PRINCIPAL EIGENVALUE FOR RANDOM WALK AMONG RANDOM TRAPS ON $\mathbb{Z}^d$

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Abstract. Let $(\tau_x)_{x\in\mathbb{Z}^d}$ be i.i.d. random variables with heavy (polynomial) tails. Given $a \in [0,1]$, we consider the Markov process defined by the jump rates $\omega_{x\to y} = {\tau_x}^{-(1-a)}{\tau_y}^a$ between two neighbours x and y in $\mathbb{Z}^d$. We give the asymptotic behaviour of the principal eigenvalue of the generator of this process, with Dirichlet boundary condition. The prominent feature is a phase transition that occurs at some threshold depending on the dimension. Our method relies mainly on results proved in the Appendix, which are of independent interest. They consist of a Gaussian-like upper bound on the transition kernel of any symmetric nearest-neighbour continuous-time random walk on $\mathbb{Z}^d$, provided its jump rates are uniformly bounded from below, together with an upper bound on the Green function when $d \geqslant 3$.

1. Introduction

For each site $x \in \mathbb{Z}^d$, pick at random $\tau_x > 0$, so that $(\tau_x)_{x\in\mathbb{Z}^d}$ are independent and identically distributed random variables. We call $\tau = (\tau_x)_{x\in\mathbb{Z}^d}$ the *environment*, and write its law $\mathbb{P}$ (and the corresponding expectation $\mathbb{E}$). Fixing $a \in [0,1]$ and an environment τ, we define the Markov process $(X_t)_{t\geqslant 0}$ by the following jump rates :

$$\omega_{x\to y} = \left| \begin{array}{ll} {\tau_x}^{-(1-a)}{\tau_y}^a & \text{if } \|x-y\| = 1 \\ 0 & \text{otherwise} \end{array} \right.$$

The associated infinitesimal generator is :

$$\mathcal{L}f(x) = \sum_{y:\|x-y\|=1} \omega_{x\to y}(f(y) - f(x))$$

The aim of this note is to investigate the behaviour of the principal eigenvalue of $\mathcal{L}$ restricted to a large box. Define the box of size n by $B_n = \{-n,\ldots,n\}^d$, and $\mathcal{L}_n$ the operator $\mathcal{L}$ restricted to this box, with Dirichlet boundary conditions. That is to say $\mathcal{L}_n f = \mathbf{1}_{B_n}\mathcal{L}f$, defined for any function $f : \mathbb{Z}^d \to \mathbb{R}$ that vanishes outside the box. Let λ_n be the smallest eigenvalue of $-\mathcal{L}_n$. We write λ_n^0 for the eigenvalue obtained in the particular case when $a = 0$.

We are particularly interested in the study of heavy tailed laws for the environment. A natural assumption (see the remark just after Theorem 1.1) is that the tail $\mathbb{P}[\tau_0 > y]$ behaves like a power of y as y goes to infinity.

Assumption 1. There exists $\alpha > 0$ such that :

$$F(y) := \mathbb{P}[\tau_0 > y] \simeq \frac{1}{y^\alpha} \qquad (y \to +\infty) \tag{1.1}$$

More precisely, we say that a function f varies regularly with index ρ at infinity, and write $f \in RV_\rho$, if for all $\kappa > 0$, $f(\kappa x)/f(x) \to \kappa^\rho$ as $x \to +\infty$ (see [BGT] for a monograph on regular variation).

Assumption 1'. There exists $\alpha > 0$ such that $F \in RV_{-\alpha}$.

Assumption 1' gives a precise sense to assumption 1, and is more general than just assuming the equality (or equivalence) in equation (1.1). Note that, for $0 < \alpha < 2$,

τ_0 belongs to the domain of attraction of an α-stable law if and only if $F \in RV_{-\alpha}$ (see [Fel, Corollary XVII.5.2]). Assumption 1' implies that for all $\varepsilon > 0$:

$$F(y)y^{\alpha+\varepsilon} \xrightarrow[y\to+\infty]{} +\infty \quad \text{and} \quad F(y)y^{\alpha-\varepsilon} \xrightarrow[y\to+\infty]{} 0 \tag{1.2}$$

and as a consequence, $\mathbb{E}[\tau_0^\beta]$ is finite for all $\beta < \alpha$, infinite for all $\beta > \alpha$ (and may be finite or infinite when $\beta = \alpha$).

Assumption 2. We will always assume that $\tau_0 \geqslant 1$, concentrating on "bad behaviours" at infinity.

We need to introduce the generalized inverse of $1/F$, defined by :

$$h(x) = \inf\{y : 1/F(y) \geqslant x\}$$

As F belongs to $RV_{-\alpha}$, one can see that $h \in RV_{1/\alpha}$ (see for instance [Res, Proposition 0.8 (v)]). Loosely speaking, $h(y) \simeq y^{1/\alpha}$. We will recall later how h is related to the asymptotic behaviour of maxima and sums of (τ_x) (see Proposition 1.2), but let us first state (and comment) our main results.

Theorem 1.1. (1) *For almost every environment, we have :*

$$\lim_{n\to\infty} -\frac{\ln(\lambda_n)}{\ln(n)} = \left| \begin{array}{ll} \max\left(2, 1+\dfrac{1}{\alpha}\right) & \text{if } d=1 \\ \max\left(2, \dfrac{d}{\alpha}\right) & \text{if } d \geqslant 2 \end{array} \right.$$

(2) *If $d \geqslant 2$ and $\alpha > d/2$ or if $d = 1$ and $\alpha > 1$, then there exist $k_1, k_2 > 0$ such that for almost every environment and n large enough :*

$$\frac{k_1}{n^2} \leqslant \lambda_n \leqslant \frac{k_2}{n^2}$$

(3) *If $\alpha < 1$ and $d \neq 2$, then for any $\varepsilon > 0$, there exist $\eta, M > 0$ such that for all n large enough :*

$$\mathbb{P}[\eta \leqslant a_n\lambda_n \leqslant M] \geqslant 1 - \varepsilon$$

where

$$a_n = \left| \begin{array}{ll} nh(n) & \text{if } d=1 \\ h(n^d) & \text{if } d \geqslant 3. \end{array} \right.$$

(4) *Let $a_n = \ln(n)h(n^2)$. If $d = 2$ and $\alpha < 1$, then for any $\varepsilon > 0$, there exist $\eta, M > 0$ such that for all n large enough :*

$$\mathbb{P}[\eta \leqslant a_n\lambda_n^0 \leqslant M] \geqslant 1 - \varepsilon$$

$$\mathbb{P}[\eta \leqslant a_n\lambda_n \leqslant \ln(n)M] \geqslant 1 - \varepsilon$$

Let us now give some heuristics about the behaviour of (X_t). If $a = 0$, the walk is in fact a time-change of the simple random walk : arriving at some site x, it waits an exponential time of mean τ_x before jumping to a neighbouring site chosen uniformly. When $a \neq 0$, things get more complicated. Suppose that the walk arrives at some deep trap, that is a site x where τ_x is very large. Compared with the $a = 0$ case, the walk will leave site x faster. On the other hand, once on a neighbouring site, it will come back to x with very high probability. These two competing effects can compensate remarkably in the limit, and indeed our main results are independent of a (as they also are in [BČ05]).

We propose to call $(X_t)_{t\geqslant 0}$ a *random walk among random traps*. It seems to us that for its relative simplicity, it should be considered one of the basic types of random walks in random environments to study, just as is the random walk among random conductances. Although one could have the feeling that theses two types are basically the same, one attaching randomness to edges of the graph and the other to sites, they exhibit very different behaviours. For instance, the reversible measure is not the uniform one in the case of random traps (it gives weight τ_x to site x). Also, if $d \geqslant 2$, the random walk in random conductances tends to avoid visiting regions where conductance is very low (and where time spent to 'get out' may be high). On the other hand, when walking among random traps, say for $a = 0$, the path is the same as for the simple random walk, and the walk is not enclined to avoid regions from which it takes a long time to get out. See [Al81] for a nice discussion about this issue.

This type of walk gained interest when J.P. Bouchaud [Bou92] proposed it as a phenomenological model to explain aging of spin glasses, and as a consequence, what we call 'random walk among random traps' is also known as *Bouchaud's model*. Later on, [RMB00] introduced the full model as presented here (including the $a \in [0, 1]$), which allows them to get more diverse aging scalings.

When $\mathbb{E}[\tau_0]$ is finite (in particular when $\alpha > 1$), one can apply results of [DFGW89] to prove that, under the averaged law, (X_t) is diffusive and converges to Brownian motion after rescaling.

In one dimension, for $a = 0$ and $\alpha < 1$, L.R.G. Fontes, M. Isopi and C.M. Newman [FIN02] proved that almost surely the process was subdiffusive and obtained convergence of the rescaled process to a singular diffusion, as well as aging. The results have been extended to general a by G. Ben Arous and J. Černý in [BČ05]. Another (also subdiffusive) scaling limit was identified when $d \geqslant 2$, $\alpha < 1$ and $a = 0$ in [BČ07]. We refer to [BČ06] for a review on the subject. To our knowledge, nothing was known in the case when $a \neq 0$ and $d \geqslant 2$.

This note comes as a partial answer to a question of [BČ06], initially directed only to the one-dimensional case :

"What is the behaviour of the edge of the spectrum for the generator of the dynamics ? This might be close to, but easier than the same question solved for Sinai's random walk by [BF08]."

Note that our method is in fact rather independent from the one used in [BF08], and the main technical problems appear only when $d \geqslant 2$. Remark also that on the complete graph and for $a = 0$, [BF05] got explicit formulas for the whole spectrum and managed to link them with aging properties.

Remark. A natural choice of (τ_x) from the statistical physics' point of view is the following : first choose independently for each site a random variable $-E_x$ with law exponential of parameter 1, and define τ_x to be $\exp(-\beta E_x)$, where β represents the inverse of the temperature. Then one can check that $F \in RV_{-1/\beta}$, and the irregularity that appears at $\beta = 1$ for $d \leqslant 2$ and at $\beta = 2/d$ for larger d can be regarded as a phase transition (the anomalous behaviour occurring for β large, that is for small temperature, or in our context, small α).

It may seem surprising that this new phase transition does not appear at the same threshold than the diffusive/subdiffusive transition (that at least for $a = 0$ occurs when $\alpha(= 1/\beta) = 1$ in any dimension). The reason for this is the following : although the principal eigenvalue will 'feel' the very deepest traps of the box (of order $n^{d/\alpha}$), the process (when $a = 0$) will exit the box after visiting only some n^2 sites, thus having seen only traps of order at most $n^{2/\alpha}$.

Before going on to show how Theorem 1.1 is a consequence of the results of the following sections, we need to recall some facts about the asymptotic behaviour of sums and maxima of (τ_x).

Proposition 1.2. (1) *For any $\varepsilon > 0$ and almost every environment :*

$$n^{-(\max(d,d/\alpha)+\varepsilon)} \sum_{x \in B_n} \tau_x \to 0 \qquad (n \to +\infty)$$

(2) *For any* $\varepsilon > 0$ *and almost every environment :*
$$n^{-(\max(d,d/\alpha)-\varepsilon)} \sum_{x \in B_n} \tau_x \to +\infty \qquad (n \to +\infty)$$

(3) *There exists a random variable* M_∞ *with values in* $(0, +\infty)$ *such that the rescaled maxima converge in law to* M_∞ *:*
$$\frac{1}{h(n^d)} \max_{x \in B_n} \tau_x \Rightarrow M_\infty \qquad (n \to +\infty)$$

(4) *If* $\alpha < 1$*, then there exists a random variable* S_∞ *with values in* $(0, +\infty)$ *such that the rescaled partial sums converge in law to* S_∞ *:*
$$\frac{1}{h(n^d)} \sum_{x \in B_n} \tau_x \Rightarrow S_\infty \qquad (n \to +\infty)$$

Proof. For the first statement, it is a consequence of the law of large numbers if $\alpha > 1$, otherwise it is an application of [Pet, Theorem 6.9]. For the second one, it comes again from the law of large numbers if $\alpha > 1$. Otherwise, observe that the sum is larger than the maximum of its terms, and
$$\mathbb{P}\left[\max_{x \in B_n} \tau_x \leqslant M n^{d/\alpha - \varepsilon}\right] = (1 - F(M n^{d/\alpha-\varepsilon}))^{(2n+1)^d}$$
Using the properties of F (see (1.2)), we see that the latter is the general term of a convergent series, and we can apply the Borel-Cantelli lemma. Now the convergence of the rescaled maxima is given in [Fel, Section VIII.8] or [Res, Proposition 1.11]. For the convergence of the partial sums, see [Fel, Section XVII.5]. □

Apart from this introduction, the paper is divided into four sections and an Appendix. In Section 2, we use the variational characterization to get bounds on λ_n^0 and λ_n that are sharp when $\alpha \leqslant 1$ or $d = 1$. We had to work harder to find a good lower bound when $d \geqslant 2$ and $\alpha > 1$. We introduce in Section 3 a time change of (X_t) for which the uniform measure is reversible. For this new walk and when $d \geqslant 3$, we can use results proved in the Appendix, giving upper bounds on the Green function, to estimate the occupation time of B_n by the initial random walk via a moments computation. For the two-dimensional case, the Green function cannot be used but we can modify the former proof using directly the estimates on the transition probabilities given in the Appendix. In Section 4, upper bounds for λ_n are computed.

Let us see how to deduce part (1) of Theorem 1.1 from the rest of the paper. Part (2) of Theorem 2.3 gives an upper bound on the exponent of the principal eigenvalue, that needs to be improved when $d \geqslant 3$ and $\alpha > 1$. This is done by Theorem 3.7. Now for the associated lower bounds on the exponent of the principal eigenvalue, they come from Theorem 4.1 and part (2) of Proposition 1.2 if $d = 1$; from part (2) of Theorem 4.2 and Theorem 4.5 if $d \geqslant 2$.

Concerning part (2) of Theorem 1.1, if $d = 1$ and $\alpha > 1$, the lower bound comes from part (3) of Theorem 2.3. If $d = 2$ and $\alpha > 1$, the lower bound is obtained in Theorem 3.13. If $d \geqslant 3$ and $\alpha > d/2$, the lower bound is given by part (2) of Theorem 3.7. In any case, Theorem 4.5 gives the desired upper bound on λ_n.

Finally, for parts (3) and (4) of Theorem 1.1, part (1) of Theorem 2.3 gives the desired result for λ_n^0 as well as a lower bound on λ_n. In dimension one, the upper estimate on λ_n is given by Theorem 4.1 and part (4) of Proposition 1.2, while if $d \geqslant 2$, it comes from part (1) of Theorem 4.2 together with part (3) of Proposition 1.2.

We show in Section 5 that the distinguished path method (see e.g. [SC97, Theorem 3.2.3]), that proved efficient in [FM06, Section 3] for random walks among random conductances, is bound to give an extra 1 in the exponent when $d \geqslant 2$ (for the one-dimensional case, [Chen, Section 3.7] proves that the method is sharp). The Appendix (Section 6) is independent from the rest of the paper and provides Gaussian-like upper estimates on the transition probabilities of symmetric random walks with transition rates bounded from below. The strategy relies on Nash inequalities and an argument that dates back to E.B. Davies [Dav87], adapted to our context.

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Notations. The operator $\mathcal{L}_n$ is self-adjoint for the scalar product $(\cdot,\cdot)$ defined by :

$$(f,g) = \sum f(x)g(x)\tau_x$$

We write $L^2(B_n)$ for the set of functions that vanish outside B_n (equipped with the former scalar product). For two points $x, y \in \mathbb{Z}^d$, we write $x \sim y$ when they are neighbours (that is, when $\|x-y\| = 1$). We define the Dirichlet form associated to $\mathcal{L}$:

$$\begin{aligned}
\mathcal{E}(f,g) = (-\mathcal{L}f,g) &= \sum_{\substack{x,y\in\mathbb{Z}^d\\ x\sim y}} \tau_x^a\tau_y^a g(x)(f(x)-f(y)) \\
&= \sum_{\substack{x,y\in\mathbb{Z}^d\\ x\sim y}} \tau_x^a\tau_y^a g(y)(f(y)-f(x)) \\
&= \frac{1}{2}\sum_{\substack{x,y\in\mathbb{Z}^d\\ x\sim y}} \tau_x^a\tau_y^a (f(y)-f(x))(g(y)-g(x))
\end{aligned}$$

(taking the half-sum of the last two expressions), and $\mathcal{E}_0$ the Dirichlet form obtained when $a = 0$. We have :

$$\lambda_n = \inf_{\substack{f\in L^2(B_n)\\ f\neq 0}} \frac{\mathcal{E}(f,f)}{(f,f)} \tag{1.3}$$

Assumption 2 gives that $\mathcal{E}(f,f) \geqslant \mathcal{E}_0(f,f)$, so it is clear that

$$\lambda_n \geqslant \lambda_n^0 \tag{1.4}$$

We further need to define the boundary of B_n, as $\partial B_n = B_{n+1} \setminus B_n$. If K is some set, $|K|$ stands for its cardinal. We write $\mathbf{P}_x^\tau$ for the law of the process starting from site x (and $\mathbf{E}_x^\tau$ for the corresponding expectation).

The real number $C > 0$ represents a generic constant that need not be the same from an occurrence to another.

2. The variational formula

We will use here the variational characterization of λ_n^0 :

$$\lambda_n^0 = \inf_{\substack{f\in L^2(B_n)\\ f\neq 0}} \frac{\mathcal{E}_0(f,f)}{(f,f)} \tag{2.1}$$

We define

$$C_n = \inf\left\{\mathcal{E}_0(f,f) \mid f \in L^2(B_n), f(0) = 1\right\}$$

Noting that B_n is a finite set, one can see by a compacity argument that the infimum is reached for some function V_n. The behaviours of C_n and λ_n^0 are related in the following way.

Proposition 2.1. *For any n and any environment, we have :*

$$\frac{C_{2n}}{\sum_{x \in B_n} \tau_x} \leqslant \lambda_n^0$$

$$\lambda_{2n+1}^0 \leqslant \lambda_{2n}^0 \leqslant \frac{C_n}{\max_{B_n} \tau}$$

Proof. Considering the homogeneity of the quotient in (2.1), we can restrict the infimum to be taken over all f with $\|f\|_\infty = 1$. Let f be such a function, and $x_0 \in B_n$ such that $|f(x_0)| = 1$. Possibly changing f in $-f$, we can assume $f(x_0) = 1$. Noting that the function $g = f(\cdot + x_0)$ is in $L^2(B_{2n})$ and satisfies $g(0) = 1$, we have :

$$\mathcal{E}_0(f, f) = \mathcal{E}_0(g, g) \geqslant C_{2n}$$

On the other hand, as $\|f\|_\infty = 1$, we have :

$$(f, f) \leqslant \sum_{x \in B_n} \tau_x$$

and these lead to the first desired inequality.

The fact that $\lambda_{2n+1}^0 \leqslant \lambda_{2n}^0$ is clear from (2.1). Now let $x_1 \in B_n$ be such that $\max_{B_n} \tau = \tau_{x_1}$, and consider the function $h = V_n(\cdot - x_1) \in L^2(B_{2n})$. We get :

$$\mathcal{E}_0(h, h) = \mathcal{E}_0(V_n, V_n) = C_n$$

But note that $h(x_1) = 1$, therefore :

$$(h, h) \geqslant \tau_{x_1} = \max_{B_n} \tau$$

and we get the second inequality. □

We now precise the asymptotic behaviour of C_n.

Proposition 2.2. *If $d = 1$, then :*

$$C_n = \frac{2}{n+1}$$

If $d = 2$, then there exist k_1, k_2 such that for all n :

$$\frac{k_1}{\ln(n)} \leqslant C_n \leqslant \frac{k_2}{\ln(n)}$$

If $d \geqslant 3$, then C_n converges to a strictly positive number.

Proof. We can regard B_{n+1} as an electrical graph (see [LP, Chapter 2]), with each edge representing a resistance of value 1. One can see that V_n is harmonic on every point that is not 0 nor a point of ∂B_n. Thus it coincides with the potential on the electrical graph, with the constraints that $V_n(0) = 1$ and $V_{n|\partial B_n} = 0$. The number C_n is the effective conductance between 0 and ∂B_n. In dimension 1, a direct computation gives the result. If $d = 2$, then we can use [LP, Proposition 2.14]. In larger dimension, the simple random walk is transient, and therefore (see [LP, Theorem 2.3]) C_n converges to a strictly positive number. □

From this, we can deduce the following.

Theorem 2.3. (1) *If $\alpha < 1$, then for any $\varepsilon > 0$, there exist $\eta, M > 0$ such that for all n large enough :*

$$\mathbb{P}\left[\eta \leqslant \frac{h(n^d)}{C_n} \lambda_n^0 \leqslant M\right] \geqslant 1 - \varepsilon$$

$$\mathbb{P}\left[\eta \leqslant \frac{h(n^d)}{C_n} \lambda_n\right] \geqslant 1 - \varepsilon$$

(2) *For almost every environment, we have :*

$$\limsup_{n\to\infty} -\frac{\ln(\lambda_n)}{\ln(n)} \leqslant \left| \begin{array}{ll} \max\left(2, 1+\dfrac{1}{\alpha}\right) & \text{if } d=1 \\ \max\left(d, \dfrac{d}{\alpha}\right) & \text{if } d\geqslant 2 \end{array} \right.$$

(3) *If $\mathbb{E}[\tau_0]$ is finite, then for almost every environment and all n large enough :*

$$\lambda_n \geqslant \frac{C_{2n}}{(2n+1)^d(\mathbb{E}[\tau_0]+1)}$$

Proof. Note first that as given by equation (1.4), we have that $\lambda_n \geqslant \lambda_n^0$. The first part of the theorem is a consequence of Propositions 2.1, 2.2 and parts (3) and (4) of Proposition 1.2. For the second part, use part (1) of Proposition 1.2 instead. The last part is an application of the law of large numbers. □

As far as lower bounds are concerned, parts (3) and (4) of Theorem 1.1 are now obtained. However, part (1) of Theorem 1.1 is proved only for $d \leqslant 2$ or $\alpha \leqslant 1$, and part (2) only for $d = 1$. The following section provides the missing lower bounds.

3. Exit times and time-changed random walk

This section aims at finding good lower bounds for λ_n when $d \geqslant 2$ and $\alpha > 1$. To do so, we will use the exit times T_n from B_n :

$$T_n = \inf\{t \geqslant 0 : X_t \notin B_n\}$$

The principal eigenvalue and the exit time from B_n are indeed related by the following (general) result :

Proposition 3.1. *For any environment τ, any $n \in \mathbb{N}$ and $t \geqslant 0$, we have*

$$e^{-t\lambda_n} \leqslant \sup_{x\in B_n} \mathbf{P}_x^\tau[T_n > t] \leqslant \frac{\sup_{x\in B_n} \mathbf{E}_x^\tau[T_n]}{t}$$

Proof. Let ψ_n be the eigenfunction associated with the principal eigenvalue λ_n such that $\sup \psi_n = 1$.

$$\mathbf{E}_x^\tau[\psi_n(X_t)\mathbf{1}_{\{T_n>t\}}] = e^{-t\lambda_n}\psi_n(x)$$

Choosing $x \in B_n$ such that $\psi_n(x) = 1$, we have :

$$\mathbf{P}_x^\tau[T_n > t] \geqslant \mathbf{E}_x^\tau[\psi_n(X_t)\mathbf{1}_{\{T_n>t\}}] = e^{-t\lambda_n}$$

□

So our objective is to find a sharp upper bound for $\sup_{x\in B_n} \mathbf{E}_x^\tau[T_n]$.

We introduce an auxiliary random walk. Let (J_i) be the jump instants of the walk X, and define (J_i') by $J_0' = 0$ and $J_{i+1}' - J_i' = (J_{i+1} - J_i)/\tau_{X_{J_i}}$. let $\hat{X}$ be the random walk defined by :

$$\hat{X}_t = X_{J_i} \quad \text{for } i \text{ such that } J_i' \leqslant t < J_{i+1}'$$

The walk $\hat{X}$ is a time change of X. More precisely, if $A(t) = \int_0^t \tau_{\hat{X}_s} \mathrm{d}s$, we have :

$$X_{A(t)} = \hat{X}_t$$

The jump rate of the walk $\hat{X}$ from site x to a neighbour y is $\tau_x^a \tau_y^a$. The advantage of considering $\hat{X}$ instead of X is that it has symmetric jump rates that are bounded away from 0. In this context and as shown in the Appendix, one can get good upper bounds on the transition probabilities (and Green function when $d \geqslant 3$) of $\hat{X}$.

To give the ideas, consider the case when $d \geqslant 3$. Let $\hat{G}(\cdot,\cdot)$ be the Green function associated to $\hat{X}$ ($\hat{G}(x, y)$ is the expected time spent on y by the walk starting from x). We can bound the expected time spent before exiting the box by the total time spent in the box. We constructed $\hat{X}$ so that the time spent by X at some site y is τ_y times the time spent by $\hat{X}$ at this same site, so we have :

$$\mathbf{E}^{\tau}_x[T_n] \leqslant \sum_{y \in B_n} \hat{G}(x,y)\tau_y \tag{3.1}$$

We will see that the expectation of this sum behaves like n^2 (we assume $\alpha > 1$), and that the probability to be far from the expectation by $n^{d/\alpha}$ is of order n^{-d}. To estimate theses fluctuations, our method will be to compute moments after a truncation and centring of the τ_x.

We need to introduce some definitions before stating the results proved in the Appendix. If x and y are neighbours, let e be the edge between x and y and define the *weight* of e by :

$$Q(e) = \frac{1}{\sqrt{\tau_x^a \tau_y^a}}$$

For a path γ, we define its *natural length* by :

$$|\gamma|_\tau = \sum_{e \in \gamma} Q(e)$$

and for any $x, y \in \mathbb{Z}^d$, the *natural distance* between x and y by :

$$\Delta_\tau(x,y) = \inf\{|\gamma|_\tau,\ \gamma \text{ simple path from } x \text{ to } y\} \tag{3.2}$$

Proposition 3.2. *If $d \geqslant 3$, then there exists C_3 such that for any environment and any $x, y \in \mathbb{Z}^d$:*

$$\hat{G}(x,y) \leqslant \frac{C_3}{(1+\Delta_\tau(x,y))^{d-2}}$$

Proof. See part (3) of Theorem 6.1. □

3.1. **The natural distance.** The first thing we need to do to make the former result effective is to compare Δ_τ with the Euclidian distance $\|\cdot\|$. To do so, we will use the fact that the law of the environment $\mathbb{P}$ is a product measure, in a percolation-like argument.

Theorem 3.3. (1) *For any environment and any $x \in \mathbb{Z}^d$:*

$$\Delta_\tau(0,x) \leqslant 2\|x\|$$

(2) *There exist $c, M_0, \kappa > 0$ such that :*

$$\mathbb{P}\left[\Delta_\tau(0,x) < c\|x\|\right] \leqslant M_0 e^{-2\kappa\|x\|}$$

(3) *Let $\mathcal{A}_n$ be the event : "$\forall x \notin B_n : \Delta_\tau(0,x) \geqslant c\|x\|$". There exists M_1 such that for all n :*

$$\mathbb{P}[\mathcal{A}_n] \geqslant 1 - M_1 e^{-\kappa n}$$

Proof. We denote by $\|\cdot\|_1$ the graph distance on $\mathbb{Z}^d$.

For any edge e, we have $Q(e) \leqslant 1$ so it is clear that $\Delta_\tau(0,x) \leqslant \|x\|_1$, and part (1) follows comparing $\|\cdot\|_1$ with $\|\cdot\|$.

Giving ourselves $\eta \in (0,1)$, we say that an edge is *heavy* if $Q(e) \geqslant \eta$. We write A_e for the event "the edge e is heavy", and $p = \mathbb{P}[A_e]$. Note that A_e is independent of $(A_{e'})$ for all edges e' that are not adjacent to the edge e (and there are $4d-1$ adjacent edges). Let γ be a simple path. We bound from above the probability that γ's natural length is small using Markov inequality : for any $k \in \mathbb{N}$ and any $\lambda \geqslant 0$, we have :

$$\mathbb{P}[\#\text{ heavy edges along } \gamma \leqslant k] = \mathbb{P}\left[\sum_{e\in\gamma} \mathbf{1}_{A_e} \leqslant k\right] \leqslant e^{\lambda k}\mathbb{E}\left[e^{-\lambda \sum_{e\in\gamma}\mathbf{1}_{A_e}}\right]$$

Thanks to the former remark, we can extract from γ a subset γ' of edges such that $(A_e)_{e\in\gamma'}$ are independent random variables and $(4d-1)|\gamma'| \geqslant |\gamma|$. It comes :

$$\mathbb{P}\left[\sum_{e\in\gamma} \mathbf{1}_{A_e} \leqslant k\right] \leqslant e^{\lambda k}(pe^{-\lambda}+1-p)^{|\gamma|/(4d-1)} \tag{3.3}$$

We can now evaluate the probability for $\Delta_\tau(0,x)$ to be much smaller than $\|x\|_1$. Take $c>0$, it comes :

$$\begin{aligned}\mathbb{P}\left[\Delta_\tau(0,x) < c\|x\|_1\right] &= \mathbb{P}[\exists\gamma \text{ from } 0 \text{ to } x \text{ s.t. } |\gamma|_\tau < c\|x\|_1] \\ &\leqslant \sum_{\gamma:0\to x} \mathbb{P}[|\gamma|_\tau < c\|x\|_1]\end{aligned}$$

For a given (simple) path γ, note that $|\gamma|_\tau < c\|x\|_1$ implies that the number of heavy edges along γ is smaller than $c\|x\|_1/\eta$. Bounding the number of paths of length l by $(2d)^l$ and using (3.3), it comes :

$$\mathbb{P}\left[\Delta_\tau(0,x) < c\|x\|_1\right] \leqslant \sum_{l=\|x\|_1}^{+\infty} (2d)^l e^{\lambda c\|x\|_1/\eta}(pe^{-\lambda}+1-p)^{l/(4d-1)}$$

Writing $B(\lambda,p) = 2d(pe^{-\lambda}+1-p)^{1/(4d-1)}$, we have that if $B(\lambda,p)<1$, then :

$$\mathbb{P}\left[\Delta_\tau(0,x) < c\|x\|_1\right] \leqslant \frac{(e^{\lambda c/\eta}B(\lambda,p))^{\|x\|_1}}{1-B(\lambda,p)}$$

and this inequality would give an exponential decay whenever $e^{\lambda c/\eta}B(\lambda,p)$ is strictly smaller than 1. Choosing η close enough to 0, we can make p as close to 1 as desired, and taking also λ large enough, we can ensure $B(\lambda,p)<1$. Then taking c small enough, the last required condition holds. The second part of the theorem is proved using the equivalence between $\|\cdot\|_1$ and $\|\cdot\|$.

Now for the last part, note that :

$$1-\mathbb{P}[\mathcal{A}_n] \leqslant \sum_{x\in\mathbb{Z}^d\setminus B_n} M_0 e^{-2\kappa\|x\|} \leqslant e^{-\kappa n}\sum_{x\in\mathbb{Z}^d} M_0 e^{-\kappa\|x\|}$$

the last sum being finite, this proves the last part of the theorem. □

3.2. **In dimension three or more.** In this part, we treat the case $d\geqslant 3$. As announced before, we want now to control the fluctuations of the sum appearing in the right-hand side of inequality (3.1), using a moment method. To do so, we need to cut and centre the random variables (τ_x).

We first pick $\alpha'<\alpha$. Remember that $\mathbb{E}[\tau_0^{\alpha'}]$ is finite (and as we will see, it is the only property we need to make this part work).

We define the following truncation of τ_x :

$$\tilde\tau_{x,n} = \left|\begin{array}{ll}\tau_x & \text{if } \tau_x \leqslant n^{d/\alpha'} \\ 0 & \text{otherwise}\end{array}\right. \tag{3.4}$$

(observe that with high probability, we have $\tau_x = \tilde\tau_{x,n}$ for every $x\in B_n$), and let $\overline{\tau}_{x,n} = \tilde\tau_{x,n} - \mathbb{E}[\tilde\tau_{x,n}]$.

When $d\geqslant 4$, we can prove the following proposition, that roughly speaking says that fluctuations of order $n^{d/\alpha'}$ of the exit time from 0 occur with probability smaller than n^{-d}. But when $d=3$, our method no longer works for large α', so we restrict ourselves to $\alpha'\leqslant 2$. But as we will see in the proof of Theorem 3.7, this restriction is of no consequence for our purpose.

Proposition 3.4. *We assume $d \geqslant 4$ or $\alpha' \leqslant 2$. For any $\beta > d/\alpha'$, there exist $\delta, C > 0$ such that for all n :*

$$\mathbb{P}\left[\left|\sum_{x \in B_n} \frac{\overline{\tau}_{x,n}}{(1+c\|x\|)^{d-2}}\right| > n^\beta\right] \leqslant \frac{C}{n^{d+\delta}}$$

Proof. Let m be an integer. We have :

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{x \in B_n} \frac{\overline{\tau}_{x,n}}{(1+c\|x\|)^{d-2}}\right)^{2m}\right] \\
&\quad = \sum_{x_1,\ldots,x_{2m}} \frac{1}{(1+c\|x_1\|)^{d-2}} \cdots \frac{1}{(1+c\|x_{2m}\|)^{d-2}} \mathbb{E}[\overline{\tau}_{x_1,n} \cdots \overline{\tau}_{x_{2m},n}] \\
&\quad = \sum_{k=1}^{m} \sum_{\substack{e_1+\cdots+e_k=2m \\ e_i \geqslant 2}} C_{e_1,\ldots,e_k} \sum_{\substack{y_1,\ldots,y_k \\ y_i \neq y_j}} \prod_{i=1}^{k} \frac{1}{(1+c\|y_i\|)^{e_i(d-2)}} \mathbb{E}[\overline{\tau}^{e_i}_{y_i,n}] \qquad (3.5) \\
&\quad \leqslant C(m) \sum_{k=1}^{m} \sum_{\substack{e_1+\cdots+e_k=2m \\ e_i \geqslant 2}} \underbrace{\prod_{i=1}^{k} \sum_{x \in B_n} \frac{1}{(1+c\|x\|)^{e_i(d-2)}} |\mathbb{E}[\overline{\tau}^{e_i}_{0,n}]|}_{=:\Pi^n_{e_1,\ldots,e_k}}
\end{aligned}$$

where, to get the second equality, we chose to decompose $x_1, \ldots, x_{2m}$ the following way : let k be the cardinal of $\{x_1, \ldots, x_{2m}\}$. We have $\{x_1, \ldots, x_{2m}\} = \{y_1, \ldots, y_k\}$. Then e_i represents then number of occurences of y_i in $x_1, \ldots, x_{2m}$. We then use the fact that the random variables $(\overline{\tau}_{x,n})_{x \in \mathbb{Z}^d}$ are independent to split the expectation in product form. Note that as $\overline{\tau}_{x,n}$ is a centred random variable, the cases when $e_i = 1$ for some i do not contribute to the sum, so it is enough to consider cases when $e_i \geqslant 2$ (and this implies $k \leqslant m$). It is a nice combinatorics exercise to check that $C_{e_1,\ldots,e_k}$ is the multinomial coefficient associated with $(e_1, \ldots, e_k)$ divided by $k!$, but the important fact is that this term does not depend on n.

Comparing with an integral, we can see that $\sum_{x \in B_n} (1+c\|x\|)^{-e_i(d-2)}$ is bounded by :

$$\left| \begin{array}{ll} C\ln(n) & \text{if } d \geqslant 4 \text{ or } e_i \geqslant 3 \\ Cn & \text{in any case} \end{array} \right.$$

On the other hand, $|\mathbb{E}[\overline{\tau}^{e_i}_{0,n}]|$ is bounded when n goes to infinity if $e_i \leqslant \alpha'$, and otherwise

$$|\mathbb{E}[\overline{\tau}^{e_i}_{0,n}]| \leqslant \mathbb{E}[|\overline{\tau}_{0,n}|^{(e_i-\alpha')+\alpha'}] \leqslant (n^{d/\alpha'})^{e_i-\alpha'} \mathbb{E}[|\overline{\tau}_{0,n}|^{\alpha'}] \leqslant C n^{e_i d/\alpha' - d} \qquad (3.6)$$

We first treat the case $d \geqslant 4$. We choose m as the smallest integer larger than (or equal to) $\alpha'/2$. All the $\Pi^n_{e_1,\ldots,e_k}$ are bounded by $C\ln(n)^m$ when n goes to infinity except :

$$\Pi^n_{2m} \leqslant C\ln(n) n^{2md/\alpha' - d}$$

It comes, using Markov inequality, that there exists C such that for all n :

$$\mathbb{P}\left[\left|\sum_{x \in B_n} (1+c\|x\|)^{2-d} \overline{\tau}_{x,n}\right| > n^\beta\right] \leqslant C n^{-d} \ln(n)^m n^{2m(d/\alpha' - \beta)}$$

which proves the desired result. We are left with the case when $d = 3$ and $\alpha' \leqslant 2$. We choose $m = 2$ in (3.5) and get :

$$\Pi^n_{2,2} \leqslant C n^2 n^{12/\alpha' - 6} \quad \text{ and } \quad \Pi^n_4 \leqslant C\ln(n) n^{12/\alpha' - 3}$$

and it comes that :

$$\mathbb{P}\left[\left|\sum_{x\in B_n}(1+\|x\|)^{2-d}\overline{\tau}_x\right| > n^\beta\right] \leqslant Cn^{-3}\ln(n)n^{4(3/\alpha'-\beta)}$$

which ends the proof of the proposition. □

The next step is to lift this estimate to the sum of $\hat{G}(0,x)\tilde{\tau}_{x,n}$.

Proposition 3.5. *We assume $d \geqslant 4$ or $\alpha' \leqslant 2$. There exists M such that for any $\beta > d/\alpha'$, there exist $\delta, C > 0$ such that for all n :*

$$\mathbb{P}\left[\sum_{x\in B_n}\hat{G}(0,x)\tilde{\tau}_{x,n} > Mn^2 + n^\beta\right] \leqslant \frac{C}{n^{d+\delta}}$$

Proof. We begin introducing the estimate given by Proposition 3.2 :

$$\sum_{x\in B_n}\hat{G}(0,x)\tilde{\tau}_{x,n} \leqslant C_3\sum_{x\in B_n}\frac{\tilde{\tau}_{x,n}}{(1+\Delta_\tau(0,x))^{d-2}} \tag{3.7}$$

Using part (3) of Proposition 3.3, we choose C' such that :

$$\mathbb{P}[\mathcal{A}_{C'\ln(n)}] \geqslant 1 - \frac{C}{n^{d+1}} \tag{3.8}$$

Now conditionally on $\mathcal{A}_{C'\ln(n)}$, the sum on the right in inequality (3.7) is bounded, up to a constant, by :

$$\sum_{x\in B_{C'\ln(n)}}\tilde{\tau}_{x,n} + \sum_{x\in B_n}\frac{\tilde{\tau}_{x,n}}{(1+c\|x\|)^{d-2}}$$

Note that for the first term, we have :

$$\sum_{x\in B_{C'\ln(n)}}\tilde{\tau}_{x,n} \leqslant C\ln(n)^d n^{d/\alpha'}$$

Comparing with an integral, there exists $\tilde{M}$ such that for all n :

$$\sum_{x\in B_n}\frac{C_3}{(1+c\|x\|)^{d-2}} \leqslant \tilde{M}n^2$$

And we have :

$$\begin{aligned}
&\mathbb{P}\left[\sum_{x\in B_n}\hat{G}(0,x)\tilde{\tau}_{x,n} > \tilde{M}\mathbb{E}[\tau_0]n^2 + n^\beta, \mathcal{A}_{C'\ln(n)}\right] \\
&\qquad\leqslant \mathbb{P}\left[C_3\sum_{x\in B_n}\frac{\tilde{\tau}_{x,n}}{(1+c\|x\|)^{d-2}} > \tilde{M}\mathbb{E}[\tau_0]n^2 + n^\beta - C\ln(n)^d n^{d/\alpha'}\right] \\
&\qquad\leqslant \mathbb{P}\left[C_3\sum_{x\in B_n}\frac{\overline{\tau}_{x,n}}{(1+c\|x\|)^{d-2}} > n^\beta - C\ln(n)^d n^{d/\alpha'}\right]
\end{aligned}$$

and we apply Proposition 3.4. What happens on the complement of $\mathcal{A}_{C'\ln(n)}$ is controlled by equation (3.8). □

We can now carry this result back to $\sup_{x\in B_n}\mathbf{E}^\tau_x[T_n]$.

Proposition 3.6. *We assume $d \geqslant 4$ or $\alpha' \leqslant 2$. There exists M' such that for any $\beta > d/\alpha'$, almost every environment and n large enough :*

$$\sup_{x\in B_n}\mathbf{E}^\tau_x[T_n] \leqslant n^\beta + M'n^2$$

Proof. We first need to relate $\mathbf{E}_x^\tau[T_n]$ with the estimates proved before (which concern only $\mathbf{E}_0^\tau[T_n]$). Let T_n^x be the exit time from $x + B_n$. Since for any $x \in B_n$, we have $B_n \subseteq x + B_{2n}$, it comes that almost surely $T_n \leqslant T_{2n}^x$, so $\mathbf{E}_x^\tau[T_n] \leqslant \mathbf{E}_x^\tau[T_{2n}^x]$, the latter having same law as $\mathbf{E}_0^\tau[T_{2n}]$ under $\mathbb{P}$.

Let $M' > 0$ and let i be an integer. We consider :

$$\mathbb{P}\left[\sup_{n \geqslant 2^i} \frac{\sup_{x \in B_n} \mathbf{E}_x^\tau[T_n]}{n^\beta + M'n^2} > 1\right] \leqslant \sum_{j=i}^{\infty} \mathbb{P}\left[\sup_{2^j \leqslant n < 2^{j+1}} \frac{\sup_{x \in B_n} \mathbf{E}_x^\tau[T_{2n}^x]}{n^\beta + M'n^2} > 1\right] \tag{3.9}$$

We bound the general term of this series by

$$\mathbb{P}\left[\sup_{x \in B_{2^{j+1}}} \mathbf{E}_x^\tau[T_{2^{j+2}}^x] > 2^{j\beta} + M'2^{2j}\right]$$

which we bound by $A_j + |B_{2^{j+1}}|A_j'$, where :

$$A_j = \mathbb{P}\left[\exists x \in B_{2^{j+2}} : \tau_x > 2^{(j+2)d/\alpha'}\right] \tag{3.10}$$

$$A_j' = \mathbb{P}\left[\sum_{x \in B_{2^{j+2}}} \hat{G}(0,x)\tilde{\tau}_{x,2^{j+2}} > 2^{j\beta} + M'2^{2j}\right]$$

We first estimate A_j. Take α'' such that $\alpha' < \alpha'' < \alpha$. It comes from assumption 1' (see (1.2)) that for all y large enough :

$$\mathbb{P}[\tau_0 > y] \leqslant y^{-\alpha''}$$

One gets that for j large enough :

$$A_j \leqslant 1 - \left(1 - 2^{-jd\alpha''/\alpha'}\right)^{|B_{2^{j+2}}|} = 1 - \exp\left(|B_{2^{j+2}}|2^{-jd\alpha''/\alpha'}(1 + o(1))\right)$$

which is the general term of a convergent series.

Now for A_j', using Proposition 3.5, we see that choosing $M' = 16M$, the term $|B_{2^{j+1}}|A_j'$ is bounded by $C2^{-j\delta}$ for some $\delta > 0$. Therefore, the series in the right-hand side of 3.9 converges (and tends to 0 when i goes to infinity), which proves the proposition. □

We now have everything in hand to conclude !

Theorem 3.7. (1) *If $d/\alpha \geqslant 2$, then for almost every environment :*

$$\limsup_{n \to \infty} -\frac{\ln(\lambda_n)}{\ln(n)} \leqslant \frac{d}{\alpha}$$

(2) *If $d/\alpha < 2$, then there exists C such that for almost every environment and all n large enough :*

$$\lambda_n \geqslant \frac{C}{n^2}$$

Proof. If $d \geqslant 4$ or $\alpha \leqslant 2$, it is a consequence of Proposition 3.6 together with Proposition 3.1 (making α' tend to α). Now if $d = 3$ and $\alpha > 2$, then in particular $\mathbb{E}[\tau_0^2]$ is finite, so we can choose $\alpha' = 2$, and as $d/2 < 2$, part (2) of the theorem still holds. □

3.3. **The two-dimensional case.** In two dimensions, the lower bound given by part (3) of Theorem 2.3 is of the following form :

$$\lambda_n \geqslant \frac{C}{\ln(n)n^2}$$

We will now show that, provided $\alpha > 1$, the $\ln(n)$ term appearing before is irrelevant, i.e. that in fact :

$$\lambda_n \geqslant \frac{C}{n^2}$$

The technique we used in Section 3.2 of computing the Green function of $\hat{X}$ cannot directly apply in this setting, so we will consider the expected time spent by $\hat{X}$ at some site $x \in B_n$ before exiting the box, say $\hat{G}_n(0,x)$ if the walk is starting from 0, and study

$$\sum_{x \in B_n} \hat{G}_n(0,x)\tau_x = \mathbf{E}_0^\tau[T_n]$$

where

$$\hat{T}_n = \inf\{t : \hat{X}_t \notin B_n\} \quad \text{and} \quad \hat{G}_n(x,y) = \mathbf{E}_x^\tau\left[\int_0^{\hat{T}_n} \mathbf{1}_{\{\hat{X}_t = y\}} \mathrm{d}t\right]$$

The problem we face is the dependence between the random variables $\hat{G}_n(0,x)$ and τ_x. When $d \geqslant 3$, it was enough to use the Gaussian-like upper bounds proved in the Appendix together with some "universal" control on the natural distance, but now $\hat{T}_n$ also comes into play. The following proposition gives a control on $\hat{T}_n$ that holds uniformly over the environment τ.

Proposition 3.8. *There exists $\hat{C} > 0$ such that for almost any τ and any $x \in B_n$:*

$$\mathbf{P}_x^\tau\left[\hat{T}_n > \frac{\hat{C}n^2}{4}\right] \leqslant \frac{1}{3}$$

Remark. Numerical factors are chosen for practical purposes, but the result would hold with 1/3 replaced by any strictly positive constant.

Proof. From Theorem 6.1 (or the comments following Proposition 6.2), we know that there exists C_2 such that for any $x, y \in \mathbb{Z}^2$, any $t > 0$ and almost any τ :

$$\mathbf{P}_x^\tau[\hat{X}_t = y] \leqslant \frac{C_2}{t} \tag{3.11}$$

It then comes that

$$\mathbf{P}_x^\tau[\hat{X}_{3C_2|B_n|} \in B_n] = \sum_{y \in B_n} \mathbf{P}_x^\tau[\hat{X}_{3C_2|B_n|} = y] \leqslant \frac{1}{3}$$

which gives the desired result. □

Now conditionally on $\hat{T}_n \leqslant \hat{C}n^2$ and using once again equation (3.11), we have :

$$\hat{G}_n(0,x) \leqslant 1 + \int_1^{\hat{C}n^2} \frac{C_2}{t} \leqslant 1 + \ln(\hat{C}n^2) \tag{3.12}$$

Such an inequality is good for x close to 0, but too weak in general, and we need to use more refined estimates. If we assume that $\hat{T}_n \leqslant \hat{C}n^2$ together with $\Delta_\tau(0,x) \geqslant$

$c\|x\|$ (note that $\Delta_\tau(0,x) \leqslant 2\|x\|$ is always true, as given by part (1) of Theorem 3.3), we get, using Theorem 6.1 :

$$\begin{aligned}
\hat{G}_n(0,x) &\leqslant \int_0^{\hat{C}n^2} \mathbf{P}_0^\tau[\hat{X}_t = x]\mathrm{d}t \\
&\leqslant 1 + \int_1^{\max(1,\frac{\|x\|}{8e^2})} \frac{C_2 e^{-3\Delta_\tau(0,x)/4}}{t}\,\mathrm{d}t + \int_1^{\hat{C}n^2} \frac{C_2 e^{-3\Delta_\tau(0,x)^2/64e^2 t}}{t}\,\mathrm{d}t \qquad (3.13)\\
&\leqslant 1 + C_2 \ln_+\left(\frac{\|x\|}{8e^2}\right) e^{-3c\|x\|/4} + \int_1^{\hat{C}n^2} \frac{C_2 e^{-3c\|x\|^2/64e^2 t}}{t}\mathrm{d}t =: g_n(x)
\end{aligned}$$

where $\ln_+(x) := \max(0, \ln(x))$. The great advantage of this estimate is that g_n is deterministic. We can now follow the same procedure as in Section 3.2 : we will prove that the random variable $\sum_{x\in B_n} g_n(x)\tau_x$ has mean of order n^2, then control its fluctuations, and then carry these results back to the exit times T_n. We begin with the statement concerning the mean, which is purely deterministic.

Proposition 3.9. *There exists M such that for all n large enough :*

$$\sum_{x\in B_n} g_n(x) \leqslant Mn^2 \qquad (3.14)$$

Proof. We recall the definition of g_n :

$$g_n(x) = 1 + C_2 \ln_+\left(\frac{\|x\|}{8e^2}\right) e^{-3c\|x\|/4} + \int_1^{\hat{C}n^2} \frac{C_2 e^{-3c\|x\|^2/64e^2 t}}{t}\mathrm{d}t$$

The "1" term gives an n^2 contribution in the sum (3.14). As $\ln_+\left(\frac{\|x\|}{8e^2}\right) e^{-3c\|x\|/4}$ tends to 0 when $\|x\|$ tends to infinity, its contribution in the sum is negligible. Now for the last part, note that if $x = (x_1, x_2) \in \{1, \ldots, n\}^2$, then :

$$e^{-3c\|x\|^2/64e^2 t} \leqslant \int_{x_1-1}^{x_1}\int_{x_2-1}^{x_2} e^{-3c(u^2+v^2)/64e^2 t}\,\mathrm{d}u\mathrm{d}v$$

We get :

$$\begin{aligned}
\sum_{x\in B_n} e^{-3c\|x\|^2/64e^2 t} &\leqslant 4n+1+4\sum_{x\in\{1,\ldots,n\}^2} e^{-3c\|x\|^2/64e^2 t} \\
&\leqslant 4n+1+4\int_0^\infty\int_0^\infty e^{-3c(u^2+v^2)/64e^2 t}\,\mathrm{d}u\mathrm{d}v \\
&\leqslant 4n+1+\frac{64\pi e^2 t}{3c}
\end{aligned}$$

From this we deduce that :

$$\sum_{x\in B_n}\int_1^{\hat{C}n^2} \frac{e^{-3c\|x\|^2/64e^2 t}}{t}\mathrm{d}t \leqslant (4n+1)\ln(\hat{C}n^2) + \frac{64\pi e^2 \hat{C}}{3c}n^2$$

which proves the proposition. □

Now let α' be such that $1 < \alpha' < \alpha$ and $\alpha' \leqslant 2$. Remember that $\mathbb{E}[\tau_0^{\alpha'}]$ is finite. Define the truncation $\tilde{\tau}$ as in (3.4) (with $d = 2$), and the centred $\overline{\tau}_{x,n} = \tilde{\tau}_{x,n} - \mathbb{E}[\tilde{\tau}_{x,n}]$.

Proposition 3.10. *For any $\beta > 2/\alpha'$, there exists $\delta, C > 0$ such that for all n :*

$$\mathbb{P}\left[\left|\sum_{x\in B_n} g_n(x)\overline{\tau}_{x,n}\right| > n^\beta\right] \leqslant \frac{C}{n^{2+\delta}}$$

Proof. Note that $g_n(\cdot) \leqslant C\ln(n)$. We proceed through the computation of moments as in (3.5) to get, for any integer m :

$$\mathbb{E}\left[\left(\sum_{x\in B_n} g_n(x)\overline{\tau}_{x,n}\right)^{2m}\right] \leqslant C\ln(n)^{2m}\sum_{k=1}^{m}\sum_{\substack{e_1+\dots+e_k=2m\\ e_i\geqslant 2}} \underbrace{n^{2k}\prod_{i=1}^{k}|\mathbb{E}[\overline{\tau}_{0,n}^{e_i}]|}_{=:\Pi^n_{e_1,\dots,e_k}}$$

Recall that (from equation (3.6) and the fact that $\alpha' \leqslant 2$)

$$|\mathbb{E}[\overline{\tau}_{0,n}^{e_i}]| \leqslant Cn^{2e_i/\alpha'-2}$$

We obtain, for any sequence $e_1,\dots,e_k$ such that $e_1+\dots+e_k = 2m$:

$$\Pi^n_{e_1,\dots,e_k} \leqslant Cn^{4m/\alpha'}$$

Now we choose m large enough so that :

$$\left(\frac{4}{\alpha'} - 2\beta\right) m < -2$$

and apply Markov inequality. □

We now lift this estimate to the sum $\sum_{x\in B_n}\hat{G}_n(0,x)\tilde{\tau}_{x,n}$. We write $\mathcal{B}_n$ for the event $\hat{T}_n \leqslant \hat{C}n^2$.

Proposition 3.11. *There exist $M',\delta,C>0$ such that for all n :*

$$\mathbb{P}\left[\sum_{x\in B_n}\hat{G}_n(0,x)\tilde{\tau}_{x,n} > M'n^2, \mathcal{B}_n\right] \leqslant \frac{C}{n^{2+\delta}}$$

Proof. First, using part (3) of Proposition 3.3, we choose C' such that :

$$\mathbb{P}[\mathcal{A}_{C'\ln(n)}] \geqslant 1 - \frac{C}{n^3} \tag{3.15}$$

On the events $\mathcal{A}_{C'\ln(n)}$ and $\mathcal{B}_n$, we get (see (3.12) and (3.13)) :

$$\sum_{x\in B_n}\hat{G}_n(0,x)\tilde{\tau}_{x,n} \leqslant C\ln(n)\sum_{x\in B_{C'\ln(n)}}\tilde{\tau}_{x,n} + \sum_{x\in B_n} g_n(x)\tilde{\tau}_{x,n}$$

The first term is bounded by $C\ln(n)^3 n^{2/\alpha'}$. We choose $M' = M\mathbb{E}[\tau_0]+1$ with M given by Proposition 3.9. It comes :

$$\mathbb{P}\left[\sum_{x\in B_n}\hat{G}_n(0,x)\tilde{\tau}_{x,n} > M'n^2, \mathcal{A}_{C'\ln(n)}, \mathcal{B}_n\right]$$
$$\leqslant \mathbb{P}\left[C\ln(n)^3 n^{2/\alpha'} + \sum_{x\in B_n} g_n(x)\overline{\tau}_{x,n} > n^2\right]$$

and we conclude using Proposition 3.10 (recalling that $2/\alpha' < 1$). The probability of non-occurrence of $\mathcal{A}_{C'\ln(n)}$ is controlled by equation (3.15). □

We then carry this result back to the exit times :

Proposition 3.12. *There exists M'' such that for almost every environment and for n large enough, we have :*

$$\sup_{x\in B_n}\mathbf{E}^\tau_x\left[T_n \mathbf{1}_{\{\hat{T}_{4n}\leqslant 4\hat{C}n^2\}}\right] \leqslant M''n^2$$

Proof. Let $\hat{T}_n^x$ be the time spent by $\hat{X}$ before exiting $x + B_n$. We have :

$$\mathbf{E}_x^\tau \left[T_n \mathbf{1}_{\{\hat{T}_{4n} \leqslant 4\hat{C}n^2\}} \right] \leqslant \mathbf{E}_x^\tau \left[T_{2n}^x \mathbf{1}_{\{\hat{T}_{2n}^x \leqslant 4\hat{C}n^2\}} \right]$$

the latter having same law under $\mathbb{P}$ as $\mathbf{E}_0^\tau \left[T_{2n} \mathbf{1}_{\mathcal{B}_{2n}} \right]$. Let $M' > 0$ and let i be an integer. We consider :

$$\begin{aligned}
(3.16) \quad \mathbb{P} &\left[\sup_{n \geqslant 2^i} \frac{\sup_{x \in B_n} \mathbf{E}_x^\tau \left[T_n \mathbf{1}_{\{\hat{T}_{4n} \leqslant 4\hat{C}n^2\}} \right]}{M'' n^2} > 1 \right] \\
&\leqslant \sum_{j=i}^{\infty} \mathbb{P} \left[\sup_{2^j \leqslant n < 2^{j+1}} \frac{\sup_{x \in B_n} \mathbf{E}_x^\tau \left[T_{2n}^x \mathbf{1}_{\{\hat{T}_{2n}^x \leqslant 4\hat{C}n^2\}} \right]}{M'' n^2} > 1 \right]
\end{aligned}$$

We can bound the general term of this series by $A_j + |B_{2^{j+1}}| A'_j$ where :

$$A_j = \mathbb{P} \left[\exists x \in B_{2^{j+2}} : \tau_x > 2^{2(j+2)/\alpha'} \right]$$

$$A'_j = \mathbb{P} \left[\sum_{x \in B_{2^{j+2}}} \hat{G}_{2^{j+2}}(0, x) \tilde{\tau}_{x, 2^{j+2}} > M'' 2^{2j}, \mathcal{B}_{2^{j+2}} \right]$$

The first term, A_j, is the general term of a convergent series (it is the same as in (3.10)). By Proposition 3.11, we see that if we choose $M'' = 16M'$, then $|B_{2^{j+1}}| A'_j$ is bounded by $C 2^{-j\delta}$. Therefore, the series in the left-hand side of 3.16 converges (and tends to 0 when i tends to infinity), which proves the proposition. $\square$

We can now conclude :

Theorem 3.13. *If $d = 2$ and $\alpha > 1$, then there exists $C > 0$ such that for almost every environment and all n large enough :*

$$\lambda_n \geqslant \frac{C}{n^2}$$

Proof. Proposition 3.1 tells us that :

$$e^{-t\lambda_n} \leqslant \sup_{x \in B_n} \mathbf{P}_x^\tau [T_n > t]$$

We decompose this the following way :

$$\mathbf{P}_x^\tau [T_n > t] \leqslant \mathbf{P}_x^\tau [\hat{T}_{4n} > 4\hat{C}n^2] + \mathbf{P}_x^\tau [T_n > t, \hat{T}_{4n} \leqslant 4\hat{C}n^2]$$

The first term is smaller than $1/3$ as given by Proposition 3.8. As for the second term, Proposition 3.12 shows that for almost every environment and all n large enough :

$$\sup_{x \in B_n} \mathbf{P}_x^\tau [T_n > 3M'' n^2, \hat{T}_{4n} \leqslant 4\hat{C}n^2] \leqslant \frac{\sup_{x \in B_n} \mathbf{E}_x^\tau \left[T_n \mathbf{1}_{\{\hat{T}_{4n} \leqslant 4\hat{C}n^2\}} \right]}{3M'' n^2} \leqslant \frac{1}{3}$$

Combining the two leads to the fact that for almost every environment and all n large enough :

$$e^{-3M'' n^2 \lambda_n} \leqslant \frac{2}{3}$$

which proves the desired result. $\square$

4. Upper bounds

We now give upper bounds on λ_n. Our method is clear from equation (1.3), that we recall here :

$$\lambda_n = \inf_{\substack{f \in L^2(B_n) \\ f \neq 0}} \frac{\mathcal{E}(f,f)}{(f,f)}$$

Picking a function in $L^2(B_n)$ gives an upper bound, and the problem is to choose the function well enough (i.e. looking more or less like the eigenfunction) to get a sharp bound.

4.1. **The one-dimensional case.**

Theorem 4.1. *We assume $d = 1$. There exists $C > 0$ such that for almost every environment and all n large enough :*

$$\lambda_n \leqslant \frac{C}{n \sum_{x \in B_{n/4}} \tau_x}$$

Proof. For $a = 0$, a "triangle function" that takes the value 0 on $-(n+1)$ and $(n+1)$, the value 1 on 0 and is piecewise linear would do well. But for general a, this function is not appropriate, and we will construct instead a function that looks like it, but is constant around deep traps.

Let $M > 0$ be such that $\mathbb{P}[\tau_0 > M] \leqslant 1/8$. Because of the law of large numbers, one gets :

$$\frac{1}{n} \, |\{k \in \{-n-1, \dots, 0\} \ : \ \tau_k > M\}| \xrightarrow[n \to \infty]{a.s.} \frac{1}{8}$$

Almost surely, for n large enough, the two following conditions are satisfied :

$$|\{k \in \{-n-1, \dots, 0\} \ : \ \tau_k > M\}| \leqslant \frac{n}{4} \tag{4.1}$$

$$|\{k \in \{0, \dots, n+1\} \ : \ \tau_k > M\}| \leqslant \frac{n}{4} \tag{4.2}$$

Let us first construct the left part of our function : let $l : -\mathbb{N} \to \mathbb{R}$ be such that $l(k) = 0$ for all $k < -n$, and for all $k \in \{-n, \dots, 0\}$:

$$l(k) - l(k-1) = \left| \begin{array}{ll} 0 & \text{if } \tau_{k-1} > M \text{ or } \tau_k > M \\ 1/n & \text{otherwise} \end{array} \right.$$

The function l is made in such a way that for all k for which it makes sense :

$$\tau_k^a \tau_{k+1}^a (l(k+1) - l(k))^2 \leqslant \frac{M^{2a}}{n^2} \tag{4.3}$$

Moreover, when (4.1) is satisfied, there are at most half of the edges on which the function is constant, so $l(0) \geqslant 1/2$. In this case, and as for any k we have $l(k) - l(k-1) \leqslant 1/n$, it comes that $l(k) \geqslant 1/4$ when $k \geqslant -n/4$.

We define in the same way a right part $r : \mathbb{N} \to \mathbb{R}$ such that $r(k) = 0$ for all $k > n$, and for all $k \in \{n, \dots, 0\}$:

$$r(k) - r(k+1) = \left| \begin{array}{ll} 0 & \text{if } \tau_k > M \text{ or } \tau_{k+1} > M \\ 1/n & \text{otherwise} \end{array} \right.$$

The function r satisfies the same small variation property as in (4.3). Similarly, when (4.2) is satisfied, we have that $r(0) \geqslant 1/2$ and $r(k) \geqslant 1/4$ for all $k \leqslant n/4$.

Now we connect the two parts l and r preserving this small variation property. Let $m = \min(l(0), r(0))$. We define $f : \mathbb{Z} \to \mathbb{R}$ by

$$f(x) = \left| \begin{array}{ll} \min(l(x), m) & \text{if } x < 0 \\ \min(r(x), m) & \text{otherwise} \end{array} \right.$$

We have therefore :

$$\mathcal{E}(f,f) \leqslant \frac{2M^{2a}}{n}$$

On the other hand, for n large enough, (4.1) and (4.2) are satisfied, and in this case $m \geqslant 1/2$ and $f(k) \geqslant 1/4$ for all k such that $-n/4 \leqslant k \leqslant n/4$. Thus :

$$(f,f) \geqslant \frac{1}{16} \sum_{-n/4 \leqslant k \leqslant n/4} \tau_k$$

and we finally obtain, for all n large enough :

$$\lambda_n \leqslant \frac{\mathcal{E}(f,f)}{(f,f)} \leqslant \frac{32M^{2a}}{n \ \sum_{x \in B_{n/4}} \tau_x}$$

□

4.2. **Large dimension, anomalous behaviour.** The results proved in this part are in fact valid in any dimension and for any $\alpha > 0$, but they are sharp only in the regime given in the title, that is for $d \geqslant 2$ and $2\alpha \leqslant d$.

Theorem 4.2. (1) *For any $\varepsilon > 0$, there exists $M > 0$ such that for all n large enough :*

$$\mathbb{P}\left[\lambda_n \max_{B_{n-1}} \tau \leqslant M\right] \geqslant 1 - \varepsilon$$

(2) *For any $\varepsilon > 0$ and almost every environment :*

$$n^{d/\alpha - \varepsilon} \lambda_n \xrightarrow[n \to \infty]{} 0$$

Proof. Let K be the set of first and second neighbours of 0, namely $K = \{x \in \mathbb{Z}^d : 1 \leqslant \|x\| \leqslant 2\}$, and c the number of edges from a point of $\{x : \|x\| = 1\}$ to a point of $\{x : \|x\| = 2\}$. Write $M_x = \max_{x+K} \tau$. If we choose the function that takes value 1 on site $x \in B_{n-1}$ and its neighbours, and 0 elsewhere, namely :

$$f(z) = \left| \begin{array}{ll} 1 & \text{if } \|z - x\| \leqslant 1 \\ 0 & \text{otherwise} \end{array} \right.$$

then we see that for any $x \in B_{n-1}$:

$$\lambda_n \leqslant \frac{c(M_x)^{2a}}{\tau_x} \tag{4.4}$$

Let $x_n \in B_{n-1}$ be such that $\tau_{x_n} = \max_{B_{n-1}} \tau$. We have :

$$\lambda_n \leqslant \frac{c(M_{x_n})^{2a}}{\max_{B_{n-1}} \tau}$$

So we get :

$$\mathbb{P}\left[\lambda_n \max_{B_{n-1}} \tau \geqslant M\right] \leqslant \mathbb{P}\left[c(M_{x_n})^{2a} \geqslant M\right]$$

Now recall that M_{x_n} is the maximum over all neighbours and second neighbours of x_n, so it should look like taking the maximum over all neighbours and second neighbours of, say, 0. More precisely, conditionally on $\max_{B_{n-1}} \tau = \tau_z$ for some fixed z, the law of $(\tau_x)_{x \in B_{n-1} \setminus \{z\}}$ is invariant under permutation. Therefore, provided $z \in B_{n-2} \setminus K$ and conditionally on $\max_{B_{n-1}} \tau = \tau_z$, the random variables M_z and M_0 have the same law. Summing over all $z \in B_{n-2} \setminus K$, we get that conditionally on the event E_n that $x_n \in B_{n-2} \setminus K$, the random variables M_0 and M_{x_n} have the same law. We obtain :

$$\mathbb{P}\left[c(M_{x_n})^{2a} \geqslant M\right] \leqslant \mathbb{P}\left[c(M_0)^{2a} \geqslant M\right] + \mathbb{P}\left[E_n^c\right]$$

The law of x_n being uniform in B_{n-1}, we have that $\mathbb{P}\left[E_n^c\right]$ goes to 0 when n goes to infinity. First part of the theorem comes choosing M large enough.

We now turn to the second assertion of the proposition. Defining :

$$\overline{M}_n = \max_{x \in B_{n-1}} \frac{\tau_x}{(M_x)^{2a}}$$

we will show that for any $\varepsilon > 0$:

$$\frac{\overline{M}_n}{n^{d/\alpha-\varepsilon}} \xrightarrow[n\to\infty]{\text{a.s.}} +\infty \tag{4.5}$$

which will prove the result via equation (4.4). There exists $k > 0$ such that $\mathbb{P}[(M_x)^{2a} > k] < 1/2$. Thus (note that M_x and τ_x are independent) :

$$\mathbb{P}\left[\frac{\tau_x}{(M_x)^{2a}} \geqslant y\right] \geqslant \frac{\mathbb{P}[\tau_x \geqslant ky]}{2} = \frac{F(ky)}{2}$$

Hence, for all $K > 0$:

$$\mathbb{P}[\overline{M}_n \leqslant n^{d/\alpha-\varepsilon}K] \leqslant \left(1 - \frac{F(kKn^{d/\alpha-\varepsilon})}{2}\right)^{(2n-1)^d}$$

and recalling that, as a consequence of assumption 1' (see (1.2)), for all $\beta < \alpha$, $F(y) \leqslant y^{-\beta}$ for all y large enough, one can see that the term on the right-hand side of the former equality is the general term of a convergent series, and thus apply the Borel-Cantelli lemma. □

4.3. **Regular behaviour.** In what follows our assumption will be that $\mathbb{E}[\tau_0^a]$ is finite. In particular, all results will be valid under the condition that $\mathbb{E}[\tau_0]$ is finite (or if $a = 0$).

We write $(e_i)_{1\leqslant i\leqslant d}$ for the canonical base of $\mathbb{R}^d$.

Proposition 4.3. *Let $f : [-1,1]^d \to \mathbb{R}$ be a continuous function. If $\mathbb{E}[\tau_0^a]$ is finite, then for all $i \in \{1,\dots,d\}$:*

$$\frac{1}{(2n+1)^d}\sum_{x\in B_n} \tau_x^a \tau_{x+e_i}^a\ f(x/n) \xrightarrow[n\to\infty]{a.s.} \mathbb{E}[\tau_0^a]^2 \int_{[-1,1]^d} f(x)\mathrm{d}x \tag{4.6}$$

Proof. If f is piecewise constant, then the limit (4.6) is proved by separating the sum over B_n into two parts B_n' and B_n'' so that $(\tau_x^a\tau_{x+e_i}^a)_{x\in B_n'}$ and $(\tau_x^a\tau_{x+e_i}^a)_{x\in B_n''}$ are two families of independent random variables, and then applying the law of large numbers. For a continuous f, one can approximate uniformly f by piecewise constant functions from above and below, and the result follows. □

For all $f : [-1,1]^d \to \mathbb{R}$ and all integer n, we define the function $f_n : \mathbb{Z}^d \to \mathbb{R}$ by $f_n(x) = f(x/n)$ if $x \in B_n$, and $f_n(x) = 0$ otherwise. Note that $f_n \in L^2(B_n)$.

Proposition 4.4. *Let $f : [-1,1]^d \to \mathbb{R}$ be a twice continuously differentiable function that takes value 0 on the boundary of $[-1,1]^d$. If $\mathbb{E}[\tau_0^a]$ is finite, then :*

$$\frac{n^2}{(2n)^d}\mathcal{E}(f_n,f_n) \xrightarrow[n\to\infty]{a.s.} \mathbb{E}[\tau_0^a]^2 \int_{[-1,1]^d} \|\nabla f(x)\|_2^2 \mathrm{d}x$$

Recall the following equality :

$$\mathcal{E}(f_n,f_n) = \sum_{i=1}^d \sum_{x\in B_n} \tau_x^a\tau_{x+e_i}^a \left(f\left(\frac{x}{n}\right) - f\left(\frac{x+e_i}{n}\right)\right)^2$$

As we assumed f to be twice continuously differentiable, it comes that for all $\varepsilon > 0$ and n large enough :

$$\forall x \in B_n : x+e_i \in B_n \Rightarrow \left|\left(f\left(\frac{x}{n}\right) - f\left(\frac{x+e_i}{n}\right)\right)^2 - \frac{1}{n^2}\frac{\partial f}{\partial x_i}\left(\frac{x}{n}\right)^2\right| \leqslant \frac{\varepsilon}{n^2}$$

and note that if $x \in B_n$ and $x + e_i \notin B_n$, then $f(x/n) = f((x + e_i)/n) = 0$, so this case does not contribute to the sum. The result follows using the previous proposition.

Theorem 4.5. *If $\mathbb{E}[\tau_0^a]$ is finite, then there exists C such that almost surely, for all n large enough :*

$$\lambda_n \leqslant \frac{C}{n^2} \frac{n^d}{\sum_{x \in B_{n/2}} \tau_x}$$

Proof. Taking $f(x) = \prod_{i=1}^d \sin\left(\frac{\pi x_i}{2}\right)$ in Proposition 4.4, we get that for almost every environment :

$$\mathcal{E}(f_n, f_n) \sim \frac{d\pi^2}{4} \frac{(2n)^d}{n^2} \mathbb{E}[\tau_0^a]^2 \qquad (n \to +\infty)$$

On the other hand, if $x \in B_{n/2}$, then $f(x) \geqslant 2^{-d/2}$, thus :

$$(f_n, f_n) \geqslant 2^{-d/2} \sum_{x \in B_{n/2}} \tau_x$$

therefore the proposition holds for any $C > 2^{3d/2-2} d\pi^2 \mathbb{E}[\tau_0^a]^2$. □

5. The distinguished path method

We present here a more direct method to get a lower bound on λ_n (close to the one presented e.g. in [SC97, Theorem 3.2.3], but adapted to treat the case of Dirichlet boundary condition), and show that it does not provide a sharp estimate when $d \geqslant 2$. Note that in dimension one, [Chen, Section 3.7] proves that this technique is always sharp, and one can verify that it gives indeed the expected lower bound. This method also proved efficient in larger dimension in [FM06, Section 3] in the context of random walks among random conductances.

For all $x \in B_n$, we give ourselves a path $\gamma_n(x)$ from some point of ∂B_n to x (that apart from the starting point, visits only points in B_n). Let $\gamma_n(x) = (x^0, \ldots, x^l)$. For an edge e, we note $e \in \gamma_n(x)$ if $e = (x^i, x^{i+1})$ for some i, and in this case, we write $\mathrm{d}f(e) = f(x^{i+1}) - f(x^i)$, and $\mathfrak{Q}(e) = \tau_{x^i}^a \tau_{x^{i+1}}^a$. Let E_n be the set of edges that go from a point of B_n to a point of $B_n \cup \partial B_n$. We give ourselves a weight function $W_n : E_n \to (0, +\infty)$. We define the W_n-length of a path γ as :

$$l_n(\gamma) = \sum_{e \in \gamma} \frac{1}{W_n(e)}$$

Note that, as we assumed that $\tau \geqslant 1$, we have that $\mathfrak{Q}(e) \geqslant 1$ (and there is equality when $a = 0$). Using Cauchy-Schwarz inequality, we get :

$$\begin{aligned} f(x)^2 &= \left(\sum_{e \in \gamma_n(x)} \mathrm{d}f(e) \right)^2 \\ &\leqslant \sum_{e \in \gamma_n(x)} \frac{1}{W_n(e)\mathfrak{Q}(e)} \sum_{e \in \gamma_n(x)} \mathrm{d}f(e)^2 W_n(e) \mathfrak{Q}(e) \\ &\leqslant l_n(\gamma_n(x)) \sum_{e \in \gamma_n(x)} \mathrm{d}f(e)^2 W_n(e) \mathfrak{Q}(e) \end{aligned}$$

$$\begin{aligned} \sum_{x \in B_n} f(x)^2 \tau_x &\leqslant \sum_{x \in B_n} l_n(\gamma_n(x)) \tau_x \sum_{e \in \gamma_n(x)} \mathrm{d}f(e)^2 W_n(e) \mathfrak{Q}(e) \\ &\leqslant \sum_{e \in E_n} \mathrm{d}f(e)^2 \mathfrak{Q}(e) W_n(e) \sum_{x : e \in \gamma_n(x)} l_n(\gamma_n(x)) \tau_x \end{aligned}$$

Note that

$$\mathcal{E}(f,f) = \sum_{e \in E_n} \mathrm{d}f(e)^2 \mathfrak{Q}(e)$$

So letting

$$\mathcal{M}_n := \max_{e \in E_n} W_n(e) \sum_{x : e \in \gamma_n(x)} l_n(\gamma_n(x)) \tau_x$$

we obtain the following lower bound on λ_n (similar to [SC97, Theorem 3.2.3]) :

$$\lambda_n \geqslant \frac{1}{\mathcal{M}_n}$$

Let us see that, however W_n and $\gamma_n(x)$ are chosen, it cannot lead to a sharp bound if $d \geqslant 2$ and $\alpha < d$. Let $z \in B_{n/2}$ be such that τ_z is maximal. The site z is such that $\tau_z \simeq n^{d/\alpha}$ and $|\gamma_n(z)| \geqslant n/2$. Now choose $e \in \gamma_n(z)$ so that $W_n(e)$ is maximal. We have :

$$\mathcal{M}_n \geqslant \sum_{e' \in \gamma_n(z)} \frac{W_n(e)}{W_n(e')} \tau_z \geqslant |\gamma_n(z)| \tau_z \gtrsim n^{1+d/\alpha}$$

where we would have hoped to find $n^{\max(2,d/\alpha)}$. So this method cannot give the appropriate exponent if $\alpha < d$.

Still, note that if one chooses W_n constant equal to 1, and the shortest paths for $(\gamma_n(x))_{x \in B_n}$, one can show using results of [BK65] that $\mathcal{M}_n$ is indeed of order $n^{\max(2,1+d/\alpha)}$, which gives an alternative proof of a lower bound for the principal eigenvalue when $\alpha \geqslant d$.

6. Appendix

In this section, we prove upper estimates on the transition kernel and Green function of any symmetric nearest-neighbour continuous-time random walk on $\mathbb{Z}^d$, provided its jump rates are uniformly bounded from below. For any pair of neighbours $x \sim y$ in $\mathbb{Z}^d$, we give ourselves ω_{xy} and consider the Markov process $(Z_t)_{t \geqslant 0}$ with jump rate between x and y given by ω_{xy}. We assume theses jump rates to be symmetric ($\omega_{xy} = \omega_{yx}$) and bounded from below by 1 ($\omega_{xy} \geqslant 1$). The generator of this process is given by :

$$\mathfrak{L} f(x) = \sum_{y \sim x} \omega_{xy} (f(y) - f(x))$$

We write $p_t(x,z)$ for the probability for Z_t to be at site z starting from x, and $\mathfrak{g}(\cdot,\cdot)$ for the associated Green function.

We define the *natural length* of a path $\gamma = (x_0, \ldots, x_l)$ by :

$$|\gamma|_\omega = \sum_{i=1}^{l} \left(\omega_{x_{i-1}x_i}\right)^{-1/2}$$

and the *natural distance* between any two points $x, y \in \mathbb{Z}^d$ by :

$$\Delta_\omega(x,y) = \inf \{ |\gamma|_\omega,\ \gamma \text{ simple path from } x \text{ to } y \} \tag{6.1}$$

Of course, the restriction on *simple* paths can be omitted without change, it is just a matter of convenience for part 3.1. We will prove the following Gaussian-like upper bounds on the transition kernel and Green function of (Z_t) :

Theorem 6.1. *There exists C_2 such that for any $x, y \in \mathbb{Z}^d$ and any $t > 0$:*

(1) *If $8de^2 t \geqslant \Delta_\omega(x,y)$, then :*

$$p_t(x,y) \leqslant \frac{C_2}{t^{d/2}} \exp\left(-\frac{3\Delta_\omega(x,y)^2}{32de^2 t} \right)$$

(2) *If* $8de^2t \leqslant \Delta_\omega(x,y)$*, then :*

$$\begin{aligned} p_t(x,y) &\leqslant \frac{C_2}{t^{d/2}} \exp\left[-\frac{1}{4}\Delta_\omega(x,y)\left(\ln\left(\frac{\Delta_\omega(x,y)}{8de^{-2}t}\right)-1\right)\right] \\ &\leqslant \frac{C_2}{t^{d/2}} \exp\left[-\frac{3}{4}\Delta_\omega(x,y)\right] \end{aligned}$$

(3) *If* $d \geqslant 3$*, then there exists* C_3 *such that for any* $x, y \in \mathbb{Z}^d$ *:*

$$\mathfrak{g}(x,y) \leqslant \frac{C_3}{(1+\Delta_\omega(x,y))^{d-2}}$$

We will use Nash inequalities to prove this result, adapting a strategy due to E.B. Davies [Dav87] generalized by E.A. Carlen, S. Kusuoka and D.W. Stroock [CKS87].

Remarks.

(1) One could be surprised to see the failure of the Gaussian upper bound to hold for all $t > 0$. Indeed, such an inequality is proved for all $t > 0$ in [Dav, Theorem 3.2.7] in a continuous-space context. The difference comes from the exponential part that appears in (6.3), which is not there in the continuous case. But it is not a technical artefact. Indeed, as t goes to 0, $p_t(x,y)$ behaves like a polynomial in t in our discrete-space context, so the Gaussian upper bound cannot hold for small times.
(2) One should not be mislead to believe that something special happens when $8de^2t = \Delta_\omega(x,y)$. We wrote the results this way because we found it more convenient, but they do not aim at being optimal (and the willing reader can improve them by a more detailed analysis). Also, note that C_2, C_3 can be made explicit, and depend only on the dimension.
(3) The previous results no longer hold (even qualitatively) without the assumption that the jump rates are bounded away from 0. We refer to [FM06] or [BBHK07] for evidence of the anomalous behaviour of the return probabilities in this case.
(4) The case of discrete-time random walks has been treated in [HS93] (or equivalently in Theorem 14.12 of the monograph [Woe]).

Due to the symmetry of the transition rates, it is clear that the uniform measure is reversible, and $\mathfrak{L}$ self-adjoint. In this Appendix, we understand scalar products and L^p norms to be taken with respect to the uniform measure.

We write $(P_t)_{t\geqslant 0}$ for the semi-group associated to Z, and $\|P_t\|_{p\to q}$ for the norm of P_t as an operator from L^p to L^q. We define the Dirichlet forms $\mathfrak{E}, \mathfrak{E}_0$:

$$\mathfrak{E}(f,g) = \frac{1}{2} \sum_{\substack{x,y\in\mathbb{Z}^d \\ x\sim y}} \omega_{xy}(f(y)-f(x))(g(y)-g(x))$$

$$\mathfrak{E}_0(f,g) = \frac{1}{2} \sum_{\substack{x,y\in\mathbb{Z}^d \\ x\sim y}} (f(y)-f(x))(g(y)-g(x))$$

Proposition 6.2. *There exists* $C_1 > 0$ *such that the following Nash inequality holds for all* f *:*

$$\|f\|_2^{2+4/d} \leqslant C_1 \mathfrak{E}(f,f) \|f\|_1^{4/d}$$

Proof. Applying [Woe, Proposition 14.1] together with [Woe, Corollary 4.12], it is clear that the inequality holds with $\mathfrak{E}_0$ instead of $\mathfrak{E}$. As $\mathfrak{E}_0(f,f) \leqslant \mathfrak{E}(f,f)$, we get the announced result. □

At this step, we can derive, following [Nash58] (or Theorem 2.1 of [CKS87]), that there exists C such that for all $t > 0$:

$$\sup_{x,y} p_t(x,y) = \|P_t\|_{1\to\infty} \leqslant \frac{C}{t^{d/2}}$$

But this is not a sharp bound when x and y are far one from the other. Davies's idea [Dav87] is the following : let $\psi : \mathbb{Z}^d \to \mathbb{R}$ be some function, let $s \in \mathbb{R}$, and consider the following semi-group :

$$Q_{s,t} f = e^{-s\psi} P_t(e^{s\psi} f)$$

The aim is to find an upper bound for

$$\|Q_{s,t}\|_{1\to\infty} = \sup_{x,y} e^{-s\psi(x)} p_t(x,y) e^{s\psi(y)}$$

From now on, we assume the following on ψ :

$$\forall x, y \quad x \sim y \Rightarrow \omega_{xy}(\psi(y) - \psi(x))^2 \leqslant 1 \tag{6.2}$$

and define

$$I(s) = ds^2(1 + e^{|s|})^2/4 \tag{6.3}$$

Proposition 6.3 ([CKS87])**.** *There exists $C_2 > 0$ such that for any $x, y \in \mathbb{Z}^d$ and any $t > 0$:*

$$p_t(x,y) \leqslant \frac{C_2}{t^{d/2}} \exp\left(3I(s)t/2 - s(\psi(y) - \psi(x))\right) \tag{6.4}$$

Proof. We follow closely [CKS87, Section 3]. In our context, we can define directly $\Gamma(\cdot,\cdot)$ of [CKS87, Theorem 3.7] as :

$$\Gamma(f,g)(x) = \frac{1}{2} \sum_{y\sim x} \omega_{xy}(f(y) - f(x))(g(y) - g(x))$$

and we can verify the Leibnitz rule :

$$\mathfrak{E}(fg, h) = \sum_{x\in\mathbb{Z}^d} f(x)\Gamma(g,h)(x) + \sum_{x\in\mathbb{Z}^d} g(x)\Gamma(f,h)(x)$$

together with the Cauchy-Schwarz inequality

$$|\Gamma(f,g)| \leqslant \Gamma(f,f)^{1/2}\Gamma(g,g)^{1/2}$$

Now a direct computation gives :

$$\begin{aligned} &\mathfrak{E}(e^{s\psi} f^{2p-1}, e^{-s\psi} f) - \mathfrak{E}(f^{2p-1}, f) \\ &= \frac{1}{2} \sum_{x\sim y} \omega_{xy} [e^{-s\psi(x)} f(x) f^{2p-1}(y) - e^{-s\psi(y)} f(y) f^{2p-1}(x)][e^{s\psi(x)} - e^{s\psi(y)}] \end{aligned} \tag{6.5}$$

which is the equivalent of [CKS87, (3.12)-(3.13)]. From this, one can follow the computations of the proof of [CKS87, Theorem 3.9] to get inequalities [CKS87, (3.10)-(3.11)] with $\Gamma(s\psi)^2$ replaced by $I(s)$, as we have :

$$\Gamma(s\psi)^2 := \max(\|e^{-2s\psi}\Gamma(e^{s\psi}, e^{s\psi})\|_\infty, \|e^{2s\psi}\Gamma(e^{-s\psi}, e^{-s\psi})\|_\infty) \leqslant I(s)$$

Indeed :

$$e^{-2s\psi(x)}\Gamma(e^{s\psi}, e^{s\psi})(x) = \frac{1}{2}\sum_{y\sim x} \omega_{xy}(1 - e^{s(\psi(y)-\psi(x))})^2$$

Note that $|1 - e^u|/(1 + e^u) \leqslant |u|/2$, which implies that

$$|1 - e^{s(\psi(y)-\psi(x))}| \leqslant |s||\psi(y) - \psi(x)|(1 + e^{s(\psi(y)-\psi(x))})/2$$

As $\omega_{xy} \geqslant 1$, property (6.2) implies that for $x \sim y$, we have $|\psi(y) - \psi(x)| \leqslant 1$. Using again property (6.2), it comes that, for all $x \in \mathbb{Z}^d$:

$$e^{-2s\psi(x)}\Gamma(e^{s\psi}, e^{s\psi})(x) \leqslant ds^2(1 + e^{|s|})^2/4 = I(s)$$

and the same inequality holds with ψ replaced by $-\psi$. (note that there is a typographical error in the last inequality of the proof of [CKS87, Theorem 3.9], where one should read $\frac{2p-1}{p^2}$ instead of $\frac{2p-1}{2}$. Also, one should read that [CKS87, (3.11)] is valid for all $p \in [2, +\infty)$ instead of $[1, +\infty)$. To prove second part of [CKS87, (3.17)], one can write $y^p - x^p$ as $p \int_x^y t^{p-1} \mathrm{d}t$ and apply Cauchy-Schwarz inequality).

Now we can use the Nash inequality we obtained in Proposition 6.2, and apply [CKS87, Theorem 3.25] (choosing $\rho = 1/2$), which proves the proposition. □

We define a distance between x and y by :

$$\Delta_\omega(x, y) = \sup\{\psi(y) - \psi(x) \mid \psi \text{ satisfies (6.2)}\} \tag{6.6}$$

We postpone the proof that this distance is indeed the natural distance introduced in equation (6.1) to the next proposition, but let us first prove Theorem 6.1.

Proof of Theorem 6.1. Let $x, y \in \mathbb{Z}^d$. We choose $\psi(\cdot) = \Delta_\omega(x, \cdot)$. Seeing (6.4), the point is to choose s in order to have $3I(s)t/2 - s(\psi(y) - \psi(x)) = 3I(s)t/2 - s\Delta_\omega(x, y)$ minimal, to get the sharpest possible bound.

To simplify a little, we first remark that $3I(s)/2 \leqslant 2ds^2e^{2|s|}$.

If $0 \leqslant s \leqslant 1$, then $3I(s)/2 \leqslant 2de^2s^2$. If $8de^2t \geqslant \Delta_\omega(x, y)$, then choosing $s = \frac{\Delta_\omega(x,y)}{8de^2t} \leqslant 1$, we have :

$$3I(s)t/2 - s\Delta_\omega(x, y) \leqslant 2de^2s^2t - s\Delta_\omega(x, y) = -\frac{3\Delta_\omega(x, y)^2}{32de^2t}$$

and part (1) of the theorem comes.

If $s \geqslant 1$, then $s^2 = \exp(2\ln(s)) \leqslant \exp(2(s - 1))$ and it comes that $3I(s)/2 \leqslant 2de^{-2}e^{4s}$. If $8de^2t \leqslant \Delta_\omega(x, y)$, then $s = \frac{1}{4}\ln\left(\frac{\Delta_\omega(x,y)}{8e^{-2}dt}\right)$ is larger than 1 and part (2) comes.

What is left is to see how to derive part (3) from the previous estimates, assuming that $d \geqslant 3$. We will show first that $\mathfrak{g}(x, y) = O(\Delta_\omega(x, y)^{2-d})$ as $\Delta_\omega(x, y)^{2-d}$ goes to infinity, and then that it is uniformly bounded. We have :

$$\begin{aligned} \mathfrak{g}(x, y) &= \left(\int_0^{\frac{\Delta_\omega(x,y)}{8de^2}} + \int_{\frac{\Delta_\omega(x,y)}{8de^2}}^{+\infty}\right) p_t(x, y)\ \mathrm{d}t \\ &= I_1 + I_2 \end{aligned}$$

where

$$I_2 \leqslant C_2 \left(\frac{3\Delta_\omega(x, y)^2}{32de^2}\right)^{1-d/2} \int_0^\infty u^{-d/2} \exp(-1/u)\ \mathrm{d}u = \frac{C}{\Delta_\omega(x, y)^{d-2}}$$

and, if $\Delta_\omega(x, y) \geqslant 2d$:

$$\begin{aligned} I_1 &\leqslant \left(\frac{\Delta_\omega(x, y)}{8de^{-2}}\right)^{1-d/2} \int_0^{e^{-4}} u^{-d/2} \exp\left(-\frac{\Delta_\omega(x, y)}{4}(\ln(1/u) - 1)\right) \mathrm{d}u \\ &\leqslant \left(\frac{\Delta_\omega(x, y)}{8de^{-2}}\right)^{1-d/2} \exp\left(-\frac{3}{4}\left(\frac{\Delta_\omega(x, y)}{4} - \frac{d}{2}\right)\right) \int_0^{e^{-4}} u^{-d/2}e^{-d/2(\ln(1/u)-1)}\mathrm{d}u \\ & O(\Delta_\omega(x, y)^{2-d}) \end{aligned}$$

We now show that $\mathfrak{g}(\cdot, \cdot)$ is bounded.

Proposition 6.4. *If $d \geqslant 3$, then there exists C such that for any $x, y \in \mathbb{Z}^d$*

$$\mathfrak{g}(x,y) \leqslant C$$

Proof. Note that for any function f :

$$\mathfrak{E}(f,f) \geqslant \mathfrak{E}_0(f,f) \geqslant 0$$

Thus we can apply [Woe, lemma 2.24] to get that

$$\mathfrak{g}(x,x) \leqslant \mathfrak{g}_0(x,x)$$

where $\mathfrak{g}_0$ is the Green function of the simple random walk. (one needs to be careful that the generators of the random walks are not invertible operators, but see the proof of [Woe, Theorem 2.25] to work things out). To conclude, note that $\mathfrak{g}_0(x,x)$ does not depend on x, and $\mathfrak{g}(x,\cdot)$ is maximal on x. □

Part (3) of the theorem now follows. □

Finally, we check that the distances introduced in (6.1) and (6.6) are indeed the same.

Proposition 6.5. *The distance defined in equation (6.6) is the natural distance :*

$$\Delta_\omega(x,y) = \inf\{|\gamma|_\omega,\ \gamma \textit{ simple path from } x \textit{ to } y\}$$

Proof. Let $\Delta'_\omega(x,y)$ be the infimum given above. Let ψ be a function that satisfies (6.2). Then for all x_1, x_2 neighbours, we have :

$$\psi(x_2) - \psi(x_1) \leqslant (\omega_{x_1x_2})^{-1/2}$$

so for any simple path γ that goes from x to y, summing the former inequality along the path, it comes that :

$$\psi(y) - \psi(x) \leqslant |\gamma|_\omega$$

and $\Delta_\omega(x,y) \leqslant \Delta'_\omega(x,y)$. On the other hand, for x fixed, choose $\psi(y) = \Delta'_\omega(x,y)$. Then ψ satisfies (6.2), so $\Delta_\omega(x,y) \geqslant \Delta'_\omega(x,y)$. □

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