-\delta)L_\lambda)\setminus B(0,\delta L_\lambda) \\ V(x) \geqslant \frac{\varepsilon}{\lambda}}} \mathbf{E}_0\left[1 - e^{-\lambda N(x) V(x)} \mid X_\sigma = y\right], \tag{3.16}$$

where $N(x)$ denotes the number of time the walk visits x before time σ. If we restrict the sum above to those x such that $V(x) \in I_j$, where $I_j = [a_j, b_j)$ is a relevant interval, then we can use Lemmas 3.8 and 3.9 to bound the sum by

$$(1+3\varepsilon)\varepsilon_0^2 L_\lambda^2 q_d \, \mathbb{P}[V(0) \in I_j] \left(1 - e^{-\lambda b_j} \sum_{r=0}^{+\infty} (1-q_d)^r q_d e^{-r\lambda b_j}\right) = (1+3\varepsilon)\varepsilon_0^2 L_\lambda^2 \, \mathbb{P}[V(0) \in I_j] \, q_d \frac{1 - e^{-\lambda b_j}}{1 - (1-q_d) e^{-\lambda b_j}}.$$

Similarly, in the right-hand side of (3.16), the sum restricted to those x such that $V(x)$ belongs to some irrelevant interval is bounded by $5\varepsilon_0^2 \varepsilon^6$. In total, the right-hand side of (3.16) is thus bounded by

$$5\varepsilon_0^2\varepsilon^6 + (1+3\varepsilon)\varepsilon_0^2 L_\lambda^2 \sum_{j : I_j \text{ is rel.}} \mathbb{P}[V(0) \in I_j] \, q_d \frac{1 - e^{-\lambda b_j}}{1 - (1-q_d) e^{-\lambda b_j}}.$$

Using (3.3), one can check that this is smaller than

$$\left(1 + \frac{7\varepsilon}{2}\right) L_\lambda^2 \varepsilon_0^2 \int_{\frac{\varepsilon}{\lambda}}^{\infty} \frac{q_d (1 - e^{-\lambda z})}{1 - (1-q_d) e^{-\lambda z}} \, \mathrm{d}\mu(z),$$

and the result is then obtained provided ε_0 is sufficiently small, since, by the definition of L_λ in (3.1),

$$L_\lambda^2 \int_{\frac{\varepsilon}{\lambda}}^{\infty} \frac{q_d (1 - e^{-\lambda z})}{1 - (1-q_d) e^{-\lambda z}} \, \mathrm{d}\mu(z) \leqslant 1.$$

□

Corollary 3.11. *For ε, ε_0 and δ small enough and then δ_1 chosen small enough, if*

$$(-4L_\lambda, (M+1)L_\lambda) \times (-3\sqrt{M} L_\lambda, 3\sqrt{M} L_\lambda)^{d-1} \tag{3.17}$$

is (L_λ,δ_1)-good, then for every $x\in(-L_\lambda/2,L_\lambda/2)\times(-\sqrt{M}L_\lambda/2,\sqrt{M}L_\lambda/2)^{d-1}$,

$$\mathbf{E}_x\left[\exp\left(-\sum_{j\in J}\lambda V(X_j)\mathbf{1}_{V(X_j)\geqslant\frac{\varepsilon}{\lambda}}\right)\;\middle|\;A(M,\varepsilon_0,\delta),\;F,\;(X_{\sigma_i})_{i\leqslant F}\right] \geqslant \exp\left(-M(1+6\varepsilon)L_\lambda^2\sqrt{d}\int_{\varepsilon/\lambda}^{\infty}f(\lambda z)\,\mathrm{d}\mu(z)\right), \tag{3.18}$$

where J is the set of times $0\leqslant j\leqslant\sigma_F$ such that if $\sigma_{i-1}\leqslant j\leqslant\sigma_i$ then $X_j\in B(X_{\sigma_{i-1}},(\varepsilon_0-\delta)L_\lambda)\setminus B(X_{\sigma_{i-1}},\delta L_\lambda)$.

Proof. By Proposition 3.10, we have that

$$\mathbf{E}_0\left[\exp\left(-\sum_{j\in J}\lambda V(X_j)\mathbf{1}_{V(X_j)\geqslant\frac{\varepsilon}{\lambda}}\right)\;\middle|\;A(M,\varepsilon_0,\delta),\;F,\;(X_{\sigma_i})_{i\leqslant F}\right] \geqslant\left(\exp\left(-(1+4\varepsilon)L_\lambda^2\varepsilon_0^2\int_{\frac{\varepsilon}{\lambda}}^{\infty}f(\lambda z)\,\mathrm{d}\mu(z)\right)\right)^F.$$

(The restriction on the index set provided by J comes from the constraint $X_s\in B(0,(\varepsilon_0-\delta)L_\lambda)\setminus B(0,\delta L_\lambda)$ appearing in (3.15).) Since on the event $A(M,\varepsilon_0,\delta)$, $F\leqslant(1+\varepsilon)M\sqrt{d}\varepsilon_0^{-2}$, this is bounded below by

$$\left(\exp\left(-(1+4\varepsilon)L_\lambda^2\varepsilon_0^2\int_{\frac{\varepsilon}{\lambda}}^{\infty}f(\lambda z)\,\mathrm{d}\mu(z)\right)\right)^{(1+\varepsilon)M\sqrt{d}\varepsilon_0^{-2}}.$$

This gives the result provided ε was fixed sufficiently small. □

We now show that in the left-hand side of (3.18), one can replace the sum over $j\in J$ by the sum of the same summands over all j, and moreover, one can remove the restriction on $V(X_j)\geqslant\frac{\varepsilon}{\lambda}$. Recall that $V(X_j)1_{V(X_j)\geqslant\frac{\varepsilon}{\lambda}}=\tilde V_\varepsilon(X_j)1_{\tilde V_\varepsilon(X_j)\geqslant\frac{\varepsilon}{\lambda}}$.

Corollary 3.12. *Under the conditions of Corollary 3.11, with probability at least $1-\delta^{1/8}$,*

$$\mathbf{E}_x\left[\exp\left(-\sum_{j\in J'}\lambda\tilde V_\varepsilon(X_j)\right)\mathbf{1}_{\sigma_F\leqslant M\sqrt{d}L_\lambda^2(1+5\varepsilon)}\;\middle|\;A(M,\varepsilon_0,\delta),\;F,(X_{\sigma_i})_{i\leqslant F}\right] \geqslant\exp\left(-M(1+7\varepsilon)L_\lambda^2\sqrt{d}\;\mathfrak{I}_{\varepsilon,\lambda}\right),$$

where J' is the collection of $j\leqslant\sigma_F$ such that $X_j\notin B(0,\delta L_\lambda)$. If in addition the probability of hitting an important site within δL_λ of 0 is less than $\delta^{1/8}$, then this bound extends to all summands j with $j\leqslant\sigma_F$.

Proof. For any event $\mathcal{A}$ and any positive random variable Z,

$$\begin{aligned}&\mathbf{E}_0[e^{-Z}\mathbf{1}_{\mathcal{A}}\mid A(M,\varepsilon_0,\delta),F,(X_{\sigma_i})_{i\leqslant F}]\\&\quad\geqslant\mathbf{E}_0[e^{-Z}\mid A(M,\varepsilon_0,\delta),F,(X_{\sigma_i})_{i\leqslant F}]-\mathbf{P}_0[\mathcal{A}^c\mid A(M,\varepsilon_0,\delta),F,(X_{\sigma_i})_{i\leqslant F}].\end{aligned}\tag{3.19}$$

We will apply this for the random variable Z equal to $\sum_{j\in J'}\lambda V(X_j)\mathbf{1}_{V(X_j)\geqslant\varepsilon/\lambda}$ and the event $\mathcal{A}$ taken to be that

- no important point within $\delta\varepsilon_0L_\lambda$ of a point X_{σ_i} for some $1\leqslant i\leqslant F$ is hit
- and $\sigma_F\leqslant M\sqrt{d}L_\lambda^2(1+5\varepsilon)$.

To begin with, we observe that on the event $\mathcal{A}$, since $\sigma_F \leqslant M\sqrt{d}L_\lambda^2(1+5\varepsilon)$, we have almost surely

$$\exp\left(-\sum_{j\in J'} \lambda \tilde{V}_\varepsilon(X_j)\mathbf{1}_{\tilde{V}_\varepsilon(X_j)<\varepsilon/\lambda}\right) \geqslant \exp\left(-M\sqrt{d}L_\lambda^2(1+5\varepsilon)\ \lambda\mathbb{E}[V(0)\mid V(0)<\varepsilon/\lambda]\right). \tag{3.20}$$

Hence,

$$\mathbf{E}_0\left[\exp\left(-\sum_{j\in J'}\lambda\tilde{V}_\varepsilon(X_j)\right)\mathbf{1}_{\mathcal{A}}\ \middle|\ A(M,\varepsilon_0,\delta),\ F,(X_{\sigma_i})_{i\leqslant F}\right] \tag{3.21}$$

is larger than

$$\exp\left(-M\sqrt{d}L_\lambda^2(1+5\varepsilon)\ \lambda\mathbb{E}[V(0)\mid V(0)<\varepsilon/\lambda]\right)$$
$$\times\,\mathbf{E}_0\left[\exp\left(-\sum_{j\in J'}\lambda V(X_j)1_{V(X_j)\geqslant\varepsilon/\lambda}\right)\mathbf{1}_{\mathcal{A}}\ \middle|\ A(M,\varepsilon_0,\delta),\ F,(X_{\sigma_i})_{i\leqslant F}\right].$$

Using (3.19) and Corollary 3.11, we get that the latter conditional expectation is larger than

$$\exp\left(-M(1+6\varepsilon)L_\lambda^2\sqrt{d}\int_{\varepsilon/\lambda}^{\infty} f(\lambda z)\ \mathrm{d}\mu(z)\right) - \mathbf{P}_0[\mathcal{A}^c\mid A(M,\varepsilon_0,\delta),F,(X_{\sigma_i})_{i\leqslant F}].$$

One can check that for λ sufficiently small,

$$(1+5\varepsilon)\lambda\mathbb{E}[V(0)\mid V(0)<\varepsilon/\lambda]\leqslant(1+6\varepsilon)\int_0^{\varepsilon/\lambda} f(\lambda z)\ \mathrm{d}\mu_\varepsilon(z),$$

so that the conditional expectation in (3.21) is larger than

$$\exp\left(-M(1+6\varepsilon)L_\lambda^2\sqrt{d}\ \mathfrak{I}_{\varepsilon,\lambda}\right) - \mathbf{P}_0[\mathcal{A}^c\mid A(M,\varepsilon_0,\delta),F,(X_{\sigma_i})_{i\leqslant F}].$$

From the proof of Proposition 3.3, we learn that reducing ε_0 if necessary,

$$\mathbf{P}_0[\mathcal{A}^c\mid A(M,\varepsilon_0,\delta)]\ \leqslant\ \delta^{1/4}$$

for δ small. As a consequence,

$$\mathbf{P}_0[\mathcal{A}^c\mid A(M,\varepsilon_0,\delta),F,(X_{\sigma_i})_{i\leqslant F}]\ \leqslant\ \delta^{1/8}$$

outside an event of probability less than $\delta^{1/8}$. Outside this event, the conditional expectation in (3.21) is larger than

$$\exp\left(-M(1+6\varepsilon)L_\lambda^2\sqrt{d}\ \mathfrak{I}_{\varepsilon,\lambda}\right)-\delta^{1/8}.$$

In view of the definition of L_λ in (3.1), it is clear that it suffices to choose δ sufficiently small to ensure that this is larger than

$$\exp\left(-M(1+7\varepsilon)L_\lambda^2\sqrt{d}\ \mathfrak{I}_{\varepsilon,\lambda}\right),$$

and the proof is complete. □

Definition: Given ε and λ, we say that a point $x\in\mathbb{Z}^d$ is $(L_\lambda,\varepsilon_0,\delta)$-*healthy* if the probability for the random walk started at x to hit an important point within $\varepsilon_0\delta L_\lambda$ of x is less than $\delta^{1/4}$.

This and Proposition 3.3 give

Proposition 3.13. *Under the conditions of Corollary 3.11,*

$$\mathbf{E}_x\left[\exp\left(-\sum_{j\leqslant\sigma}\lambda\tilde{V}_\varepsilon(X_j)\right)\mathbf{1}_{A(M,\varepsilon,\delta)}\right] \geqslant \exp\left(-M(1+8\varepsilon)L_\lambda^2\sqrt{d}\ \mathfrak{I}_{\varepsilon,\lambda}\right)\frac{c_1}{3}e^{-M\sqrt{d}/2},$$

provided that x is $(L_\lambda,\varepsilon_0,\delta)$-healthy.

4. The block argument

We have established Proposition 3.13 above which basically states that (with appropriate probability) a random walk starting in $(-L_\lambda\sqrt{M}/2, L_\lambda\sqrt{M}/2)^d$ will arrive at $\{ML_\lambda\}\times(L_\lambda\sqrt{M}/2, 3L_\lambda\sqrt{M}/2)\times(-L_\lambda\sqrt{M}/2, L_\lambda\sqrt{M}/2)^{d-2}$ without leaving the designated bounded area provided the environment is (L_λ,δ_1)-good. This motivates an oriented percolation approach to show that with high probability as L_λ tends to infinity, there is the possibility of the random walk travelling from $(0,0,\cdots 0)$ to $(n,0,\cdots 0)$ without entering bad environments and by essentially having the first component increase by ML_λ in time intervals of length $ML_\lambda^2 d^{1/2}$.

For $i,j\in\mathbb{Z}$, let $B_{i,j}$ denote the set

$$\big(iML_\lambda-4L_\lambda,(iM+M+1)L_\lambda\big)\times\big((j-3)\sqrt{M}L_\lambda,(j+3)\sqrt{M}L_\lambda\big) \times\big(-3\sqrt{M}L_\lambda,3\sqrt{M}L_\lambda\big)^{d-2}.$$

The set $B_{i,j}$ is nothing but a translation of $B_{0,0}$, more precisely,

$$B_{i,j}=(iML_\lambda,j\sqrt{M}L_\lambda,0\cdots,0)+B_{0,0},$$

and $B_{0,0}$ is the set appearing in (3.17).

Let $G=(V,E)$ be the oriented graph with vertex set

$$V=\{(i,j)\in\mathbb{Z}^2, i\geqslant 0, i+j\equiv 0 \bmod 2\},$$

and edge set

$$E=\big\{[(i,j),(i+1,j+1)],\ [(i,j),(i+1,j-1)],\ \text{for } (i,j)\in V\big\}.$$

We consider a site percolation process on G by declaring the vertex (i,j) to be *open* if $B_{i,j}$ is (L_λ,δ_1)-good.

Lemma 4.1. *Let A_N be the event that there exist $j_0=0,j_1,\dots,j_N$ such that the sequence of sites $(0,j_0),\dots,(N,j_N)$ is a directed open path in G. We have*

$$\inf_N\mathbb{P}[A_N]\xrightarrow[\lambda\to 0]{}1.$$

Proof. It follows from Lemma 3.1 that the probability for a given site to be open tends to 1 as λ tends to 0. Moreover, the percolation process has a finite range of dependence. The lemma is then a direct consequence of [Li, Theorem B26] (or of [LSS97]). $\square$

For the random walk $(X_r)_{r\geqslant 0}$, we define the stopping times σ_j^i recursively in the following way. We let $\sigma_0^0=0$, and for all $i\geqslant 0$ and $j>0$,

$$\sigma_j^i\ =\ \inf\{r>\sigma_{j-1}^i:|X_r-X_{\sigma_{j-1}^i}|\geqslant\varepsilon_0L_\lambda\},$$

where for $i\geqslant 0$, the stopping time σ_0^{i+1} equals $\sigma_{F(i)}^i$ with

$$F(i):=\inf\{j:X_{\sigma_j^i}\in[(i+1)ML_\lambda-\varepsilon_0L_\lambda,\infty)\times\mathbb{Z}^{d-1}\}. \tag{4.1}$$

On the event that A_N is realized, we pick (in some arbitrary way) $j_0 = 0, j_1, \ldots, j_N$ such that $(0, j_0), \ldots, (N, j_N)$ is a directed open path in G. We define the (random walk) event B_n as the conjunction over all $i \leqslant N-1$ of:

(i) the random walk $(X_{\sigma^i_j})_{j \geqslant 0}$ hits

$$\big((iM + M - \varepsilon_0)L_\lambda, ML_\lambda\big) \times \left(\left(j_{i+1} - \frac{1}{2}\right)\sqrt{M}L_\lambda, \left(j_{i+1} + \frac{1}{2}\right)\sqrt{M}L_\lambda\right) \times \left(-\frac{\sqrt{M}L_\lambda}{2}, \frac{\sqrt{M}L_\lambda}{2}\right)^{d-2}$$

before leaving

$$\big(iML_\lambda - 3L_\lambda, (i+1)ML_\lambda\big) \times \big((j_{i+1} - 2)\sqrt{M}L_\lambda, (j_{i+1} + 2)\sqrt{M}L_\lambda\big) \times \big(-2\sqrt{M}L_\lambda, 2\sqrt{M}L_\lambda\big)^{d-2};$$

(ii) $F(i)$ defined in (4.1) satisfies

$$F(i) \leqslant (1+\varepsilon)M\varepsilon_0^{-2}\sqrt{d};$$

(iii) for every $j \leqslant F(i)$, the pair $(X_{\sigma^i_j}, X_{\sigma^i_{j+1}})$ is generic.

This event is a conjunction of events similar to the event $A(M, \varepsilon_0, \delta)$. Since with probability tending to 1 as N tends to infinity, the origin is healthy, we can apply Proposition 3.13 iteratively and get that on this event and on A_N,

$$\begin{aligned}
&\mathbf{E}_0\left[\exp\left(\sum_{i \leqslant \tau_{NML_\lambda}} \lambda \tilde{V}_\varepsilon(X_i)\right)\right] \\
&\quad \geqslant \mathbf{E}_0\left[\exp\left(\sum_{i \leqslant \tau_{NML_\lambda}} \lambda \tilde{V}_\varepsilon(X_i)\right)\mathbf{1}_{B_N}\right] \\
&\quad \geqslant \left(\frac{c_1}{3}\exp\left(-M(1+8\varepsilon)L_\lambda^2\sqrt{d}\,\mathfrak{I}_{\varepsilon,\lambda}\right)e^{-M\sqrt{d}/2}\right)^N.
\end{aligned}$$

Up to multiplicative corrections that can be taken as close to 1 as desired, the cost of travel to the hyperplane in the direction $(1, 0, \cdots, 0)$ at distance n to the origin is thus no more than n times

$$\frac{1}{L_\lambda}\left(L_\lambda^2\sqrt{d}\,\mathfrak{I}_{\varepsilon,\lambda} + \frac{\sqrt{d}}{2}\right) = L_\lambda\sqrt{d}\,\mathfrak{I}_{\varepsilon,\lambda} + \frac{\sqrt{d}}{2L_\lambda},$$

as n tends to infinity. In view of the definition of L_λ in (3.1), this is equal to $\sqrt{2d\,\mathfrak{I}_{\varepsilon,\lambda}}$, and the proof is complete.

5. The remaining case

We now treat the case where the contribution to $\mathfrak{I}_\lambda$ from the mass on $[\varepsilon/\lambda, \infty)$ is small, i.e. when (3.2) does not hold, that is,

$$2\,\mathfrak{I}_{\varepsilon,\lambda} \geqslant \frac{1}{\varepsilon^4}\mathbb{P}(V \geqslant \varepsilon/\lambda).$$

Long but elementary calculations yield that under this condition we have

$$\mathfrak{I}_\lambda(1 - K\varepsilon) \leqslant \int_0^{\varepsilon/\lambda} f(\lambda z)\,\mathrm{d}\mu(x) \leqslant \mathfrak{I}_\lambda(1 + K\varepsilon)$$

for universal K and equally for universal K'

$$\mathbb{P}(V(0) \geqslant \varepsilon/\lambda) \;\leqslant\; K'\varepsilon^4 \int_0^{\varepsilon/\lambda} f(\lambda z)\, \mathrm{d}\mu(x).$$

We now define (a bit simpler than before) a subset $A \subseteq \mathbb{Z}^d$ to be (L_λ, δ_1)-good (for an environment) if for every cube of the form $[i_1\delta_1 L_\lambda, (i_1+1)\delta_1 L_\lambda) \times [i_2\delta_1 L_\lambda, (i_2+1)\delta_1 L_\lambda) \times [i_d\delta_1 L_\lambda, (i_d+1)\delta_1 L_\lambda)$ we have that the number of important points is less than $2L_\lambda^{d-2}\delta^d K'\varepsilon^4$. We clearly have

Lemma 5.1. *There exists ε_F such that for all $\varepsilon < \varepsilon_F$, all $\delta_1 > 0$ and all $\gamma > 0$, if λ is sufficiently small, then the cube $[0, L_\lambda]^d$ is (L_λ, δ_1)-good with probability greater than $1-\gamma$.*

We can now pursue the arguments of Sections 2 and 3 (with the stopping times σ_i as before). In particular given $0 < \varepsilon_0 \ll \varepsilon$, we say that a point $x \in \mathbb{Z}^d$ is (ε_0, δ) *-good* (for the environment given) if

$$\mathbf{P}_x(\exists r \leqslant \sigma : V(X_r) \geqslant \varepsilon/\lambda) \leqslant \varepsilon_1\delta,$$

with σ defined as in (3.10).

Proposition 5.2. *Given M, ε_0 and δ, if δ_1 is fixed sufficiently small and if the environment in $[-3L_\lambda, ML_\lambda] \times (-3\sqrt{M}L_\lambda, 3\sqrt{M}L_\lambda)^{d-1}$ is (L_λ, δ_1)-good, then*

$$\mathbf{P}_x\left[A(M, \varepsilon_0, \delta)\right] \geqslant \frac{c_1}{3} e^{-M\sqrt{d}/2},$$

uniformly over initial points $x \in (-\sqrt{M}L_\lambda/2, \sqrt{M}L_\lambda/2)^d$ which are (ε_1, δ)-good, where the event $A(M, \varepsilon_0, \delta)$ is the intersection of

(i) *the random walk hits $\{ML_\lambda\} \times (\sqrt{M}L_\lambda/2, 3\sqrt{M}L_\lambda/2) \times (\sqrt{M}L_\lambda/2, \sqrt{M}L_\lambda/2)^{d-2}$ without leaving $[-3L_\lambda, ML_\lambda] \times (-\sqrt{M}L_\lambda, 2\sqrt{M}L_\lambda) \times (-\sqrt{M}L_\lambda, \sqrt{M}L_\lambda)^{d-2}$*

(ii) *the random walk hits $\{ML_\lambda\} \times (\sqrt{M}L_\lambda, 3\sqrt{M}L_\lambda) \times (\sqrt{M}L_\lambda/2, \sqrt{M}L_\lambda/2)^{d-2}$ at time $\chi_M \leqslant Md^{3/2}$ without having hit an important point.*

(iii) *The hitting point X_{χ_M} is (ε_0, δ)-good.*

From this the argument proceeds as in the preceding sections.

References

[BČ07] G. Ben Arous, J. Černý. Scaling limit for trap models on $\mathbb{Z}^d$. *Ann. Probab.* **35** (6), 2356-2384 (2007).

[BK93] J. van den Berg, H. Kesten. Inequalities for the time constant in first-passage percolation. *Ann. Appl. Probab.* **3** (1), 56–80 (1993).

[Fl07] M. Flury. Large deviations and phase transition for random walks in random nonnegative potentials. *Stochastic Process. Appl.* **117** (5), 596–612 (2007).

[Ko13] E. Kosygina. Crossing speeds of random walks among "sparse" or "spiky" Bernoulli potentials on integers. *J. Stat. Phys.* **152** (2), 213–236 (2013).

[KMZ11] E. Kosygina, T. Mountford, M.P.W. Zerner. Lyapunov exponents of Green's functions for random potentials tending to zero. *Probab. Theory Related Fields* **150** (1-2), 43–59 (2011).

[La] G. Lawler. *Intersections of random walks.* Probability and its applications, Birkhäuser (1991).

[Li] T. Liggett, *Stochastic interacting systems: contact, voter and exclusion processes.* Grundlehren der mathematischen Wissenschaften **324**, Springer (1999).

[LSS97] T. Liggett, R. Schonmann, A. Stacey. Domination by product measures. *Ann. Probab.* **25**, 71-95 (1997).

[Mo12] J.-C. Mourrat. Lyapunov exponents, shape theorems and large deviations for the random walk in random potential. *ALEA Lat. Am. J. Probab. Math. Stat.* **9**, 165-211 (2012).

[MM13] T. Mountford, J.-C. Mourrat. Lyapunov exponents for the random walk in random potential: the lower bound. *Comm. Math. Phys.* **323** (3), 1071–1120 (2013).

[RY] D. Revuz, M. Yor. *Continuous martingales and brownian motion.* Grundlehren der Mathematischen Wissenschaften **293**, Springer (1991).

[Ze98] M.P.W. Zerner. Directional decay of the Green's function for a random nonnegative potential on $\mathbf{Z}^d$. *Ann. Appl. Probab.* **8** (1), 246–280 (1998).

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