ON THE RATE OF CONVERGENCE IN THE MARTINGALE CENTRAL LIMIT THEOREM

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Abstract. Consider a discrete-time martingale, and let V^2 be its normalized quadratic variation. As V^2 approaches 1 and provided some Lindeberg condition is satisfied, the distribution of the rescaled martingale approaches the Gaussian distribution. For any $p \geqslant 1$, [Ha88] gives a bound on the rate of convergence in this central limit theorem, that is the sum of two terms, say $A_p + B_p$, where up to a constant, $A_p = \|V^2 - 1\|_p^{p/(2p+1)}$. We discuss here the optimality of this term, focusing on the restricted class of martingales with bounded increments. In this context, [Bo82] sketches a strategy to prove optimality for $p = 1$. Here, we extend this strategy to any $p \geqslant 1$, thus justifying the optimality of the term A_p. As a necessary step, we also provide a new bound on the rate of convergence in the central limit theorem for martingales with bounded increments that improves on the term B_p, generalizing another result of [Bo82].

MSC 2010: 60F05, 60G42.

Keywords: Martingale, central limit theorem, rate of convergence.

1. Introduction

Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a sequence of square-integrable random variables such that for any i, X_i satisfies $\mathbb{E}[X_i \mid \mathcal{F}_{i-1}] = 0$, where $\mathcal{F}_i$ is the σ-algebra generated by $(X_1, \ldots, X_i)$. In other words, $\mathbf{X}$ is a square-integrable martingale difference sequence. Following the notation of [Bo82], we write M_n for the set of all such sequences of length n, and introduce

$$s^2(\mathbf{X}) = \sum_{i=1}^n \mathbb{E}[X_i^2],$$

$$V^2(\mathbf{X}) = s^{-2}(\mathbf{X}) \sum_{i=1}^n \mathbb{E}[X_i^2 \mid \mathcal{F}_{i-1}],$$

$$S(\mathbf{X}) = \sum_{i=1}^n X_i.$$

One may call $V^2(\mathbf{X})$ the normalized quadratic variation of $\mathbf{X}$. Let $(\mathbf{X}_n)_{n\in\mathbb{N}}$ be such that for any n, $\mathbf{X}_n \in M_n$. It is well known (see for instance [Du, Section 7.7.a]) that if

$$V^2(\mathbf{X}_n) \xrightarrow[n\to+\infty]{\text{(prob.)}} 1 \tag{1.1}$$

and some Lindeberg condition is satisfied, then the rescaled sum $S(\mathbf{X}_n)/s(\mathbf{X}_n)$ converges in distribution to a standard Gaussian random variable, that is to say:

$$\forall t \in \mathbb{R}, \quad \mathbb{P}[S(\mathbf{X}_n)/s(\mathbf{X}_n) \leqslant t] \xrightarrow[n\to+\infty]{} \Phi(t), \tag{1.2}$$

where $\Phi(t) = (2\pi)^{-1/2} \int_{-\infty}^t e^{-x^2/2}\, \mathrm{d}x$.

We are interested in bounds on the speed of convergence in this central limit theorem. Several results have been obtained under a variety of additional assumptions. One natural way to strengthen the convergence in probability (1.1) is to change it for a convergence in L^p for some $p \in [1, +\infty]$. Quantitative estimates in terms of $\|V^2 - 1\|_p$ seem indeed particularly convenient when one wants to apply the result to practical situations. Let us write

$$D(\mathbf{X}) = \sup_{t \in \mathbb{R}} \left| \mathbb{P}[S(\mathbf{X})/s(\mathbf{X}) \leqslant t] - \Phi(t) \right|,$$

and

$$\|\mathbf{X}\|_p = \max_{1 \leqslant i \leqslant n} \|X_i\|_p \qquad (p \in [1, +\infty]).$$

[Ha88] proves the following result.

Theorem 1.1 ([Ha88])**.** *Let $p \in [1, +\infty)$. There exists a constant $C_p > 0$ such that, for any $n \geqslant 1$ and any $\mathbf{X} \in M_n$, one has*

$$D(\mathbf{X}) \leqslant C_p \left(\|V^2(\mathbf{X}) - 1\|_p^p + s^{-2p}(\mathbf{X}) \sum_{i=1}^n \|X_i\|_{2p}^{2p} \right)^{1/(2p+1)}. \tag{1.3}$$

In [Jo89], Theorem 1.1 is generalized to the following.

Theorem 1.2. *Let $p \in [1, +\infty]$ and $p' \in [1, +\infty)$. There exists $C_{p,p'} > 0$ such that for any $n \geqslant 1$ and any $\mathbf{X} \in M_n$, one has*

$$D(\mathbf{X}) \leqslant C_{p,p'} \left[\|V^2(\mathbf{X}) - 1\|_p^{p/(2p+1)} + \left(s^{-2p'}(\mathbf{X}) \sum_{i=1}^n \|X_i\|_{2p'}^{2p'} \right)^{1/(2p'+1)} \right]. \tag{1.4}$$

One should unerstand that $p/(2p+1) = 1/2$ for $p = +\infty$. In fact, a stronger, non-uniform bound is given ; we refer to [Jo89, Theorem 2.2] (or equivalently, to [Jo93]) for details.

The main question that is addressed here concerns the optimality of the term $\|V^2(\mathbf{X}) - 1\|_p^{p/(2p+1)}$ appearing in the r.h.s. of (1.3) or (1.4). About this, [Ha88] constructs a sequence of elements $\mathbf{X}_n \in M_n$ such that

- $s(\mathbf{X}_n) \simeq \sqrt{n}$,
- $D(\mathbf{X}_n) \simeq \log^{-1/2}(n)$,
- $\|V^2(\mathbf{X}) - 1\|_p^p \simeq s^{-2p}(\mathbf{X}) \, \|\mathbf{X}\|_{2p}^{2p} \simeq s^{-2p}(\mathbf{X}) \sum_{i=1}^n \|X_i\|_{2p}^{2p} \simeq \log^{-(2p+1)/2}(n)$,

where we write $a_n \simeq b_n$ if there exists $C > 0$ such that $a_n/C \leqslant b_n \leqslant C a_n$ for all large enough n. The example demonstrates that one cannot improve the exponent $1/(2p+1)$ appearing on the outer bracket of the r.h.s. of (1.3). But as the two terms of the r.h.s. of (1.3) are of the same order, one cannot draw any conclusion about the optimality of the term $\|V^2(\mathbf{X}) - 1\|_p^{p/(2p+1)}$ alone. Most importantly, it is rather disappointing that in the example, $\|\mathbf{X}\|_{2p}^{2p}$ and $\sum_{i=1}^n \|X_i\|_{2p}^{2p}$ are of the same order, if the typical martingales that one has in mind have increments of roughly the same order.

Using a similar construction, but imposing also that $V^2(\mathbf{X}) = 1$ a.s., [Jo89, Example 2.4] proves the optimality of the exponent $1/(2p'+1)$ appearing on the second term of the sum in the r.h.s. of (1.4), but does not discuss the optimality of the first term $\|V^2(\mathbf{X}) - 1\|_p^{p/(2p+1)}$.

For $1 \leqslant p \leqslant 2$, Theorem 1.1 is in fact already proved in [HB70]. In [HH, Section 3.6], the authors could only show that the bound on $D(\mathbf{X})$ can be no better than $\|V^2(\mathbf{X}) - 1\|_1^{1/2}$.

The proof of Theorem 1.1 given in [Ha88] is inspired by a method introduced in [Bo82]. There, the following results are proved.

Theorem 1.3 ([Bo82])**.** *Let $\gamma \in (0, +\infty)$. There exists a constant $C_\gamma > 0$ such that, for any $n \geqslant 2$ and any $\mathbf{X} \in M_n$ satisfying $\|\mathbf{X}\|_\infty \leqslant \gamma$ and $V^2(\mathbf{X}) = 1$ a.s., one has*

$$D(\mathbf{X}) \leqslant C_\gamma \frac{n \log(n)}{s^3(\mathbf{X})}.$$

In typical instances, $s(\mathbf{X})$ is of order $\sqrt{n}$ when $\mathbf{X} \in M_n$. Under such a circumstance, Theorem 1.3 thus gives a bound of order $\log(n)/\sqrt{n}$. Moreover, [Bo82] provides an example of a sequence of elements $\mathbf{X}_n \in M_n$ satisfying the conditions of Theorem 1.3, such that $s^2(\mathbf{X}_n) = n$ and for which

$$\limsup_{n\to+\infty} \sqrt{n} \log^{-1}(n) D(\mathbf{X}_n) > 0,$$

so the result is optimal.

Relaxing the condition that $V^2(\mathbf{X}) = 1$ a.s., [Bo82] then shows the following.

Corollary 1.4 ([Bo82])**.** *Let $\gamma \in (0, +\infty)$. There exists a constant $\overline{C}_\gamma > 0$ such that, for any $n \geqslant 2$ and any $\mathbf{X} \in M_n$ satisfying $\|\mathbf{X}\|_\infty \leqslant \gamma$, one has*

$$D(\mathbf{X}) \leqslant \overline{C}_\gamma \left[\frac{n \log(n)}{s^3(\mathbf{X})} + \min\left(\|V^2(\mathbf{X}) - 1\|_1^{1/3}, \|V^2(\mathbf{X}) - 1\|_\infty^{1/2} \right) \right]. \tag{1.5}$$

We refer to [Jo89, Theorem 3.2] for a non-uniform version of this result. A strategy is sketched in [Bo82] to prove that the bound $\|V^2(\mathbf{X}) - 1\|_1^{1/3}$ is indeed optimal, even on the restricted class considered by Corollary 1.4 of martingales with bounded increments. This example provides a satisfactory answer to our question of optimality for $p = 1$. The aim of the present paper is to generalize Corollary 1.4 and the optimality result to any $p \in [1, +\infty)$. We begin by proving the following general result.

Theorem 1.5. *Let $p \in [1, +\infty)$ and $\gamma \in (0, +\infty)$. There exists a constant $C_{p,\gamma} > 0$ such that, for any $n \geqslant 2$ and any $\mathbf{X} \in M_n$ satisfying $\|\mathbf{X}\|_\infty \leqslant \gamma$, one has*

$$D(\mathbf{X}) \leqslant C_{p,\gamma} \left[\frac{n \log(n)}{s^3(\mathbf{X})} + \left(\|V^2(\mathbf{X}) - 1\|_p^p + s^{-2p}(\mathbf{X}) \right)^{1/(2p+1)} \right]. \tag{1.6}$$

Note that, somewhat surprisingly, the term $s^{-2p}(\mathbf{X}) \sum_{i=1}^n \|X_i\|_{2p}^{2p}$ appearing in inequality (1.3) is no longer present in (1.5), and is changed for the smaller $s^{-2p}(\mathbf{X})$ in (1.6).

Finally, we justify the optimality of the term $\|V^2(\mathbf{X}) - 1\|_p^{p/(2p+1)}$ appearing in the r.h.s. of (1.6).

Theorem 1.6. *Let $p \in [1, +\infty)$ and $\alpha \in (1/2, 1)$. There exists a sequence of elements $\mathbf{X}_n \in M_n$ such that*

- $\|\mathbf{X}_n\|_\infty \leqslant 2$,
- $s(\mathbf{X}_n) \simeq \sqrt{n}$,
- $\|V^2(\mathbf{X}_n) - 1\|_p^{p/(2p+1)} = O\left(n^{(\alpha-1)/2}\right)$,
- $\limsup_{n\to+\infty} n^{(1-\alpha)/2} D(\mathbf{X}_n) > 0$.

Our strategy to prove Theorem 1.6 builds up on the one sketched in [Bo82] for the case when $p = 1$. Interestingly, Theorem 1.5 is used in the proof of Theorem 1.6.

The question of optimality of the term $\|V^2(\mathbf{X}) - 1\|_p^{p/(2p+1)}$, now settled by Theorem 1.6, arises naturally in the problem of showing a quantitative central limit theorem for the random walk among random conductances on $\mathbb{Z}^d$ [Mo11]. There, one approximates the random walk by a martingale. The martingale increments are stationary and "almost bounded" for $d \geqslant 3$, in the sense that they have bounded L^p norm for every $p < +\infty$. Roughly speaking, it is shown that for $d \geqslant 3$, the variance of the rescaled quadratic variation up to time t decays like t^{-1}. This bound is optimal, and leads to a Berry-Esseen bound of order $t^{-1/5}$. Theorem 1.6 thus demonstrates that there is no way to obtain a better exponent of decay than 1/5 if one relies only on information about the variance of the quadratic variation.

Theorem 1.5 is proved in Section 2, and Theorem 1.6 in Section 3.

2. Proof of Theorem 1.5

The proof of Theorem 1.5 is essentially similar to the proof of Corollary 1.4 given in [Bo82], with the additional ingredient of a Burkholder inequality.

Let $\mathbf{X} = (X_1, \ldots, X_n) \in M_n$ be such that $\|\mathbf{X}\|_\infty \leqslant \gamma$. The idea, which probably first appeared in [Dv72], is to augment the sequence to some $\hat{\mathbf{X}} \in M_{2n}$ such that $V^2(\hat{\mathbf{X}}) = 1$ almost surely, while preserving the property that $\|\hat{\mathbf{X}}\|_\infty \leqslant \gamma$, and apply Theorem 1.3 to this enlarged sequence. Let

$$\tau = \sup \left\{ k \leqslant n : \sum_{i=1}^{k} \mathbb{E}[X_i^2 \mid \mathcal{F}_{i-1}] \leqslant s^2(\mathbf{X}) \right\}.$$

For $i \leqslant \tau$, we define $\hat{X}_i = X_i$. Let r be the largest integer not exceeding

$$\frac{s^2(\mathbf{X}) - \sum_{i=1}^{\tau} \mathbb{E}[X_i^2 \mid \mathcal{F}_{i-1}]}{\gamma^2}.$$

As $\|X\|_\infty \leqslant \gamma$, it is clear that $r \leqslant n$. Conditionally on $\mathcal{F}_\tau$ and for $1 \leqslant i \leqslant r$, we let $\hat{X}_i$ be independent random variables such that $\mathbb{P}[\hat{X}_{\tau+i} = \pm\gamma] = 1/2$. If $\tau + r < 2n$, we let $\hat{X}_{\tau+r+1}$ be such that

$$\mathbb{P}\left[\hat{X}_{\tau+r+1} = \pm \left(s^2(\mathbf{X}) - \sum_{i=1}^{\tau} \mathbb{E}[X_i^2 \mid \mathcal{F}_{i-1}] - r\gamma^2 \right)^{1/2} \right] = \frac{1}{2},$$

the sign being decided independently of everything else. Finally, if $\tau + r + 1 < 2n$, we let $\hat{X}_{\tau+r+i} = 0$ for $i \geqslant 2$.

Possibly enlarging the σ-fields, we can assume that $\hat{X}_i$ is $\mathcal{F}_i$-measurable for $i \leqslant n$, and define $\mathcal{F}_i$ to be the σ-field generated by $\mathcal{F}_n$ and $\hat{X}_{n+1}, \ldots, \hat{X}_{n+i}$ if $i > n$. By construction, one has

$$\sum_{i=\tau+1}^{2n} \mathbb{E}[\hat{X}_i^2 \mid \mathcal{F}_{i-1}] = s^2(\mathbf{X}) - \sum_{i=1}^{\tau} \mathbb{E}[X_i^2 \mid \mathcal{F}_{i-1}],$$

which can be rewritten as

$$\sum_{i=1}^{2n} \mathbb{E}[\hat{X}_i^2 \mid \mathcal{F}_{i-1}] = s^2(\mathbf{X}).$$

As a consequence, $s^2(\hat{\mathbf{X}}) = s^2(\mathbf{X})$ and $V^2(\hat{\mathbf{X}}) = 1$ almost surely. The sequence $\hat{\mathbf{X}}$ thus satisfies the assumptions of Theorem 1.3, so

$$D(\hat{\mathbf{X}}) \leqslant 4C_\gamma \frac{n \log(n)}{s^3(\mathbf{X})}. \tag{2.1}$$

For any $x > 0$, we have

$$\begin{aligned}
\mathbb{P}\left[\frac{S(\mathbf{X})}{s(\mathbf{X})} \leqslant t\right] &\leqslant \mathbb{P}\left[\frac{S(\mathbf{X})}{s(\mathbf{X})} \leqslant t, \frac{|S(\mathbf{X}) - S(\hat{\mathbf{X}})|}{s(\mathbf{X})} \leqslant x\right] + \mathbb{P}\left[\frac{|S(\mathbf{X}) - S(\hat{\mathbf{X}})|}{s(\mathbf{X})} \geqslant x\right] \\
&\leqslant \mathbb{P}\left[\frac{S(\hat{\mathbf{X}})}{s(\mathbf{X})} \leqslant t + x\right] + \frac{1}{x^{2p}}\mathbb{E}\left[\left|\frac{S(\mathbf{X}) - S(\hat{\mathbf{X}})}{s(\mathbf{X})}\right|^{2p}\right]. \quad (2.2)
\end{aligned}$$

Due to (2.1), the first term in the r.h.s. of (2.2) is smaller than

$$\Phi(t+x) + 4C_\gamma \frac{n\log(n)}{s^3(\mathbf{X})} \leqslant \Phi(t) + \frac{x}{\sqrt{2\pi}} + 4C_\gamma \frac{n\log(n)}{s^3(\mathbf{X})}. \quad (2.3)$$

To control the second term, note first that

$$S(\mathbf{X}) - S(\hat{\mathbf{X}}) = \sum_{i=\tau+1}^{2n} (X_i - \hat{X}_i), \quad (2.4)$$

where we put $X_i = 0$ for $i > n$. As $\tau + 1$ is a stopping time, conditionally on τ, the $(X_i - \hat{X}_i)_{i \geqslant \tau+2}$ still forms a martingale difference sequence. We can thus use Burkholder's inequality (see for instance [HH, Theorem 2.11]), which states that

$$\begin{aligned}
&\frac{1}{C}\mathbb{E}\left[\left|\sum_{i=\tau+2}^{2n} (X_i - \hat{X}_i)\right|^{2p}\right] \quad (2.5) \\
&\qquad \leqslant \mathbb{E}\left[\left(\sum_{i=\tau+2}^{2n} \mathbb{E}[(X_i - \hat{X}_i)^2 \mid \mathcal{F}_{i-1}]\right)^p\right] + \mathbb{E}\left[\max_{\tau+2 \leqslant i \leqslant 2n} \left|X_i - \hat{X}_i\right|^{2p}\right],
\end{aligned}$$

and we can safely discard the summand indexed by $\tau+1$ appearing in (2.4), that is uniformly bounded. The maximum on the r.h.s. of (2.5) is also bounded by $2\gamma^{2p}$. As for the other term, X_i and $\hat{X}_i$ being orthogonal random variables, we have

$$\begin{aligned}
\sum_{i=\tau+1}^{2n} \mathbb{E}[(X_i - \hat{X}_i)^2 \mid \mathcal{F}_{i-1}] &= \sum_{i=\tau+1}^{2n} \mathbb{E}[X_i^2 \mid \mathcal{F}_{i-1}] + \sum_{i=\tau+1}^{2n} \mathbb{E}[\hat{X}_i^2 \mid \mathcal{F}_{i-1}] \\
&= s^2(\mathbf{X})V^2(\mathbf{X}) + s^2(\mathbf{X}) - 2\underbrace{\sum_{i=1}^{\tau} \mathbb{E}[X_i^2 \mid \mathcal{F}_{i-1}]}. \quad (2.6)
\end{aligned}$$

Now, if $\tau = n$, then by definition the sum underbraced above is $s^2(\mathbf{X})V^2(\mathbf{X})$. Otherwise, $\sum_{i=1}^{\tau+1} \mathbb{E}[X_i^2 \mid \mathcal{F}_{i-1}]$ exceeds $s^2(\mathbf{X})$, but as the increments are bounded, the sum underbraced is necessarily larger than $s^2(\mathbf{X}) - \gamma^2$. In any case, we thus have

$$\sum_{i=1}^{\tau} \mathbb{E}[X_i^2 \mid \mathcal{F}_{i-1}] \geqslant \min(s^2(\mathbf{X})V^2(\mathbf{X}), s^2(\mathbf{X}) - \gamma^2).$$

As a consequence, we obtain from (2.6) that

$$\sum_{i=\tau+1}^{2n} \mathbb{E}[(X_i - \hat{X}_i)^2 \mid \mathcal{F}_{i-1}] \leqslant \left|s^2(\mathbf{X})V^2(\mathbf{X}) - s^2(\mathbf{X})\right| + 2\gamma^2.$$

Combining this with equations (2.5), (2.4), (2.3) and (2.2), we finally obtain that

$$\mathbb{P}\left[\frac{S(\mathbf{X})}{s(\mathbf{X})} \leqslant t\right] - \Phi(t) \leqslant 4C_\gamma \frac{n\log(n)}{s^3(\mathbf{X})} + \frac{x}{\sqrt{2\pi}} + \frac{C}{x^{2p}}\left(\|V^2(\mathbf{X}) - 1\|_p^p + \frac{\gamma^{2p}}{s^{2p}(\mathbf{X})}\right).$$

Optimizing this over $x > 0$ leads to the correct estimate. The lower bound is obtained in the same way.

3. Proof of Theorem 1.6

Let $p \geqslant 1$ and $\alpha \in (1/2, 1)$ be fixed. We let $(X_{ni})_{1 \leqslant i \leqslant n-n^\alpha}$ be independent random variables with $\mathbb{P}[X_{ni} = \pm 1] = 1/2$. The subsequent $(X_{ni})_{n-n^\alpha < i \leqslant n}$ are defined recursively. Let

$$\lambda_{ni} = \sqrt{n - i + \kappa_n^2},$$

where $\kappa_n = n^{1/4}$ (in fact, any n^β with $1 - \alpha < 2\beta < \alpha$ would be fine). Assuming that $X_{n,1}, \ldots, X_{n,i-1}$ have already been defined, we write $\mathcal{F}_{n,i-1}$ for the σ-algebra that they generate, and let

$$S_{n,i-1} = \sum_{j=1}^{i-1} X_{nj}.$$

For any i such that $n - n^\alpha < i \leqslant n$, we construct X_{ni} such that

$$\mathbb{P}[X_{ni} \in \cdot \mid \mathcal{F}_{n,i-1}] = \left| \begin{array}{ll} \delta_{-\sqrt{3/2}} + \delta_{\sqrt{3/2}} & \text{if } S_{n,i-1} \in [\lambda_{ni}, 2\lambda_{ni}], \\ \delta_{-\sqrt{1/2}} + \delta_{\sqrt{1/2}} & \text{if } S_{n,i-1} \in [-2\lambda_{ni}, -\lambda_{ni}], \\ \delta_{-1} + \delta_1 & \text{otherwise,} \end{array} \right. \tag{3.1}$$

where δ_x is the Dirac mass at point x. One can view $(S_{ni})_{i \leqslant n}$ as an inhomogeneous Markov chain. We write $\mathbf{X}_n = (X_{n1}, \ldots, X_{nn})$, and $\mathbf{X}_{ni} = (X_{n1}, \ldots, X_{ni})$ for any $i \leqslant n$. Let

$$\delta(i) = \sup_{n \geqslant i} D(\mathbf{X}_{ni}). \tag{3.2}$$

Proposition 3.1. *One has, uniformly over n,*

$$\|V^2(\mathbf{X}_{ni}) - 1\|_p = O\big(i^{(\alpha-1)(1+1/2p)}\big) \qquad (i \to +\infty), \tag{3.3}$$

and

$$\delta(i) = O\left(i^{(\alpha-1)/2}\right) \qquad (i \to +\infty). \tag{3.4}$$

The proof goes the following way: first, we bound $\|V^2(\mathbf{X}_{ni}) - 1\|_p$ in terms of $(\delta(j))_{j \leqslant i}$ in Lemma 3.2. This gives an inequality on the sequence $(\delta(i))_{i \in \mathbb{N}}$ through Theorem 1.5, from which we deduce (3.4), and then (3.3).

Lemma 3.2. *Let $K_i = \max_{j \leqslant i} \delta(j) j^{(1-\alpha)/2}$. For any n and i, the following inequalities hold:*

$$\big|\mathbb{E}[X_{ni}^2] - 1\big| \leqslant \left| \begin{array}{ll} 0 & \text{if } i \leqslant n - n^\alpha, \\ 2\delta(i-1) & \text{if } n - n^\alpha < i \leqslant n, \end{array} \right. \tag{3.5}$$

$$\big|s^2(\mathbf{X}_{ni}) - i\big| \leqslant \left| \begin{array}{ll} 0 & \text{if } i \leqslant n - n^\alpha, \\ C i^{(3\alpha-1)/2} K_i \leqslant C i^\alpha & \text{if } n - n^\alpha < i \leqslant n, \end{array} \right. \tag{3.6}$$

$$\|V^2(\mathbf{X}_{ni}) - 1\|_p \leqslant \left| \begin{array}{ll} 0 & \text{if } i \leqslant n - n^\alpha, \\ C i^{(\alpha-1)(1+1/2p)} (1 + K_i)^{1/p} + C i^{(3\alpha-3)/2} K_i & \text{otherwise.} \end{array} \right. \tag{3.7}$$

Proof of Lemma 3.2. Inequality (3.5) is obvious for $i \leqslant n - n^\alpha$. Otherwise, from the definition (3.1), we know that

$$\mathbb{E}[X_{ni}^2] = 1 + \frac{1}{2}\mathbb{P}[S_{n,i-1} \in I_{ni}^+] - \frac{1}{2}\mathbb{P}[S_{n,i-1} \in I_{ni}^-],$$

where we write

$$I_{ni}^+ = [\lambda_{ni}, 2\lambda_{ni}] \quad \text{and} \quad I_{ni}^- = [-2\lambda_{ni}, -\lambda_{ni}]. \tag{3.8}$$

The random variable $S_{n,i-1}/s(\mathbf{X}_{n,i-1})$ is approximately Gaussian, up to an error controlled by $\delta(i-1)$. More precisely,

$$\left|\mathbb{P}[S_{n,i-1} \in I_{ni}^+] - \int_{I_{ni}^+/s(\mathbf{X}_{n,i-1})} \mathrm{d}\Phi\right| \leqslant 2\delta(i-1).$$

We obtain (3.5) using the fact that

$$\int_{I_{ni}^+/s(\mathbf{X}_{n,i-1})} \mathrm{d}\Phi = \int_{I_{ni}^-/s(\mathbf{X}_{n,i-1})} \mathrm{d}\Phi.$$

As a by-product, we also learn that

$$\left|s^2(\mathbf{X}_{ni}) - i\right| \leqslant \left| \begin{array}{ll} 0 & \text{if } i \leqslant n - n^\alpha, \\ 2\sum_{n-n^\alpha<j\leqslant i} \delta(j-1) & \text{if } n - n^\alpha < i \leqslant n. \end{array}\right.$$

Recalling that $\alpha < 1$, we obtain (3.6) noting that, for $n - n^\alpha < i \leqslant n$,

$$\sum_{n-n^\alpha<j\leqslant i} \delta(j-1) \leqslant n^\alpha (n-n^\alpha)^{(\alpha-1)/2} K_i.$$

In particular, it follows that

$$s^2(\mathbf{X}_{ni}) = i(1+o(1)). \tag{3.9}$$

Turning now to (3.7), $\|V^2(\mathbf{X}_{ni}) - 1\|_p$ is clearly equal to 0 for $i \leqslant n - n^\alpha$, so let us assume the contrary. We have:

$$\begin{aligned}
\|V^2(\mathbf{X}_{ni}) - 1\|_p &= s^{-2}(\mathbf{X}_{ni}) \left\| \sum_{j=1}^{i} \mathbb{E}[X_{nj}^2|\mathcal{F}_{n,j-1}] - s^2(\mathbf{X}_{ni}) \right\|_p \\
&\leqslant \frac{1}{s^2(\mathbf{X}_{ni})} \sum_{j=1}^{i} \left\|\mathbb{E}[X_{nj}^2|\mathcal{F}_{n,j-1}] - 1\right\|_p + \frac{\left|s^2(\mathbf{X}_{ni}) - i\right|}{s^2(\mathbf{X}_{ni})} \\
&\leqslant \frac{1}{2s^2(\mathbf{X}_{ni})} \sum_{n-n^\alpha<j\leqslant i} \left(\mathbb{P}[S_{n,j-1} \in I_{nj}^+ \cup I_{nj}^-]\right)^{1/p} + \frac{\left|s^2(\mathbf{X}_{ni}) - i\right|}{s^2(\mathbf{X}_{ni})}. \qquad (3.10)
\end{aligned}$$

We consider the two terms in (3.10) separately. First, by the definition of δ, we know that

$$\left|\mathbb{P}[S_{n,j-1} \in I_{nj}^+ \cup I_{nj}^-] - \int_{(I_{nj}^+ \cup I_{nj}^-)/s(\mathbf{X}_{n,j-1})} \mathrm{d}\Phi\right| \leqslant 2\delta(j-1).$$

Equation (3.9) implies that, uniformly over $j > n - n^\alpha$,

$$\int_{(I_{nj}^+ \cup I_{nj}^-)/s(\mathbf{X}_{n,j-1})} \mathrm{d}\Phi = (2\pi)^{-1/2} \frac{2\lambda_{nj}}{s(\mathbf{X}_{n,j-1})}(1+o(1)) \leqslant C n^{(\alpha-1)/2},$$

so the first term of (3.10) is bounded by

$$\begin{aligned}
&\frac{C}{i} \sum_{n-n^\alpha<j\leqslant i} (n^{(\alpha-1)/2} + 2\delta(j-1))^{1/p} \\
&\leqslant \frac{C}{i} \sum_{n-n^\alpha<j\leqslant i} (n^{(\alpha-1)/2} + 2(n-n^\alpha)^{(\alpha-1)/2} K_i)^{1/p} \qquad (3.11) \\
&\leqslant C i^{(\alpha-1)(1+1/2p)} (1+K_i)^{1/p}.
\end{aligned}$$

The second term in (3.10) is controlled by (3.6), and we obtain inequality (3.7). □

Proof of Proposition 3.1. Applying Theorem 1.5 with the information given by Lemma 3.2, we obtain that, up to a multiplicative constant that does not depend on n and $i \leqslant n$, $D(\mathbf{X}_{ni})$ is bounded by:

$$\frac{\log(i)}{\sqrt{i}} + i^{(\alpha-1)/2}(1+K_i)^{1/(2p+1)} + i^{-3(1-\alpha)p/(4p+2)} K_i^{p/(2p+1)} + i^{-p/(2p+1)}. \tag{3.12}$$

The first term can be disregarded, as it is dominated by $i^{-p/(2p+1)}$. Note also that, as $p \geqslant 1$, we have

$$\frac{3(1-\alpha)p}{4p+2} \geqslant \frac{1-\alpha}{2},$$

and as $\alpha > 1/2 > 1/(2p+1)$, we also have

$$\frac{p}{2p+1} \geqslant \frac{1-\alpha}{2}.$$

Multiplying (3.12) by $i^{(1-\alpha)/2}$, we thus obtain

$$K_i \leqslant C(1+K_i)^{1/(2p+1)} + CK_i^{p/(2p+1)},$$

where we recall that the constant C does not depend on i. Observing that the set $\{x \geqslant 0 : x \leqslant C(1+x)^{1/(2p+1)} + Cx^{p/(2p+1)}\}$ is bounded, we obtain that K_i is a bounded sequence, so (3.4) is proved. The relation (3.3) then follows from (3.4) and (3.7). □

Proposition 3.3. *We have*

$$\limsup_{i \to +\infty} \ i^{(1-\alpha)/2} \ \delta(i) > 0.$$

Proof. Our aim is to contradict, by reductio ad absurdum, the claim that

$$\delta(i) = o\left(i^{(\alpha-1)/2}\right) \qquad (i \to +\infty). \tag{3.13}$$

Let $Z_1, \ldots, Z_n$ be independent standard Gaussian random variables, and ξ_n be an independent centred Gaussian random variable with variance κ_n^2, all being independent of $\mathbf{X}_n$. Assuming (3.13), we will contradict the fact that

$$D(\mathbf{X}_n) = o\left(n^{(\alpha-1)/2}\right). \tag{3.14}$$

Let $W_{ni} = \sum_{j=i+1}^n Z_j + \xi_n$. Noting that $n^{-1/2}\sum_{j=1}^n Z_j$ is a standard Gaussian random variable, and with the help of [Bo82, Lemma 1], we learn that

$$\left|\mathbb{P}[W_{n0} \leqslant 0] - \frac{1}{2}\right| \leqslant C\frac{\kappa_n}{\sqrt{n}},$$

and similarly,

$$\left|\mathbb{P}[S_{nn} + \xi_n \leqslant 0] - \frac{1}{2}\right| \leqslant C\left(D(\mathbf{X}_n) + \frac{\kappa_n}{s(\mathbf{X}_n)}\right).$$

Combining these two observations with (3.6), we thus obtain that

$$\mathbb{P}[S_{nn} + \xi_n \leqslant 0] - \mathbb{P}[W_{n0} \leqslant 0] \leqslant C\left(D(\mathbf{X}_n) + \frac{\kappa_n}{\sqrt{n}}\right). \tag{3.15}$$

As $\kappa_n = n^{1/4}$ and $\alpha > 1/2$, we know that $\kappa_n/\sqrt{n} = o(n^{(\alpha-1)/2})$. We decompose the l.h.s. of (3.15) as

$$\sum_{i=1}^n \mathbb{P}[S_{n,i-1} + X_{ni} + W_{ni} \leqslant 0] - \mathbb{P}[S_{n,i-1} + Z_i + W_{ni} \leqslant 0].$$

The random variable W_{ni} is Gaussian with variance $\lambda_{ni}^2 = n - i + \kappa_n^2$, and independent of $\mathbf{X}_n$, so the sum can be rewritten as

$$\sum_{i=1}^{n} \mathbb{E}\left[\Phi\left(-\frac{S_{n,i-1}+X_{ni}}{\lambda_{ni}}\right) - \Phi\left(-\frac{S_{n,i-1}+Z_i}{\lambda_{ni}}\right)\right], \tag{3.16}$$

Let $\varphi(x) = (2\pi)^{-1/2}\ e^{-x^2/2}$. We can replace

$$\Phi\left(-\frac{S_{n,i-1}+X_{ni}}{\lambda_{ni}}\right) \tag{3.17}$$

by its Taylor expansion

$$\Phi\left(-\frac{S_{n,i-1}}{\lambda_{ni}}\right) - \frac{X_{ni}}{\lambda_{ni}}\varphi\left(-\frac{S_{n,i-1}}{\lambda_{ni}}\right) + \frac{X_{ni}^2}{2\lambda_{ni}^2}\varphi'\left(-\frac{S_{n,i-1}}{\lambda_{ni}}\right), \tag{3.18}$$

up to an error that is bounded by

$$\frac{|X_{ni}|^3}{6\lambda_{ni}^3}\|\varphi''\|_\infty. \tag{3.19}$$

Step 1. We show that the error term (3.19), after integration and summation over i, is $o(n^{(\alpha-1)/2})$. As X_{ni} is uniformly bounded, it suffices to show that

$$\sum_{i=1}^{n} \frac{1}{\lambda_{ni}^3} = o\left(n^{(\alpha-1)/2}\right). \tag{3.20}$$

The sum above equals

$$\sum_{i=1}^{n} \frac{1}{(n-i+\kappa_n^2)^{3/2}} \leqslant n^{-1/2}\int_{(\kappa_n^2-1)/n}^{(\kappa_n^2+n)/n} x^{-3/2}\ \mathrm{d}x = O\left(\kappa_n^{-1}\right).$$

As we defined κ_n to be $n^{1/4}$ and $\alpha > 1/2$, equation (3.20) is proved.

Step 2. For the second part of the summands in (3.16), the same holds with X_{ni} replaced by Z_i, and similarly,

$$\sum_{i=1}^{n} \frac{\mathbb{E}[|Z_i|^3]}{\lambda_{ni}^3} = o\left(n^{(\alpha-1)/2}\right). \tag{3.21}$$

Step 3. Combining the results of the two previous steps, we know that up to a term of order $o\left(n^{(\alpha-1)/2}\right)$, the sum in (3.16) can be replaced by

$$\sum_{i=1}^{n} \mathbb{E}\left[\frac{Z_i - X_{ni}}{\lambda_{ni}}\varphi\left(-\frac{S_{n,i-1}}{\lambda_{ni}}\right) + \frac{X_{ni}^2 - Z_i^2}{2\lambda_{ni}^2}\varphi'\left(-\frac{S_{n,i-1}}{\lambda_{ni}}\right)\right].$$

Conditionally on $S_{n,i-1}$, both Z_i and X_{ni} are centred random variables, so the first part of the summands vanishes, and there remains only

$$\sum_{i=1}^{n} \mathbb{E}\left[\frac{X_{ni}^2 - Z_i^2}{2\lambda_{ni}^2}\varphi'\left(-\frac{S_{n,i-1}}{\lambda_{ni}}\right)\right] = \sum_{i=1}^{n} \mathbb{E}\left[\frac{\mathbb{E}[X_{ni}^2 - 1 \mid S_{n,i-1}]}{2\lambda_{ni}^2}\varphi'\left(-\frac{S_{n,i-1}}{\lambda_{ni}}\right)\right]. \tag{3.22}$$

From the definition of X_{ni}, we learn that $\mathbb{E}[X_{ni}^2 - 1 \mid S_{n,i-1}]$ is 0 if $i \leqslant n - n^\alpha$, and otherwise equals

$$\left|\begin{array}{ll} 1/2 & \text{if } S_{n,i-1} \in I_{ni}^+, \\ -1/2 & \text{if } S_{n,i-1} \in I_{ni}^-, \\ 0 & \text{otherwise,} \end{array}\right.$$

where I_{ni}^{+} and I_{ni}^{-} were defined in (3.8). As a consequence, it is clear that the contribution of each summand in the r.h.s. of (3.22) is positive. Moreover, for $i > n - n^{\alpha}$ and on the event $S_{n,i-1} \in I_{ni}^{-} \cup I_{ni}^{+}$, we have

$$\mathbb{E}[X_{ni}^2 - 1 \mid S_{n,i-1}] \; \varphi'\left(-\frac{S_{n,i-1}}{\lambda_{ni}}\right) \geqslant \frac{1}{2} \inf_{[1,2]} |\varphi'| > 0.$$

Let us assume temporarily that, uniformly over n and i such that $n - n^{\alpha} < i \leqslant n - (n^{\alpha})/2$, we have

$$\mathbb{P}[S_{n,i-1} \in I_{ni}^{-} \cup I_{ni}^{+}] \geqslant C \frac{\lambda_{ni}}{\sqrt{n}}. \tag{3.23}$$

Then the sum in the r.h.s. of (3.22) is, up to a constant, bounded from below by

$$\sum_{n-n^{\alpha} < i \leqslant n-(n^{\alpha})/2} \frac{1}{\lambda_{ni}\sqrt{n}} \geqslant C n^{\alpha} \frac{1}{n^{\alpha/2}\sqrt{n}} = C n^{(\alpha-1)/2}.$$

This contradicts (3.14) via inequality (3.15), and thus finishes the proof of the Proposition.

Step 4. There remains to show (3.23), for $n - n^{\alpha} < i \leqslant n - (n^{\alpha})/2$. We have

$$\left| \mathbb{P}[S_{n,i-1} \in I_{ni}^{+}] - \int_{I_{ni}^{+}/s(\mathbf{X}_{n,i-1})} \mathrm{d}\Phi \right| \leqslant 2\delta(i-1).$$

Using inequality (3.6), it follows that

$$\int_{I_{ni}^{+}/s(\mathbf{X}_{n,i-1})} \mathrm{d}\Phi \geqslant C \frac{\lambda_{ni}}{\sqrt{n}}.$$

As we choose i inside $[n - n^{\alpha}, n - (n^{\alpha})/2]$, λ_{ni} is larger than $C n^{\alpha/2}$, while $\delta(i-1) = o\left(i^{(\alpha-1)/2}\right)$ by assumption (3.13). This proves (3.23). □

Remark. To match the example proposed in [Bo82], one should choose instead $\alpha = 1/3$ and $\kappa_n = 1$ in the definition of the sequences $(\mathbf{X}_n)$. In this case, Propositions 3.1 and 3.3 are still true. While the proof of Proposition 3.1 can be kept unchanged, Proposition 3.3 requires a more subtle analysis. First, one needs to choose ξ_n of variance $\overline{\kappa}_n^2 \neq 1$, thus requiring to change the λ_{ni} appearing in (3.16) by, say, $\overline{\lambda}_{ni} = \sqrt{n - i + \overline{\kappa}_n^2}$. The sequence $\overline{\kappa}_n^2$ should grow to infinity with n, while remaining $o(n^{\alpha})$. In step 1, bounding the difference between (3.17) and (3.18) by (3.19) is too crude. Instead, one can bound it by

$$\frac{C}{\overline{\lambda}_{ni}^3} \Psi\left(-\frac{S_{n,i-1}}{\overline{\lambda}_{ni}}\right),$$

where $\Psi(x) = \sup_{|y| \leqslant 1} |\varphi''(x+y)|$. One can then appeal to [Bo82, Lemma 2] and get through this step, using the fact that $\overline{\kappa}_n$ tends to infinity. Step 2 is similar, with some additional care required by the fact that Z_i is unbounded. The rest of the proof then applies, taking care of the discrepancy between λ_{ni} and $\overline{\lambda}_{ni}$ when necessary.

Acknowledgments. I would like to thank an anonymous referee for his careful review, and for his mention of the reference [Jo89], which I was not aware of.

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