On Novikov homology

Jean-Claude Sikorav, December 22, 2017

Introduction

Novikov homology is a homology with twisted coefficients whose values belong to a suitable completion of a group ring, "in the direction" of a given homomorphism from the group to $\mathbb{R}$. Usually, the group is the fundamental group of some topological space, in particular a smooth manifold. In the latter case, the homomorphism can be identified with the cohomology class of a closed differential one-form. This homology was introduced by S.P. Novikov [No 1981] [No 1982] as the right object for the generalization of the Morse inequalities from critical points of functions to critical points of circle-valued functions or more generally zeros of closied one-forms.

Since for this generalization one needs a principal ring (or a field), Novikov considered only an Abelian version, which geometrically corresponds to working on an Abelian covering, for instance the integration covering associated to the kernel of the homomorphism. The general version, which amounts to working on the universal covering (or any intermediate covering) was introduced in [Sik] in 1987. It appeared that the the first-dimensional Novikov homology is very closely related to the sigma-invariants of groups defined at the same time by R. Bieri, W.D. Neumann and R. Strebel [BiNeRe], and the higher dimension Novikov homology is similarly related to the higher invariants of Bieri and B. Renz [Bi-Re]. These invariants measure finiteness properties such as the finite generation or properties $FP(n)$ of the kernel of the homomorphism (finite presentation being a more delicate question). There have been many papers devoted to these invariants since their definitions, but very few taking the point of view of Novikov homology (in part because of the restricted distribution of my 1987 thesis).

Another application of Novikov homology is to the problem of fibering a (closed) manifold over the circle, or more generally (by the "Abel-Tischler" theorem [Ti]) the existence of a non singular closed one-form in a given cohomology class. The first problem was solved in high dimensions by L.C. Siebenmann [Sie], and the general one was solved in dimension three by W.P. Thurston [Thu], in terms of his norm on the one-dimensional cohomology. The general problem in high dimension was solved by F. Latour [Lat] using Novikov homology, and the case of dimension three was reinterpreted in terms of sigma-invariants by Bieri-Neumann-Strebel and in terms of Novikov homology in my thesis. But many questions remain in the three-dimensional case, especially the relation with the Thurston norm and various notions of twisted Alexander polynomials. There are also famous questions conerning three-manifolds fibering over the circle, for which Novikov homology seems to be useful. One was recently solved by S. Friedl and S. Vidussi [Fr-Vi 2011], the equivalence between classes in $H^1(M^3;\mathbb{R})$ containing a nonsingular closed one-form and those in $H^2(M\times S^1;\mathbb{R})$ containing a symplectic form (sometimes called "Taubes' conjecture"), the other is of course the famous one of Thurston whether most closed (irreducible) three-manifolds have a finite covering which fibers over the circle.

Let us now describe the contents of this paper.

In Section 1, we define the Novikov ring $\mathbb{Z}[G]_\xi$ associated to a group G equipped with a nonzero homomorphism $\xi : G \to \mathbb{R}$, as a suitable completion of the group ring $\mathbb{Z}[G]$. We study some of its properties, in particular the description of special invertible elements and invertible matrices. Also, we prove that it is *stably finite*, ie every matrix with coefficients in $\mathbb{Z}[G]_\xi$ which is right and left invertible is square, and every square matrice which is right or left invertible is invertible.

In Section 2, we define the Novikov homology $H(C_*, \xi)$, where C_* is a chain complex which is free over $\mathbb{Z}[G]$ and G is equipped with a nonzero homomorphism $\xi : G \to \mathbb{R}$. In particular, taking the standard resolution of G we obtain the homology $H(G, \xi)$, which is the homology of the group G with coefficients in $\mathbb{Z}[G]_\xi$. More generally, one considers a chain complex equivalent to the singular complex of a topological space, obtaining thus the topological Novikov homology $H(X, \xi)$, with $\xi \in H^1(X, \mathbb{R})$ nonzero. We give a computational criterion for the vanishing of Novikov homology.

In Section 3, we define the notion of geometric closed one-forms and study topological Novikov homology.

In section 4, we characterize computationally classes with vanishing Novikov homology. In degree one, this takes a special form which will be often used.

In Section 5, we study homology of degree one, more precisely we give a large number of properties equivalent to its vanishing: group-theoretic, geometric, dynamical etc.

Section 6 is the heart of this paper, relating Novikov homology to the sigma-invariants of groups and thus to finiteness properties.

In Section 7, we describe the Morse-Novikov (the "Thom-Smale-Witten" version) complex which for Novikov was the starting point.

In Section 8, 9 and 10 we study the Abelian case, in particular the relation with the Alexander polynomial.

In Section 11, we study the case of three-manifolds.

In Section 12, we study a problem about units in group rings and Novikov rings, whose solution has some nice applications to group theory and three-dimensional topology.

Contents

1 Novikov rings

1.1 Definitions. Let G be a group. We denote by $\mathbb{Z}[[G]]$ the additive group of formal series

$$\lambda = \sum_{g \in G} n_g(\lambda) g \ , \ n_g(\lambda) \in \mathbb{Z}.$$

The *support* of λ is $\operatorname{supp}(\lambda) = \{g \in G \mid n_g(\lambda) \neq 0\}$. The elements of finite support form the *group ring* (or group algebra) $\mathbb{Z}[G]$.

The homomorphims $\xi : G \to \mathbb{R}$ form a vector space $\operatorname{Hom}(G, \mathbb{R}) = H^1(G, \mathbb{R})$. When it it finite-dimensional, which is the case if G is finitely generated, we denote by $b_1(G)$ its dimension (first Betti number), and we equip it with the canonical vector space topology.

We note $\mathcal{N}(G) = \operatorname{Hom}(G, \mathbb{R}) \setminus \{0\}$, and $S(G) = \mathcal{N}(G)/\mathbb{R}_+^*$, the action being by homotheties. If $H^1(G, \mathbb{R})$ is finite-dimensional, it is a sphere dimension $b_1(G) - 1$. The class of ξ in $S(G)$ is denoted by $[\xi]$.

If $\xi \in \mathcal{N}(G)$, the *Novikov ring* associated to (G, ξ), denoted by $\mathbb{Z}[G]_\xi$, is by definition the subgroup of all series $\lambda \in \mathbb{Z}[[G]]$ whose support is "infinite only in the direction of ξ":

$$\begin{aligned}\mathbb{Z}[G]_\xi &:= \{\lambda \in \mathbb{Z}[[G]] \mid (\forall r \in \mathbb{R}) \operatorname{supp}(\lambda) \cap \xi^{-1}([-\infty, r[) \text{ is finite}\} \\ &= \{\sum_{i=0}^{+\infty} a_i g_i \mid a_i \in \mathbb{Z} \ , \ g_i \in G \ , \ i \leq N \text{ or } \lim_{i \to \infty} \xi(g_i) = +\infty\}.\end{aligned}$$

It clearly depends only on the class $[\xi] \in S(G)$. Thus we can write $\mathbb{Z}[G]_\xi = \mathbb{Z}[G]_{[\xi]}$.

Lemma. *Let $A, B \subset G$ be such that $(\forall r \in \mathbb{R})$ A and B intersect $\xi^{-1}([-C, +\infty[)$ in a finite subset. Then $(\forall r \in \mathbb{R})$, $\{(a,b) \in A \times B \mid \xi(ab) \leq r\}$ is finite.*

Proof of the lemma. If A or B is empty, it is obvious. If not, $\xi(A)$ and $\xi(B)$ have a smallest element α and β. If $(a,b) \in A \times B$ and $\xi(ab) \leq r$, $\xi(a) \leq r - \beta$ and $\xi(b) \leq r - \alpha$, thus there are only finitely many possibilities for (a,b).

Corollary. *The group $\mathbb{Z}[G]_\xi$ can be equipped with the obvious mulitplication*

$$\lambda_1.\lambda_2 = \sum_{g \in G} \sum_{\substack{g_i \in \operatorname{supp}(\lambda_i) \\ g_1 g_2 = g}} n_{g_1}(\lambda_1) n_{g_2}(\lambda_2).$$

It becomes a unital ring, which contains $\mathbb{Z}[G]$ as a subring.

1.2 Example: $G = \mathbb{Z}$**.** The group ring $\mathbb{Z}[\mathbb{Z}]$ can be identified with the Laurent polynomials $\mathbb{Z}[t, t^{-1}]$. We have $S(\mathbb{Z}) = \{[\mathrm{Id}], -[\mathrm{Id}]\}$, with both elements giving clearly isomorphic Novikov rings. If $\xi(t) > 0$, one has

$$\mathbb{Z}[\mathbb{Z}]_\xi = \mathbb{Z}[[t]][t^{-1}] = \{\sum_{i=-i_0}^{+\infty} n_i t^i \mid n_i \in \mathbb{Z}\}.$$

It is easy to adapt the Euclidean division algorithm to prove that is a Euclidean ring (see below a more general version). In particular, it is a principal ideal domain, which implies that free finitely generated complexes over $\mathbb{Z}[\mathbb{Z}]_\xi$ satisfy the Morse inequalities (see below).

1.3 Valuation, minimal part

Let $\lambda = \sum n_g g$ be an element of $\mathbb{Z}[G]_\xi$. Its *ξ-valuation* is $v_\xi(\lambda) := \min(\xi|\mathrm{supp}(\lambda))$, with the convention $v_\xi(0) = +\infty$. Its *ξ-minimal part* and its *ξ-tail* are

$$m_\xi(\lambda) := \sum_{g|\xi(g)=v_\xi(\lambda)} n_g g$$
$$t_\xi(\lambda) := \lambda - m_\xi(\lambda),$$

so that $v_\xi(t_\xi(\lambda)) > v_\xi(\lambda)$ if $\lambda \neq 0$. We shall say that λ is *ξ-simple* if $m_\xi(\lambda) = ng$ with $n \in \mathbb{Z}^*$ and $g \in G$. In that case, we define $n_\xi(\lambda) := n$, $\gamma_\xi(\lambda) := g$. We extend n_ξ by setting $n_\xi(0) = 0$. Clearly, $n_\xi(\lambda\mu) = n_\xi(\lambda)n_\xi(\mu)$ when $n_\xi(\lambda)$ and $n_\xi(\mu)$ are defined.

For $E \subset \mathbb{R}$, the subgroup of all $\lambda \in \mathbb{Z}[G]_\xi$ with support in E will be denoted $\mathbb{Z}[G]_{\xi,E}$. For $r \in \mathbb{R}$, one has a decomposition

$$\mathbb{Z}[G]_\xi = \mathbb{Z}[G]_{\xi,]-\infty,r]} \oplus \mathbb{Z}[G]_{\xi,]C,+\infty[},$$

the first subgroup being contained in $\mathbb{Z}[G]$. We write $x = x_{\xi\leq r} + x_{\xi>r}$ the decomposition of $x \in \mathbb{Z}[G]_\xi$.

Converging series. If (λ_n) is a sequence of elements of $\mathbb{Z}[G]_\xi$ (in particular of $\mathbb{Z}[G]$) such that $v_\xi(\lambda_n) \to +\infty$, $\sum_{n=0}^{+\infty} \lambda_n$ is clearly well-defined as an element of $\mathbb{Z}[G]_\xi$.

1.4 Trivial invertible elements in Novikov rings and matrix rings

Let λ be a nonzero element of $\mathbb{Z}[G]_\xi$. If $m_\xi(\lambda)$ is invertible in $\mathbb{Z}[G]$, λ is invertible in $\mathbb{Z}[G]_\xi$, with inverse

$$\begin{aligned}\lambda^{-1} &= (m_\xi(\lambda) + t_\xi(\lambda))^{-1} = (m_\xi(\lambda)(1 + m_\xi(\lambda)^{-1}t_\xi(\lambda))^{-1} \\ &= \Big(\sum_{n=0}^{+\infty}(-m_\xi(\lambda)^{-1}t_\xi(\lambda))^n\Big)m_\xi(\lambda)^{-1}.\end{aligned}$$

This is in particular the case if λ is ξ-simple and $n_\xi(\gamma) = \pm 1$. In that case, λ will be called a *trivial invertible*. The elements of $\mathbb{Z}[G]$ which are trivial invertibles in $\mathbb{Z}[G]_\xi$ form a multiplicative subset of the unit group $\mathbb{Z}[G]_\xi^*$, which will be denoted by S_ξ.

Example: if $\xi(g) \neq 0$, $g - 1$ is a trivial invertible in $\mathbb{Z}[G]_\xi$, with, $(g-1)^{-1} = -\sum_{i=0}^{\infty} g^i$ if $\xi(g) > 0$, and $(g-1)^{-1} = g^{-1}\sum_{i=0}^{\infty} g^{-i}$ if $\xi(g) < 0$.

Special case. Assume that $\mathbb{Z}[G]$ has only trivial invertibles, ie of the form $\pm g$. This is true for instance if G is right-orderable, and conjectured to be true if G is torsion free (cf. [Seh 2003]). Then all the invertibles of $\mathbb{Z}[G]_\xi$ are such that $m_\xi(\lambda)$ is invertible in $\mathbb{Z}[G]$, thus are trivial.

Trivially invertible matrices. If S is a matrix in $\mathrm{M}_n(\mathbb{Z}[G]_\xi)$ such that $m_\xi(S) = \mathrm{Id}_n$, ie A can be written $S = \mathrm{Id}_n + A$ with every coefficient of A in $\mathbb{Z}[G]_{\xi,]0,+\infty[}$ (or $\mathrm{supp}(A) \subset]0,+\infty[$), then S is invertible in $\mathrm{M}_n(\mathbb{Z}[G]_\xi)$ with inverse $S^{-1} = \sum_{k=0}^{+\infty}(-1)^k A^k$ We define

$$(\Sigma_\xi)_n = \{S \in \mathrm{M}_n(\mathbb{Z}[G]) \mid \mathrm{supp}(S - \mathrm{Id}_n) \subset]0,+\infty[\} \ , \ \Sigma_\xi = \bigcup_{n\in\mathbb{N}} (\Sigma_\xi)_n.$$

1.5 Well-definedness of the dimension of modules

Proposition. *The Novikov ring $\mathbb{Z}[G]_\xi$ always admits a unital ring homomorphism to a field. Thus every matrix $A \in \mathrm{M}_{p,q}(\mathbb{Z}[G]_\xi)$ which is right invertible satisfies $p \leq q$, and every matrix which is right and left invertible is square. Equivalently, the dimension of a free and finitely generated module over $\mathbb{Z}[G]_\xi$ is well defined.*

Proof. The group $G/\ker\xi$ is nonzero, Abelian and torsion free, thus the ring $\mathbb{Z}[G/\ker\xi]$ is an integral domain. The natural ring homomorphism $f : \mathbb{Z}[G]_\xi \to F = \mathrm{Frac}(\mathbb{Z}[G/\ker\xi])$ proves the first assertion. This homomorphism extends obviously to matrices, and if $AB = \mathrm{Id}_p \in \mathrm{M}_p(\mathbb{Z}[G]_\xi)$ we have $f(A)f(B) = \mathrm{Id}_p \in \mathrm{M}_p(F)$, thus $p \leq q$.

Remark. This property is also true for any group ring $\mathbb{Z}[G]$, but the proof is not so easy. It is a consequence of the fact that $\mathbb{Z}[G]$ is stably finite, see the next section.

1.6 Direct and stable finiteness of Novikov rings

Let R be a unital ring. One says (cf. Lam 1999) that it is

- *directly finite*, or *von Neumann-finite*, or *Dedekind-finite*, if every element of R which is right invertible is invertible. Equivalently, $xy = 1$ implies $yx = 1$.
- *stably finite*, or *weakly finite*, if every matrix ring $\mathrm{M}_n(R)$ is directly finite.

I. Kaplansky has shown that every group ring is stably finite ([Kap] p.122-123, [Pas 1977] 1977 p.33-38, [E]). D.H. Kochloukova [Ko 2006] has shown that $\mathbb{Z}[G]_\xi$ is stably finite if G is finitely generated and ξ is of rank one. In fact, we have the

Proposition. *The Novikov ring $\mathbb{Z}[G]_\xi$ is always stably finite.*

Proof. We adapt the proof of Kaplansky. For $K > 1$, define

$$\mathbb{Z}[G]_\xi^K = \{\lambda = \sum_g n_g g \in \mathbb{Z}[G]_\xi \mid (\exists C)\ (\forall r \in \mathbb{N}) \sum_{r-1 \leq \xi(g) \leq r} |n_g| \leq CK^r\}.$$

Lemma. *(i) The subgroup $\mathbb{Z}[G]_\xi^K$ is a subring of $\mathbb{Z}[G]_\xi$.*

(ii) If $S = \mathrm{I}_n + A \in (\Sigma_\xi)_n \subset \mathrm{M}_n(\mathbb{Z}[G])$, ie all the coefficients of X are in $(\xi > 0)$, then $(\mathrm{I}_n - X)^{-1} = \sum_{k=0}^{\infty}(-1)^k A^k$ is in $\mathbb{Z}[G]_\xi^K$ for some $K > 1$.

Proof of the lemma. (i) If $\lambda = \sum n_g g$ and $\mu = \sum m_g g$ are in $\mathbb{Z}[G]_\xi^K$, their product $\lambda\mu = \sum p_g g$ with $p_g = \sum_{hk=g} n_h m_k$. Also, $n_g = m_g = 0$ if $\xi(g) \leq -N$ for some N. Thus

$$\begin{aligned}
\sum_{r-1 \leq \xi(g \leq r} |p_g| &\leq \sum_{s=-N}^{r} \sum_{rs-1 \leq \xi(h) \leq s} |p_h| \sum_{r-s-1 \leq \xi(k) \leq r-s+1} |q_k| \\
&\leq \sum_{s=-N}^{r} C_1 K^s (2C_2 K^{r-s+1}) \\
&= 2C_1C_2(r+N+1)K.K^r.
\end{aligned}$$

In fact $\mathbb{Z}[G]_\xi^K$ is not a subring, example with $G=\mathbb{Z}$: $\sum_{r=0}^{\infty}(2t)^r \notin \mathbb{Z}[G]_\xi^2$. One should either consider directly $\mathbb{Z}[G]_\xi^{exp} = \bigcup_{K>1} \mathbb{Z}[G]_\xi^K$, or redefine

$$\mathbb{Z}[G]_\xi^K = \{\lambda = \sum_g n_g g \in \mathbb{Z}[G]_\xi \mid (\exists C, m)\ (\forall r \in \mathbb{N}) \sum_{r-1\leq \xi(g) \leq r} |n_g| \leq C r^m K^r\}.$$

(ii)

The stable finiteness of $\mathbb{Z}[G]_\xi$ can be reduced to the stable finiteness of $\mathbb{Z}[G]_\xi^K$ for every K. Indeed, consider an identity $XY = \mathrm{Id}_n$ in $\mathrm{M}_n(\mathbb{Z}[G]_\xi)$. Without loss of generality, we can assume that the coefficients of Y belong to $\mathbb{Z}[G]_{\xi>0}$. For $r \in \mathbb{R}$, we have

$$X_{\xi\leq C} Y_{\xi \leq C} = \mathrm{Id}_n - X_{\xi\leq r}Y_{\xi>r} + X_{\xi>r}Y_{\xi\leq r} + X_{\xi>r}Y_{\xi>r}.$$

For C large enough, the right-hand-side belongs to Σ_ξ, thus is invertible in $\mathbb{Z}[G]_\xi$. Thus $X_{\xi\leq r}$ has a right inverse Z in $\mathrm{M}_n(\mathbb{Z}[G]_\xi)$, which moreover belongs to $\mathbb{Z}[G]_\xi^K$ for some K, and also has coefficients in $\mathbb{Z}[G]_{\xi>0}$. If $\mathbb{Z}[G]_\xi$ is stably finite, one also has $ZX_{\xi\leq r} = \mathrm{Id}_n$, thus $ZX - \mathrm{Id}_n$ has coefficients in $\mathbb{Z}[G]_{\xi>0}$, thus ZX is invertible in $\mathrm{M}_n(\mathbb{Z}[G]_\xi)$, and X is left invertible

Define the trace $\tau_n : \mathrm{M}_n(\mathbb{C}[G]_\xi^K) \to \mathbb{C}$ which associates to $A = (a_{i,j})$ the coefficient λ_1 in the decomposition $\mathrm{tr}(A) = \sum_{g\in G} \lambda_g$. The key point is "Kaplansky Positivity theorem":

Lemma. *A nontrivial idempotent $E \in \mathrm{M}_n(\mathbb{C}[G]_\xi^K)$ satisfies $0 < \mathrm{tr}(E) < n$.*

Proof of the lemma. We define the conjugacy $\overline{\sum_g \lambda_g g} = \sum_g \overline{\lambda} g^{-1}$, and the adjoint $(a_{i,j})^* = (\overline{a}_{j,i})$. Every matrix in $\mathrm{M}_n(\mathbb{C}[G]_\xi^K)$ can be uniquely written $\sum_{g\in G} A_g g$ with $A_g \in \mathrm{M}_n(\mathbb{C})$. We fix $C > K$, and we equip $\mathcal{H} = \mathrm{M}_n(\mathbb{C}[G]_\xi^K)$ with the Hermitian inner product

$$(\sum_g A_g g, \sum_g B_g g) = \sum_g \tau_n(AB^*) C^{-2\xi(g)},$$

the convergence being guaranteed by the exponential growth condition. This makes it a Hilbert space. Note that $\tau_n(\lambda) = (\lambda, 1)$. We denote by

$$||A|| = (A, A)^{\frac{1}{2}} = (AA^*, 1)^{\frac{1}{2}} = \tau_n(A)^{\frac{1}{2}},$$

the associated norm.

It suffices to prove $\tau_n(E) > 0$, since this will also apply to the nontrivial idempotent $\mathrm{Id}_n - E$. The subspace $E\mathcal{H}$ is complete, thus closed, thus there exists a unique $F = EA \in E\mathcal{H}$ which minimizes $||\mathrm{Id}_n - F||$. Since $E \neq \mathrm{Id}_n$, $\mathrm{Id}_n \notin E\mathcal{H}$ thus $F \neq 0$. Also, $(X, \mathrm{Id}_n - F) = 0$ for every $x \in E\mathcal{H}$. For $X = F$, this gives $\tau_n(F) = ||F||^2$. For $X = (\mathrm{Id}_n - F)EE^* = E(E^* - AE^*))$, we obtain

$$||E - FE||^2 = ((\mathrm{Id}_n - F)E, (\mathrm{Id}_n - F)E) = (E(\mathrm{Id}_n - F)EE^*, \mathrm{Id}_n - F) = 0.$$

Thus $E = FE$, and

$$\begin{aligned} \tau_n(E) &= \tau_n(FE) = \tau_n(EF) \\ &= \tau_n(F) \text{ since } F = EA \text{ and } E^2 = E \\ &= ||F||^2 > 0. \end{aligned}$$

Corollary. *Every subring $R \subset \mathbb{Z}[G]_\xi$ is stably finite.*

Question. Is $\Sigma_\xi^{-1}\mathbb{Z}[G]$ stably finite ? This would follow from the injectivity of φ. Note that, at least, the dimension of modules on $\Sigma_\xi^{-1}\mathbb{Z}[G]$ which are free of finite rank is well defined, because of the ring homomorphism $\Sigma_\xi^{-1}\mathbb{Z}[G]$ to $\mathbb{Z}[G]_\xi$.

Remark. This implies that a (left or right) linear map $u : (\mathbb{Z}[G]_\xi)^m \to (\mathbb{Z}[G]_\xi)^n$ can be onto only if $m \geq a$. Actually, it suffices to remark that the Abelianization $u^{ab} : (\mathbb{Z}[G/G']_{\bar{\xi}})^m \to (\mathbb{Z}[G/G']_{\bar{\xi}})^n$ is still surjective, and that $\mathbb{Z}[G/G']_{\bar{\xi}}$ is an commutative integral domain, thus contained in a field K, thus we get a linear surjective map $K^m \to K^n$.

1.7 Rings equivalent to the Novikov ring

Certain rings appear naturally which are in some sense (to be defined below) equivalent to the Novikov ring. The first three below are subrings, the last one maps to it:

- the subring $\mathbb{Z}[G]_\xi^\Sigma$ generated by $\mathbb{Z}[G]$ and coefficients of matrices in Σ_ξ and their inverses
- the *stable* Novikov ring, defined when $H^1(G,\mathbb{R})$ has finite dimension: equip its dual $H_1(G,\mathbb{R}) = G^{ab}/\text{Torsion}$ with a norm associated to a finite generating system, and define

$$\mathbb{Z}[G]_\xi^{stable} = \{\lambda \in \mathbb{Z}[[G]] \mid (\exists\ C_1, C_2 > 0)\ \xi(g) \geq r_1||[g]|| - C_2 \text{ if } g \in \text{supp}(\lambda)\}$$

  Then ($\lambda \in \mathbb{Z}[G]_\xi^{stable}$) is equivalent to ($\lambda \in \mathbb{Z}[G]_\eta$ for η close enough to ξ), a property that Novikov included in his original definition.
- the *exponential* Novikov ring $\mathbb{Z}[G]_\xi^{exp} = \bigcup_{K>1} \mathbb{Z}[G]_\xi^K$
- the *Cohn localization*, introduced in this context by [Fa]. Consider all Σ_ξ*-inverting* rings, ie pairs (R,ρ) where R is a ring and $\rho : \mathbb{Z}[G] \to R$ is a ring homomorphism sending every matrix in Σ_ξ to an invertible matrix. Examples for R: $\mathbb{Z}[G]_\xi$, $\mathbb{Z}[G]_\xi^\Sigma$, $\mathbb{Z}[G]_\xi^{stable}$, $\mathbb{Z}[G]_\xi^{exp}$, ρ being the inclusion. By Cohn 1985 (chapter 7, Theorem 2.1), there exists a unique (up to a unique isomorphism) universal Σ_ξ-inverting ring $(\Sigma_\xi^{-1}\mathbb{Z}[G], \rho_0)$, and for every Σ-inverting ring (R,ρ), ρ factors through ρ_0. In particular, we have a commutative diagram

$$\begin{array}{ccccc} \mathbb{Z}[G] & & \xrightarrow{\ i\ } & & \mathbb{Z}[G]_\xi \\ & \rho_0 \searrow & & \nearrow \varphi & \\ & & \Sigma_\xi^{-1}\mathbb{Z}[G] & & \end{array}$$

  Since i is injective, ρ_0 is injective, thus $\Sigma_\xi^{-1}\mathbb{Z}[G]$ can be viewed as an extension of $\mathbb{Z}[G]$. However, it is not known whether φ is injective. This is obviously the case if G is Abelian, since then it coincides with the usual localization $S_\xi^{-1}\mathbb{Z}[G]$. It is also true if $\mathbb{Z}[G]$ embeds into a skew field, in particular if G is right-orderable, and thus conjecturally if G is torsion free.

Properties. *One has*

$$\varphi(\Sigma_\xi^{-1}\mathbb{Z}[G]) = \mathbb{Z}[G]_\xi^\Sigma \subset \mathbb{Z}[G]_\xi^{stable} \cap \mathbb{Z}[G]^{\text{exp}}.$$

Proof. The inclusion on the right is obvious. For the equality on the left, extending i, ρ_0 and φ to matrices, $\rho_0(\Sigma_\xi)$ is contained in the invertible matrices of $\Sigma_\xi^{-1}\mathbb{Z}[G]$, thus $\varphi(\Sigma_\xi^{-1}\mathbb{Z}[G])$ contains the coefficients of matrices in $\varphi \circ \rho_0(\Sigma_\xi) = \Sigma_\xi$ and their inverses, ie $\mathbb{Z}[G]^\Sigma_\xi$. Also, $\varphi^{-1}(\mathbb{Z}[G]^\Sigma_\xi)$ is a universal Σ_ξ-inverting ring, thus $\varphi^{-1}(\mathbb{Z}[G]^\Sigma_\xi) = \Sigma_\xi^{-1}\mathbb{Z}[G]$, thus $\varphi(\Sigma_\xi^{-1}\mathbb{Z}[G]) = \mathbb{Z}[G]^\Sigma_\xi$.

The following Proposition gives the sense it which all Σ_ξ-inverting rings are equivalent to the Novikov ring.

Proposition. *Let (R, ρ) be a Σ_ξ-inverting ring, with $\varphi : R \to \mathbb{Z}[G]_\xi$ the natural ring homomorphism. Let C_* be a free and finitely generated R-complex such that $\mathbb{Z}[G]_\xi \otimes_\varphi C_*$ is acyclic up to degree m. Then C_* is acyclic up to degree m.*

Proof. By freeness, the acyclicity is equivalent to the existence of a $\mathbb{Z}[G]_\xi$-linear homotopy $h : \mathrm{Id}_{C_*} \simeq 0$, ie a degree one linear map h, defined up to degree m, ie an equation $\partial h + h\partial = \mathrm{Id}$.

Choose a basis B_i of C_i for all $i \in \mathbb{N}$. Then $\partial_i : C_i \to C_{i-1}$ is given by right multiplication with a matrix $D_i \in \mathrm{M}_{B_i,B_{i-1}}(R)$. (This matrix is finite if $i \leq m$, and has almost null columns always). The homotopy is given by right multiplication with $H_i \in \mathrm{M}_{B_i,B_{i+1}}(\mathbb{Z}[G]_\xi)$ (finite for $i < m$), with almost null columns always) such that

$$(\forall i) \qquad H_i D_{i+1} + D_i H_{i-1} = \mathrm{Id}_{B_i}.$$

By convention, H_i and D_i are zero if $i < 0$. We want to find $\widetilde{H}_i$ with the same properties as H_i but with coefficients in R. Denote $(H_i)_{\leq a} \in \mathrm{M}_{B_i,B_{i+1}}(\mathbb{Z}[G])$ a truncation of H_i below some level $\xi = a$. By finiteness, if a is large, we have

$$(\forall i \leq a) \qquad (H_i)_{\leq a} D_{i+1} + D_i (H_{i-1})_{\leq a} = A_i \in (\Sigma_\xi)_{B_i}.$$

Since R is Σ_ξ-inverting, A_i is invertible over R. Define

$$\widetilde{H}_i = A_i^{-1} \widehat{H}_i \in \mathrm{M}_{B_i,B_{i+1}}(R).$$

Then, using $D_{i+1}D_i = D_i D_{i-1} = 0$, we have $A_i D_i = D_i \widehat{H}_{i-1} D_i = D_i A_{i-1}$, thus $D_i A_{i-1}^{-1} = A_i^{-1} D_i$, so that

$$H_i D_{i+1} + D_i \widetilde{H}_{i-1} = A_i^{-1}\big((H_i)_{\leq a} D_{i+1} + D_i (H_{i-1})_{\leq a}\big) = \mathrm{Id}_{B_i}.$$

This proves the Proposition.

Remark. The restriction to finitely generated complexes is necessary for the proof, as one easily sees. If $G = \mathbb{Z}$ so that $\mathbb{Z}[\mathbb{Z}] \approx \mathbb{Z}[t, t^{-1}]$, $\xi = \mathrm{Id}$, consider the length-one complex $C_1 = \mathbb{Z}[\mathbb{Z}]_\xi^{(\mathbb{N}^*)} \to \mathbb{Z}[\mathbb{Z}]_\xi^{(\mathbb{N}^*)} = C_0$ with ∂ represented by the matrix $D = \mathrm{diag}(1 - t^n)$. The D is invertible over $\mathbb{Z}[\mathbb{Z}]_\xi$, but no matter how high we truncate $H_0 = D^{-1}$, $(H_0)_{\leq a} D$ is n

However, in this example D is invertible over R if R is Σ_ξ-inverting.

1.8 Faithful flatness

Recall that if R is a subring of S, S is *faithfully flat* over S if the following $S \otimes_R$ preserves exact sequences and non-exact sequences (cf. [Bou] p.44, [Mat] p.45). This is equivalent to the *linear extension property* ([Bou] p.54): every solution over R of an linear system with coefficients in S, is the sum of a solution over S and a linear combination over R of solutions over S of the associated homogeneous system.

Question. If R is a Σ_ξ-inverting ring and $\varphi : R \to \mathbb{Z}[G]_\xi$ is injective, is $\mathbb{Z}[G]_\xi$ faithfully flat over R ? This is a much stronger property than the one described in the proposition of the preceding section. In section 9.6, we shall see that this true if G is Abelian.

2 Novikov complex, Novikov homology

2.1 Definitions

Let (G, ξ) be as in section 1, and let $C_* = \bigoplus_{i=0}^{+\infty} C_i$ be a chain complex of free left $\mathbb{Z}[G]$-modules. By definition, the *Novikov complex* is obtained by extending the coefficients from $\mathbb{Z}[G]$ to $\mathbb{Z}[G]_\xi$:

$$C_*(\xi) := \mathbb{Z}[G]_\xi \otimes_{\mathbb{Z}[G]} C = \bigoplus_{i=0}^{+\infty} C_i(\xi)\ ,\ \partial_i^\xi = \mathrm{Id}_{\mathbb{Z}[G]_\xi} \otimes_{\mathbb{Z}[G]} \partial_i.$$

Its homology is the *Novikov homology*:

$$H(C_*, \xi) := H(C(\xi)) = \bigoplus_{i=0}^{+\infty} H_i(C_*, \xi)\ ,\ H_i(C_*, \xi) = \frac{\ker \partial_i^\xi}{\mathrm{im}\ \partial_{i+1}^\xi}.$$

If we fix the basiss B_i, C_i is isomorphic to $\mathbb{Z}[G]^{(B_i)}$. Every element of C_i is represented by a line indexed by B_i, which is almost null. The differential $\partial_i : C_i \to C_{i-1}$ is the multiplication on the right by a matrix $D_i \in \mathrm{M}_{B_i, B_{i-1}}(\mathbb{Z}[G])$ with every column almost null. Then $H(C_*, \xi)$ is the homology of the complex $\bigoplus \mathbb{Z}[G]_\xi^{(B_)}$ with the same differentials, viewed as matrices with coefficients in $\mathbb{Z}[G]_\xi$.

It is clear that $H_j(C_*, \xi)$ is naturally a left $\mathbb{Z}[G]_\xi$-module. And also that $H(C_*, \xi)$ only depends on the homotopy type of C_* as a $\mathbb{Z}[G]$-complex. More generally, $H_i(C_*, \xi)$ only depends on the homotopy type of C_* truncated below the degree $i + 1$.

2.2 Novikov homology of a group

Let G be a group and $\xi \in \mathrm{Hom}(G, \mathbb{R}) \setminus \{0\}$. Let $C_* \to \mathbb{Z}$ be a partial free resolution of $\mathbb{Z}$ over $\mathbb{Z}[G]$ of length $m+1$. Since all such resolutions are homotopy equivalent, for $j \leq m$, $H_j(C_*, \xi)$ only depends on (G, ξ) up to a unique isomorphism. By definition, $H_j(G, \xi) = H_i(C_*, \xi)$. Equivalently (cf. [Br 1982]), $H(G, \xi)$ is the homology of G with coefficients in the ring $\mathbb{Z}[G]_\xi$ viewed as a left $\mathbb{Z}[G$-module :

$$H(G, \xi) = H(G; \mathbb{Z}[G]_\xi).$$

One can for instance use the standard (or bar) resolution ($C_n(G) = \mathbb{Z}[G]^{(G^n)}$), with generators $[g_1 \mid \cdots \mid g_n]$ and differentials

$$\partial_n[g_1 \mid \cdots |g_n] = g_1[g_1| \cdots |g_n] + \sum_{i=1}^{n-1} (-1)^i [g_1| \cdots |g_i g_{i+1}| \cdots g_n] + (-1)^n [g_1| \cdots |g_{n-1}].$$

In particular, identifying $C_0(G) = \mathbb{Z}[G]$, one has $\partial_1[g] = g - 1$ and $\partial_2[g|h] = g[h] - [gh] + [g]$.

Then $H(G, \xi)$ is the homology of the complex ($C_n(G)_\xi = \mathbb{Z}[G]_\xi^n$), with the same basiss and the same differentials. In particular:

$$H_0(G, \xi) = \mathbb{Z}[G]_\xi / \langle g - 1 | g \in G \rangle.$$

Since there exists g such that $\xi(g) \neq 0$, $g - 1$ is invertible in $\mathbb{Z}[G]_\xi$ thus $H_0(G, \xi) = 0$.

2.3 Intermediate Novikov homologies

Let R be a ring equipped with a ring homomorphism $\rho : \mathbb{Z}[G]_\xi \to R$. One can define the complex

$$(C_R)_* = R \otimes_{\mathbb{Z}[G]_\xi} C(\xi) = R \otimes_{\mathbb{Z}[G]} C_*$$

and its homology $H_*((C_R)_*)$. Clearly, if $H_*(X,\xi) = 0$ one has $H_*((C_R)_*) = 0$, but not conversely.

In particular, this applies if $R = \mathbb{Z}[(G/H)]_{\overline{\xi}}$, where H is a subgroup of G containing $\ker\xi$ and $\overline{\xi}$ denotes the induced homomorphism $G/\ker\xi \to \mathbb{R}$. The first example is the original Novikov homology, which is obtained by taking 1) $H = \ker\xi$ or 2) H the inverse image of the torsion by the homomorphism $G \to G/G'$.

In case 1), we shall call $H_*((C_R)_*)$ the *Abelian Novikov homology* and denote it by $H_*^{ab}(C_*,\xi)$. Note that since $\overline{\xi}$ is injective, if G is finitely generated $\mathbb{Z}[G/\ker\xi]_{\overline{\xi}}$ is a principal ideal ring. In case 2), we shall call $H(C_R)$ the *universal Abelian Novikov homology* and denote it by $H_*^{ab,univ}(C_*,\xi)$.

2.4 Support of chains and interpretation with inverse limits

Assume that C_* is a free complex over $\mathbb{Z}[G]$ of length $m+1$ for some $m \in \mathbb{N}$, such that each C_j has a finite basis B_j. Using these basiss, we define the *support* $\mathrm{supp}(c) \subset G$ of a chain $c \in C_j$ by induction on j as follows (slightly modifying [Bi-Re], 2.5):

- if $c \in C_0$, $c = \sum_{g\in G, e\in B_0} n_{g,e}e$, $\mathrm{supp}(c) := \{g \mid (\exists e \in B_0)\ n_{g,e} \neq 0\}$
- if $j > 0$ and supp has already been defined on C_{j-1}, we set

$$\mathrm{supp}(e) := \{1\} \cup \mathrm{supp}(\partial e) \text{ if } e \in B_j$$
$$\mathrm{supp}\Big(\sum_{g\in G, e\in B_j} n_{g,e}e\Big) := \bigcup_{n_{g,e}\neq 0} g\ \mathrm{supp}(e).$$

This definition implies that for every $E \subset G$, the chains with support in E form a subcomplex $C(E)$, over the ring $\mathbb{Z}$ or $\mathbb{Z}[\ker\pi]$. This complex is free over $\mathbb{Z}[\ker\pi]$, and finite if E is finite.

In particular, $C(\{\xi \geq a\})$ is a subcomplex over $\mathbb{Z}$ or $\mathbb{Z}[\ker\pi]$. In the topological case $C* = C_*(\widetilde{X})$, $C(\{\xi \geq a\}) = C_*(\widetilde{X}_{\xi\geq a})$. Then $(C/C(\{\xi \geq a\}))$, $a \in \mathbb{R}$, is naturally an inverse system of complexes, and there are natural isomorphisms (over $\mathbb{Z}$ or $\mathbb{Z}[\ker\pi]$):

$$C(\xi) \simeq \lim_{a\to+\infty}{}^0 (C/C(\{\xi \geq a\})).$$

Since the maps $C/C(\{\xi \geq a\}) \to C/C(\xi \geq b)$ are onto if $a \geq b$, a general result on inverse limits (cf [Mas], p. 411) gives the

Corollary. *For every $j \leq m$, there is a short exact sequence*

$$0 \to \lim_{a\to+\infty}{}^1 H_{j+1}(C, C(\{\xi \geq a\})) \to H_j(C_*,\xi) \to \lim_{a\to+\infty}{}^0 H_j(C_* C(\{\xi \geq a\})) \to 0.$$

We recall that, if $(A_a)_{a\to+\infty}$, $f_{a,b} : A_a \to A_b, a \geq b$ is an inverse system of Abelian groups, we have

$$\lim_{a\to+\infty}{}^1 A_a = \lim_{n\to+\infty}{}^1 A_n = \Big(\prod_{n\in\mathbb{Z}} A_n\Big)_{(x_n - f_{n+1}^n(x_{n+1}))_n},$$

and that if $0 \to \{A_a\} \to \{B_a\} \to \{C_a\} \to 0$ is a short exact sequence, one has an exact sequence

$$0 \to \lim^0 A_a \to \lim^0 B_a \to \lim^0 C_a \to \lim^1 A_a \to \lim^1 B_a \to \lim^1 C_a \to 0.$$

Special cases. 1) Assume that C_* is a partial resolution of length $m+1$ of $\mathbb{Z}$ over $\mathbb{Z}[G]$, ie $H_0(C) = \mathbb{Z}$ and $H_j(C) = 0$ for $1 \leq j < m$. Then for all $j = 1, \cdots, m$ and $r \in \mathbb{R}$, we have $H_j(C_* C(\{\xi \geq a\})) \simeq \widetilde{H}_{j-1}(C(\{\xi \geq a\}))$. Thus for every $j \leq m$ we get the exact sequence

$$0 \to \lim_{a\to+\infty}{}^1 \widetilde{H}_j(C(\{\xi \geq a\})) \to H_j(C_*, \xi) = H_j(G, \xi) \to \lim_{a\to+\infty}{}^0 \widetilde{H}_{j-1}(C(\{\xi \geq a\})) \to 0.$$

In fact, we have in this case a more concrete characterization of the acyclicity of $C(\xi)$:

Proposition. *Assume that $C_{\leq m} \to \mathbb{Z}$ is a partial resolution of length $m+1$ of $\mathbb{Z}$ over $\mathbb{Z}[G]$ and that C_j is free and finitely generated for $j \leq m$. Then the following assertions are equivalent:*

(i) $H_j(G, \xi) = 0$ *for* $j \leq m$

(ii) There exists $r > 0$ such that the inclusion $C(\{\xi \geq r\}) \to C(\xi \geq 0\})$ induces zero in homology of degree $\leq m-1$.

Proof. $(i) \Rightarrow (ii)$. (i) is equivalent to the existence of a $\mathbb{Z}[G]$-homotopy $h_\xi : \mathrm{Id} \simeq \varphi_\xi$ such that $\xi|\mathrm{supp}\ \varphi_\xi e > \max(\xi|\mathrm{supp}(e))$ **(cf. ?)**. Set $r = \max\{||h_\xi e|| : e \in B_0 \cup \cdots \cup B_m\}$ and let z be a cycle in $C_j(\{\xi \geq r\}$ for some $j \leq m-1$. If $j = 0$, we assume that $\varepsilon(z) = 0$. Since $C_{\leq m} \to \mathbb{Z}$ is a partial resolution, there exists $c \in C_j$ such that $\partial c = z$. Then for every $i \in \mathbb{N}$ we have $\partial h_\xi \varphi_\xi^i = -h_\xi \partial \varphi_\xi^i + \varphi_\xi^i - \varphi_\xi^{i+1}$, thus for every $n \in \mathbb{N}$ we have

$$\partial(\varphi_\xi^n c + h_\xi(\mathrm{Id} + \varphi_\xi + \cdots + \varphi_\xi^{n-1})z) = \varphi_\xi^n z - h_\xi \partial(\mathrm{Id} + \varphi_\xi + \cdots + \varphi_\xi^{n-1})z) + (z - \varphi_\xi^n z) = z.$$

For n large enough, $\varphi_\xi^n c \in C_j(\{\xi \geq 0\})$, thus $\varphi_\xi^n c + h_\xi(\mathrm{Id} + \varphi_\xi + \cdots + \varphi_\xi^{n-1})z \in C_j(\{\xi \geq 0\})$, thus $[z] = 0$ in $H_j(C(G_{\xi \geq 0})$, qed.

$(ii) \Rightarrow (i)$. We construct h_ξ by induction on j, such that, for $\varphi_\xi = \mathrm{Id} - \partial h_\xi - h_\xi \partial$, we have

$$(\forall j \leq m)(\forall j \in B_j) \ \ \xi|\mathrm{supp}\ \varphi_\xi > \max(\xi|\mathrm{supp}(e)) + (m-j)r.$$

We can assume that $h_\xi | C_{k<j}$ has already been constructed, with $h_\xi | C_{-1} = 0$. Let $z \in B_j$, then $\partial(e - h_\xi \partial e) = \varphi_\xi \partial e$ is a cycle in $C_{j-1}(\{\xi > \max(\xi|\mathrm{supp}(e) + (m-(j-1)r\})$, thus it admits a primitive $c \in C_j$ such that

$$\xi|\mathrm{supp}(c) > \max(\xi|\mathrm{supp}(e)) + m - (j-1)r - r = \max(\xi|\mathrm{supp}(e)) + (m-j)r.$$

Then $e - h_\xi \partial e - c$ is a cycle, and since $H_j(C) = 0$ it has a primitive in C_{j+1}. We define $h_\xi e$ to be a such a primitive. Then

$$\varphi_\xi e = e - \partial h_\xi e - h_\xi \partial e = e - (e - h_\xi \partial e - c) - h_\xi \partial e = c,$$

which finishes the proof.

3 Geometric closed one-forms and topological Novikov homology

3.1 Definition. Let (X, x_0) be a connected pointed CW-complex, with universal covering $\widetilde{X}$. We denote $\pi_1(X, x_0) = G$ and identify it with $\mathrm{Aut}(\widetilde{X}|X) = G$, acting on $\widetilde{X}$ on the left. Let $\xi \in \mathrm{Hom}(G, \mathbb{R}) \setminus \{0\}$. Following Levitt 1994, we say that a function $\widetilde{f} : \widetilde{X} \to \mathbb{R}$, defined modulo a constant, is a *geometric closed one-form [non exact] on* X*, with cohomology class* ξ, if $\widetilde{f}(g.x) = \widetilde{f}(x) + \xi(g)$.

If ξ is given, it is easy to produce such a function. One first defines a section $s : X^{(0)} \to \widetilde{X}^{(0)}$ and defines $\widetilde{f}$ arbitrarily on $s(X^{(0)})$. Then one extends it by equivariance on $\widetilde{X}^{(0)}$, then one extends ξ equivariantly to each $\widetilde{X}^{(k)}$.

Harmonic one-forms. A geometric closed one-form $\widetilde{f}$ is *harmonic* if $\widetilde{f} \circ \sigma : \Delta^q \to \mathbb{R}$ is a Harvmonic function for every parameterized q-cell $\sigma : \Delta^q \to \widetilde{X}$. If $q = 1$ this is the same as affine. Clearly, every morphism ξ is represented by a harmonic closed one-form, which is uniquely defined by its restriction to $s(X^{(0)})$. In particular, if X has a unique vertex this form is unique up to an additive constant.

In particular, if $X = X(\mathcal{P})$ is the two-complex associated to a presentation $\mathcal{P} = \langle S \mid R \rangle$, so that $\widetilde{X}^{(1)}$ can be identified with the Cayley graph $\Gamma(G, S)$, there is a well-defined harmonic one-form defined on X, which coincides on $\Gamma(G, S)$ with the extension of ξ given before.

3.2 Topological Novikov homology

Let $(X, G, \widetilde{f}, \xi)$ be a geometric closed one-form. Consider $C_*(\widetilde{X})$, the cellular chain complex of the universal covering. It is a left $\mathbb{Z}[\pi_1(X)]$-complex, which is free with generators of dimension i in one-to-one correspondence with the i-cells of X.

Applying the definitions of Section 2 to $G = \pi_1(X)$, $C* = C_*(\widetilde{X})$, we obtain the group ring $\mathbb{Z}[\pi_1(X)]$, the Novikov ring $\mathbb{Z}[\pi_1(X)]_\xi$, the Novikov complex and the Novikov homology:

$$C(X, \xi) := \mathbb{Z}[\pi_1(X)]_\xi \otimes_{\pi_1(X)} C_*(\widetilde{X})$$
$$H(X, \xi) := H(C(X, \xi)).$$

3.3 First properties

(i) $H(X, \xi)$ *only depends on the homotopy type of* (X, ξ).

(ii) $H_0(X, \xi) = 0$.

(iii) $H_1(X, \xi) = H_1(G, \xi)$.

(iv) If $\pi_j(X) = 0$ *for* $2 \leq j \leq i$*, then* $H_i(X, \xi) = H_i(G, \xi)$.

Proof. (i) This follows from the fact that $\pi_1(X)$ and the homotopy type of $C_*(\widetilde{X})$ as a complex over $\mathbb{Z}[\pi_1(X)]$ only depen on the homotopy type of X.

(ii) We can assume that X has only one vertex x_0. The map ∂_1 sends the generator e_σ associated to the oriented edge σ to $g_\sigma - 1$, where $g_\sigma \in G$ is such that the lift of σ starting from $\widetilde{x}_0$ arrives at $g.\widetilde{x}_0$. Since $\xi \neq 0$ and the g_σ generate G, there exists σ such that $\xi(g_\sigma) \neq 0$, thus $\partial_1 e_\sigma$ is invertible in $\mathbb{Z}[G]_\xi$. Thus ∂_1 is onto, which proves $H_0(X, \xi) = 0$.

(iii) This follows from the fact that $C_2(\widetilde{X}) \xrightarrow{\partial_2} C_1(\widetilde{X}) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z}$ is a partial free resolution of length 2 of $\mathbb{Z}$ over $\mathbb{Z}[\pi_1(X)]$.

(iv) Similarly, this follows from the fact that, by Hurewicz,

$$C_{m+1}(\widetilde{X}) \xrightarrow{\partial_{m+1}} \cdots \xrightarrow{\partial_2} C_1(\widetilde{X}) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z}$$

is a partial free resolution of length $m + 1$ of $\mathbb{Z}$ over $\mathbb{Z}[\pi_1(X)]$.

3.4 Novikov homology and Poincaré duality

Let X be a Poincaré duality complex of dimension n. This means that X is a finite CW-complex of dimension n and there exists a homotopy equivalence

$$\varphi : C_*(\widetilde{X}) \to C^* = \mathrm{Hom}_{\mathbb{Z}[\pi_1(X)]}(C_*(\widetilde{X}), \mathbb{Z}[\pi_1(X)]).$$

Proposition.

Question: is there a relation between Novikov homology and the ℓ^1-norm on $H_{n-1}(C, \mathbb{R}) \simeq H^1(C, \mathbb{R})$? Note that if $n = 3$ the answer is (indirectly) yes in view of section 8 and the equality (up to a factor 1/2) of Thurston and Gromov norms on $H_2(M, \mathbb{R})$ ([G]).

4. Classes with vanishing Novikov homology

4.1 Computational characterizations

Recall that if C_*, C'_* are complexes over the same ring R, a *homotopy* $h : \psi \simeq \varphi$ (defined up degree m) is an equation $\psi - \varphi = h \circ \partial + \partial \circ h$ where ψ, φ are chain maps of degree zero (defined up degree m) and h is a linear map of degree one (defined up degree m). Note that if ψ is a chain map, such an equation implies that φ is also a chain map. We shall use mostly the case where $\psi = \mathrm{Id}$: then any linear map $h : C \to C'$ of degree one gives rise to a homotopy $\mathrm{Id} \simeq \varphi$ where $\varphi = \mathrm{Id} - (\partial h + h\partial)$.

If C_* is a free complex, its homology vanishes if and only there exists a homotopy $\mathrm{Id}_{C_*} \simeq 0$, ie $\mathrm{Id} = h\partial + \partial h$. Taking basiss of the C_i, this translates into matrix equalities. Thus we get the following proposition:

Proposition 1. *The following are equivalent:*

(i) $H_i(C_*, \xi) = 0$ *for* $i \leq m$.

(ii) There exists a $\mathbb{Z}[G]_\xi$*-linear homotopy* $h : \mathrm{Id} \simeq 0$ *defined in degrees up to* m*, ie there exists* $\mathbb{Z}[G]_\xi$*-linear maps* $h_i : C_i \to C_{i+1}$ *such that*

$$\partial_1 h_0 = \mathrm{Id}_0$$
$$\partial_{i+1} h_i + h_{i-1}\partial_i = \mathrm{Id}_{C_i} \ , \ l \leq i \leq m.$$

(ii') Let B_i *a basis of* C_i*, so that the left* $\mathbb{Z}[G]$*-linear map* $\partial_i : C_i \to C_{i-1}$ *is represented by the multiplication on the right with the matrix* $D_i \in \mathrm{M}_{B_i,B_{i-1}}(\mathbb{Z}[G])$ *(with almost null lines). Then there exist matrices*

$$H_i \in \mathrm{M}_{B_i,B_{i+1}}(\mathbb{Z}[G]_\xi) \ , \ i \leq m,$$

such that

$$H_0 D_1 = \mathrm{Id}_0$$
$$H_i D_{i+1} + D_i H_{i-1} = \mathrm{Id}_{B_i} \ , \ l \leq i \leq m.$$

In the case where the basiss are finite up to degree m, we give a version of this proposition which does not involve the Novikov ring. Recall that $(\Sigma_\xi)_n \subset \mathrm{M}_n(\mathbb{Z}[G])$ is the set of matrices $\mathrm{Id}_n + \{\xi > 0\}$, and that they are invertible over $\mathbb{Z}[G]_\xi$.

Proposition 2. *Assume that* C_i *has a finite basis* B_i *for* $0 \leq i \leq m$*. The following are equivalent:*

(i) $H_i(C_*, \xi) = 0$ *for* $i \leq m$.

(ii) There exists a $\mathbb{Z}[G]$*-linear homotopy* $\mathrm{Id} \simeq \varphi$ *such that* $\xi_{|\mathrm{supp}(\varphi_v(e))} > \max(\xi_{|\mathrm{supp}(e)})$ *for every* $i \leq m$ *and every* $e \in B_i$.

(ii') Let $D_i \in \mathrm{M}_{B_i,B_{i-1}}(\mathbb{Z}[G])$ *be the matrix representing* ∂_i*. There exist matrices*

$$\widehat{H}_i \in \mathrm{M}_{B_i,B_{i+1}}(\mathbb{Z}[G]) \ , \ 0 \leq i \leq m,$$

and $A_i \in (\Sigma_\xi)_{B_i}$ *such that*

$$H_0 D_1 = A_0$$
$$\widehat{H}_i D_{i+1} + D_i \widehat{H}_{i-1} = A_i \ , \ 0 \leq i \leq m.$$

Proof. Note that (ii') is clearly a matrix translation of (ii), thus $(ii) \Leftrightarrow (ii')$. The equivalence between (i) and (ii') uses the argument of the proof of the Proposition in 1.7.

$(i) \Rightarrow (ii')$. Let H_i be given by Proposition 1. Let $\widehat{H}_i = (H_i)_{\xi \leq a}$ the truncation up to level a. For a large enough, all coefficients of $D_{i+1}(\widehat{H}_i - H_i)$ and $(\widehat{H}_{i-1} - H_{i-1})D_i$ belong to $\mathbb{Z}[G]_{\xi>0}$, thus (ii') is satisfied.

$(ii') \Rightarrow (i)$. The proof of 1.7 shows that if we set $H_i = A_i^{-1}\widehat{H}_i$, (ii) is satisfied:

$$\begin{aligned} H_i D_{i+1} + D_i H_{i-1} &= A_i^{-1}\widehat{H}_i D_{i+1} + D_i A_{i-1}^{-1}\widehat{H}_i \\ &= A_i^{-1}\widehat{H}_i D_{i+1} + A_i^{-1} D_i \widehat{H}_i \\ &= A_i^{-1}(\widehat{H}_i D_{i+1} + D_i \widehat{H}_i) = \mathrm{Id}_{B_i}. \end{aligned}$$

4.2 The topological case : computation in low degrees

Definition. Let X be a connected CW-complex, with cells $C* = \bigcup_{i\geq 0} C^{(i)}$ and let $\pi : \widetilde{X} \to X$ be its universal covering, with $C_*(\widetilde{X})$ its cellular chain complex. We identify $\pi_1(X) = \mathrm{Aut}(\widetilde{X}|X)$, thus we view $C_*(\widetilde{X})$ as a complex over $\mathbb{Z}[\pi_1 X]$. We also identify $H^1(X;\mathbb{R}) = \mathrm{Hom}(\pi_1(X), \mathbb{R})$.

Let ξ be a nonzero element of $H^1(X;\mathbb{R})$. Applying the definitions of section 1 to $G = \pi_1(X)$, $C* = C_*(\widetilde{X})$, we obtain the group ring $\mathbb{Z}[\pi_1(X)]$, the Novikov ring $\mathbb{Z}[\pi_1(X)]_\xi$, the Novikov complex and the *topological Novikov homology*:

$$\begin{aligned} C_*(\widetilde{X},\xi) &:= \mathbb{Z}[\pi_1(X)]_\xi \otimes_{\pi_1(X)} C_*(\widetilde{X}) \\ H(X,\xi) &:= H(C_*(\widetilde{X},\xi)). \end{aligned}$$

Property. By standard homological algebra, the Novikov homology $H(X,\xi)$ only depends on the homotopy type of (X,ξ).

Computations. Orienting the cells, the set B_i of i-cells becomes a basis (e_i) of $C_i(\widetilde{X})$ over $\mathbb{Z}[\pi_1(X)]$. Then ∂_i is represented by right multiplication with a matrix

$$D_i \in \mathrm{M}_{B_i,B_{i-1}}(\mathbb{Z}[G])$$

(with almost-null columns). Thus we have

$$H_i(X,\xi) = \frac{\{L \in \mathrm{M}_{1,B_i}(\mathbb{Z}[\pi_1 X]_\xi) \mid LD_i = 0\}}{\mathrm{M}_{B_{i+1},B_i}(\mathbb{Z}[\pi_1 X]_\xi)D_{i+1}}.$$

Without loss of generality, we can assume that X has a unique vertex, thus $B_0 = \{e_0\}$.

1) Let us first compute D_0. To each one-cell $e_i \in B_1$ of X we associate a generator s_i of a free group $F = F(s_i)_{i\in I}$, with a natural surjection $\pi : F \to G$. We denote also s_i the image $\pi(s_i) \in G$, even if $\pi|S$ is not injective. We denote also by π the induced morphism $\mathbb{Z}[F] \to \mathbb{Z}[G]$.

Identifying $C_0 = \mathbb{Z}[G]$, we have $\partial_1(e_i) = x_i - 1$, thus $D_1 = (s_i - 1)_{i\in B_1}$ (line matrix with columns indexed by B_1).

Corollary. *(i) One always have $H_0(X,\xi) = 0$.*

(ii) Let $s \in S$ be such that $\xi(s) \neq 0$. Then $H_1(X,\xi) = 0$ if and only if the matrix $(D_2)_s$ obtained by deleting the line corresponding to s is right invertible.

Proof. (i) There exists i such that $\xi(s_i) \neq 0$, thus $s_i - 1$ is invertible in $\mathbb{Z}[G]_\xi$, thus D_1 is left invertible over $\mathbb{Z}[G]_\xi$.

(ii) The kernel of $(\partial_1)_\xi$ consists of vectors $\sum_{t\in S} \lambda_t e_t$ such that $\sum_t \lambda_t(t-1) = 0$ ie $\lambda_s = -\sum_{t\in S\setminus\{s\}} \lambda_t(t-1)(s-1)^{-1}$. It is thus generated by the vectors $e_t - (t-1)(s-1)^{-1}e_s$, $t \in S \setminus \{s\}$. Thus $H_1(X,\xi) = 0$ if and only if every such vector is in the image of $(\partial_2)_\xi$. Finally, $e_t - (t-1)(s-1)^{-1}e_s = \partial_2 \widehat{c}$ is equivalent to $e_t = (\partial_2)_s \widehat{c}$ where $(\partial_2)_s$ is the linear map with matrix $(D_2)_s$. This proves (ii).

Remark. (i) is related to the fact (first proved in [Ar-Le] that every cohomology class of degree 1 is represented by a Morse form without points of index zero (and also without points of index n), see section 6.

2) Lets us compute D_2, using Fox differential calculus (cf. [Br 1982]). By definition, if $w = x_1 \cdots x_n$, $x_i \in \{s_1^{\pm 1}, \cdots, s_p^{\pm 1}\}$, is an element of F written as a reduced word, we have

$$\frac{\partial w}{\partial s_i} = \pi\Big(\sum_{k, x_k = s_i} x_1 \cdots x_{k-1} - \sum_{k, x_k = s_i^{-1}} x_1 \cdots x_k\Big) \in \mathbb{Z}[G].$$

This implies

$$\pi(w) - 1 = \sum_{i=1}^{p} \frac{\partial w}{\partial s_i}(s_i - 1).$$

In particular, if $w \in \ker \pi$ ie w is a relation, the right-hand side is zero.

If $f_j \in B_2$ is a 2-cell, it defines a relation $r_j \in \ker \pi$. Then the matrix D_2 is given by $(D_2)_{i,j} = \dfrac{\partial r_j}{\partial s_i}$.

Remark. (cf. [Br 1982]) The map $r \in \mathcal{R} \mapsto \sum_{s\in S} \frac{\partial r}{\partial s} e_s$ induces an isomorphism from $\mathcal{R}/[\mathcal{R},\mathcal{R}]$ to $Z_1(C)$, where $\mathcal{R} \subset F(S)$ is the normal subgroup generated by the relations, ie $\mathcal{R} = \ker \pi$. We shall denote it also ∂_2.

5 Homology of degree one

In this section we fix a presentation $G = \langle S|R\rangle$, from which we deduce a partial resolution of length two:

$$C_2 = \mathbb{Z}[G]^{(R)} \to C_1 = \mathbb{Z}[G]^{(S)} \to C_0 = \mathbb{Z}[G] \to \mathbb{Z}.$$

We denote $(e_s, s \in S)$ the basis of C_1.

5.1 Representation of $H_1(G,\xi)$ as a cokernel

Proposition. *Let $s \in S$ be such that $\xi(s) \neq 0$, so that $s-1$ is invertible in Λ_ξ. Denote by $(\partial_2)_s$ the restriction of ∂_2 to the subspace* $\mathrm{vect}(e_t, t \in S \setminus \{s\})$. *Then one has natural identifications*

$$H_1(G,\xi) = H_1(C_*,\xi) = \mathrm{coker}(\partial_2)_s.$$

Proof. An element $v = \sum_{s\in S} \lambda_s e_s \in C_2(\xi)$ is in $\ker \partial_1^\xi$ if and only if $\sum \lambda_s(s-1) = 0$. This is equivalent to $\lambda_s = \sum_{t \in S\setminus\{s\}} \lambda_t(t-1)(1-s)^{-1}$, thus $\ker \partial_1^\xi$ is generated by $e'_t := e_t + (t-1)(1-s)^{-1}e_s$, $t \in S \setminus \{s\}$. These generators are clearly free, and

$$\sum_{t\in S\setminus\{s\}} \lambda_t e'_t \in \mathrm{im}(\partial_2^\xi) \Leftrightarrow \sum_{t\in S\setminus\{s\}} \lambda_t e_t \in \mathrm{im}(\partial_2)_s,$$

thus the natural projection $e'_t \mapsto e_t$ induces an isomorphism $H_1(C_*,\xi) \to \mathrm{coker}(\partial_2)_s$.

Corollary. *Assume that G has a finite presentation $\langle x_1, \cdots, x_p \mid r_1, \cdots, r_q\rangle$. If $H_1(G,\xi) = 0$, $q \geq p-1$. Thus if $p - q \geq 2$, $H_1(G,\xi)$ never vanishes.*

Proof. There exists $i \in \{1, \cdots, p\}$ such that $\xi(x_i) \neq 0$. If $H_1(G,\xi) = 0$, by the proposition the map $(\partial_2)_{x_i}$ is surjective from $(\mathbb{Z}[G]_\xi)^q$ to $(\mathbb{Z}[G]_\xi)^{p-1}$. Thus $q \geq p-1$ since a Novikov ring maps to a field (cf 1.7).

5.2 Criteria for the vanishing of $H_1(G,\xi)$ when G is finitely generated

Here we assume that S is finite, and view G a subset of its Cayley graph $\Gamma(G,S)$, on which we extend ξ affinely. We define $\Gamma(G,S)_{\geq r}$ as the smallest subgraph containing $\{\xi \geq r\}$: it is the union of all edges $e = (g,h)$ such that $\xi(g), \xi(h) \geq r$.

Let X be the cellular 2-complex associated to a presentation $\langle S|R\rangle$, and C_* the chain complex of $\widetilde{X}$, so that $C^{\leq 1}$ is the chain complex of $\Gamma(G,S)$. Then the subcomplex associated to $\Gamma(G,S)_{\xi\geq r}$ is $C^{\leq 1}(\{\xi \geq r\})$, the subcomplex generated by cells with support in $\{\xi \geq r\}$ (where the support is defined as in 2.4). Thus $\widetilde{H}_0(\Gamma(G,S)_{\xi\geq r}) = \widetilde{H}_0(C(\{\xi \geq r\}))$.

Proposition 5.2. *Let (G,S) be a group equipped with a finite generating set. The following are equivalent:*

(i) $H_1(G,\xi) = 0$.

(ii) $\Gamma(G,S)_{\xi\geq 0}$ *is connected.*

(iii) Every g such that $\xi(g) \geq 0$ can be written $g = x_1 \cdots x_n$ with $x_i \in S \cup S^{-1}$ and $\xi(x_1 \cdots x_k) \geq 0$ for $k = 1, \cdots, n$.

Proof. (ii) is equivalent to the fact that every $g \in G_{\xi\geq 0}$ is connected to 1 in $\Gamma(G,S)_{\xi\geq 0}$, ie (iii). Thus it remains to prove $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (i)$.

(i) ⇒ *(ii)*. It suffices to prove that for every $g \in G$, $g-1$ bounds a chain in $C(G_{\xi\geq 0})$. Choose $\gamma \in C_1$ such that $\partial_1\gamma = g-1$. Let s be an element of S such that $\xi(s) \neq 0$. Then $\partial e_s = s-1$ is invertible in $\mathbb{Z}[G]_\xi$, with $\mathrm{supp}((s-1)^{-1}-1) \subset \{\xi > 0\})$. Define

$$\widetilde{\gamma} = \gamma + (1-g)(s-1)^{-1}e_s \in C_1(\xi).$$

Then $\partial_1\widetilde{\gamma} = 0$, thus by hypothesis there exists $\widetilde{c} \in C_2(\xi)$ such that $\partial_2\widetilde{c} = \widetilde{\gamma}$. By truncating, we find $c \in C_2$ such that $\widetilde{\gamma} - \partial_2 c \in C(\{\xi > 0\})$. Moreover, since $\mathrm{supp}((s-1)^{-1}-1) \subset \{\xi > 0\})$ and $\xi(g) \geq 0$, we have $\widetilde{\gamma} - \gamma \in C(G_{\xi\geq 0})$, thus

$$\gamma - \partial_2 c = (\widetilde{\gamma} - \gamma) + (\widetilde{\gamma} - \partial_2 c) \in C(G_{\xi\geq 0}).$$

Thus $\gamma - \partial_2 c$ is a ∂_1-primitive of $g-1$ in $C(G_{\xi\geq 0})$, qed.

(ii) ⇒ *(i)*. Fix $s \in S$ such that $\xi(s) \neq 0$. By 5.1 (or the corollary in 3.2), it suffices to find a right inverse to $(D_2)_s$. It suffices to find a matrix such that $(D_2)_s M \in \Sigma_\xi$. Equivalently to find, for every $t \in S \setminus \{s\}$, $c \in C_2(\xi)$ and $\lambda \in \mathbb{Z}[G]_\xi$ such that $\partial_2 c - e_t - \lambda e_s \in C(\xi > 0)$. Actually, we shall find c in C_2 and $\lambda \in \mathbb{Z}[G]$.

Replacing s by s^{-1} we can assume that $\varepsilon = 1$. For n large enough, $\xi(ts^n) = \xi(s^n t) > 0$. Thus $s^n t$ and ts^n are connected in $\Gamma(G,S)_{\xi>0}$, ie $s^n t - ts^n = \partial_1\gamma$ with $\gamma \in C_{\xi>0}$. Thus

$$\begin{aligned}\partial_1\Big((1-s^n)e_t + (t-1)(1+s+\cdots+s^{n-1})e_s\Big) &= (1-s^n)(t-1) + (t-1)(s^n-1)\\ &= ts^n - s^n t = \partial_1\gamma.\end{aligned}$$

Since $\ker\partial_1 = \mathrm{im}\,\partial_2$, there exists $c \in C_2$ such that

$$(1-s^n)e_t + (t-1)(1+s+\cdots+s^{n-1})e_s = \gamma + \partial_2 c.$$

Thus $\partial_2 c - e_t - (t-1)(1+s+\cdots+s^{n-1})e_s \in C_{\xi>0}$. Setting $\lambda = (t-1)(1+s+\cdots+s^{n-1})$, this finishes the proof of the Proposition.

Remark. 1) One can define *initial ξ-track* of a word $w = x_1\cdots x_n \in F(S)$, $x_i \in S \cup S^{-1}$ as

$$\mathrm{IT}^S_\xi(w) = \{1, \xi(w_1), \xi(w_1w_2)\cdots, \xi(w_1\cdots w_n)\},$$

and *minimal initial ξ-level* as $\mu^S_\xi(w) = \min \mathrm{IT}^S_\xi(w)$. Then (ii) is equivalent to the fact that every g with $\xi(g) \geq 0$ can be written $\pi(w)$ with $\mu^S_\xi(w) \geq 0$. Compare [Bi-Ne-St] and [Br 1987] who define instead the final ξ-track and the minimal find ξ-track, in keeping with the fact that their groups act on the right. Note that if $\xi(w) = 0$, the minimal final ξ-level is the opposite of the minimal initial ξ-level.

Remark. If w is a non-reduced word with $\mu^S_\xi(w) \geq 0$, the initial ξ-tarck of the reduced word $\overline{w}$ is contained in that of w, thus also $\mu_\xi(\overline{w}) \geq 0$.

Other equivalent properties. *The following are properties equivalent to $H_1(X,\xi) = 0$:*

(iv) Every $g \in \ker\xi$ can be written $g = \pi(w)$ with $\mu^S_\xi(w) \geq 0$. It suffices that it be true for every $g \in G'$.

(v) (Meigniez, 1991) There is only one end in the direction $\{\xi \to +\infty\}$, ie: if (g_n) and (g'_n) are two sequences in G such that $\xi(g_n) \to +\infty$, $\xi(g'_n) \to +\infty$, there exists a sequence of constants (K_n) tending to $+\infty$ such that g_n and g'_n are connected in $\Gamma(G,S)_{\xi\geq K_n}$.

Proof. (iv) In view of (iii), it suffices to prove that (for any ξ) every $g \in G_{\xi \geq 0}$ can be connected in $G_{\xi \geq 0}$ to an element of G'. Write $g = x_1 \cdots x_n$, with $x_i \in S \cup S^{-1}$. Since $\xi(g) \geq 0$, there exists a permutation $\sigma \in \mathfrak{S}_n$ such that $\xi(x_{\sigma(1)} \cdots x_{\sigma(k)} \geq 0$ for $k = 1, \cdots, n$. Thus $h = x_{\sigma(1)} \cdots x_{\sigma(n)}$ is connected to 1 in $\{\xi \geq 0$. Multiplying on the left by gh^{-1}, we join g to gh^{-1}, which belongs to G' as desired.

(v) If $\Gamma(G, S)_{\xi \geq 0}$ is connected, then $\Gamma(G, S)_{\xi \geq r}$ is connected for every $r \in \text{Im}\, \xi$. Thus if (g_n) and (g'_n) are two sequences in G such that $\xi(g_n) \to +\infty$, $\xi(g'_n) \to +\infty$, one can join g_n to g'_n in $G_{\xi \geq r}$ with $r = \min(\xi(g_n), \xi(g'_n))$, thus there is only one end in the direction $\xi \to +\infty$.

Conversely, assume that there is only one end in the direction $\xi \to +\infty$. Let $s \in S$ be such that $\xi(s) \neq 0$, thus $\xi(s^\varepsilon) > 0$ for a suitable $\varepsilon \in \{1, -1\}$. By hypothesis, for n large enough $s^{\varepsilon n}$ is connected to $gs^{\varepsilon n}$ in $G_{\xi \geq 0}$. Thus g is connected to 1 in $G_{\xi \geq 0}$, via $gs^{\varepsilon n}$ and $s^{\varepsilon n}$. Thus $H_1(G, \xi) = 0$.

Remark. In (iv), one can replace $\ker \xi$ or G' by any H which is contained in $\ker \xi$, normal in G, and such that $H_1(G/N, \bar{\xi}) = 0$, for instance if G/N does not contain a nontrivial free semigroup.

Property (cf ?). *If $H_1(G, \xi) \neq 0$, G contains a non trivial free semigroup.*

Proof. By hypothesis, there exists $g \in G_{\xi \geq 0}$ which is not connected to 1 in $G_{\xi \geq 0}$. Write $g = x_1 \cdots x_n$ with $x_i \in S \cup S^{-1}$. Let k be such that $\xi(x_1 \cdots x_k)$ minimal, by hypothesis it is negative. Define $x = (x_1 \cdots x_k)^{-1}$, $y = x_{k+1} \cdots x_n$. Then $\xi(y) \geq \xi(x) > 0$ and x, y are connected to 1 in $G_{\xi \geq 0}$, but not in $\{\xi \geq \xi(x)\}$.

We claim that the semigroup $\langle x, y \rangle$ is free. Indeed, assume that we have a relation $w_1 = w_2$ where w_1, w_2 are nonnegative and distinct words in x, y. Assuming the length of w_1 to be minimal, we can assume that $w_1 = xw'_1$, $w_2 = yw'_2$ where w'_1, w'_2 are nonnegative words in x, y. Since x, y are connected to 1 in $G_{\xi \geq 0}$, x is connected to xw'_1 [or rather its image in G] in $\{\xi \geq \xi(x)\}$, and y is connected to yw'_2 in $\{\xi \geq \xi(x)\} \subset \{\xi \geq \xi(x)\}$. Since $xw'_1 = yw'_2$ [as elements of G], we have connected x and y in $\{\xi \geq \xi(x)\}$, contradiction.

Remarks. 1) The equivalences between (i)-(v) are essentially due to Bieri-Neumann-Strebel.

2) The equivalence with (v) is in [Mei 1991]. In the topological case where $C_* = C_*(\widetilde{X})$, the singular chain complex of a path-connected space X such that $\pi_1(X) = G$, one can formulate a variant of (v), cf Latour 1994): *the space $\Gamma(X, \xi)$ of paths $\gamma : ([0, +\infty[, 0) \to (X, x_0)$ such that $\lim_{t \to +\infty} \widetilde{f}(\widetilde{\gamma}(t)) = -\infty$, is connected.*

5.3 Vanishing of $H_1(G, \xi)$ and the set $\Sigma(G)$ of Bieri-Neumann-Strebel

Notation. If $x, y \in G$, we denote $x^y = y^{-1}xy$. This is the most natural since then $x^{yz} = (x^y)^z$, but it means that the natural action by G on itself by conjugation is on the right.

Definition Bieri-Neumann-Strebel. Assume that G is finitely generated. By definition, $\Sigma(G)$ [$= \Sigma_{G'}(G)$] is the set of all $[\xi] \in \Sigma(G)$ such that, for the right action by conjugation of G on itself, G' is finitely generated over some finitely generated submonoid of $G_{\xi \geq 0}$. In other words:

$$[\xi] \in \Sigma(G) \Leftrightarrow \begin{cases} \text{there exists finite subsets } A \subset G_{\xi \geq 0} \textit{ and } B \subset \ker \xi \\ \text{such that every element of } G' \text{ can be written } \prod b_i^{g_i} \\ \text{with } b_i \in B \text{ and } g_i \text{ represented by a nonnegative word on } A. \end{cases}$$

Using yxy^{-1} instead, we get

$$[\xi] \in \Sigma(G) \Leftrightarrow \begin{cases} \text{there exists finite subsets } A \subset G_{\xi \leq 0} \textit{ and } B \subset \ker \xi \\ \text{such that every element of } G' \text{ can be written } \prod g_i b_i g_i^{-1} \\ \text{with } b_i \in B \text{ and } g_i \text{ represented by a nonnegative word on } A. \end{cases}$$

Remark. Since $\ker \xi / G'$ is finitely generated as a group, one gets the same set if one replaces G' by $\ker \xi$.

Example. If G is the Baumslag-Solitar group $BS(1,2) = \langle a, b \mid aba^{-1}b^{-2} \rangle$ and ξ is the homomorphism such that $\xi(a) = 1$, then as a monoid $\ker \xi$ is generated by $a^{-q}ba^q = b^{\frac{1}{2^q}}$ for $q \in \mathbb{N}$, thus one can take $A = \{a\}$, $B = \{b\}$, showing that $[\xi] \in \Sigma(G)$. On the other hand, a finite subset $B \subset \ker \xi$ is contained in $\langle a^{-q}ba^q \rangle$ for some $q \in \mathbb{N}$, and $\alpha M \alpha^{-1}$ is contained in $\langle a^{-q}ba^q \rangle$ if $\xi(\alpha) \leq 0$, thus $\ker \xi$ is not even generated by B over $G_{\xi \leq 0}$, thus $[-\xi] \notin \Sigma(G)$, thus $\Sigma(G) = \{[\xi]\}$.

The relation with Novikov homology is given by the next proposition (which is very close to Prop. 2.1 and Prop. 3.4 in Bieri-Neumann-Strebel).

Proposition 5.3 *Assume that G is finitely generated. The following are equivalent:*

(i) $H_1(G, \xi) = 0$.

(ii) $-\xi \in \Sigma(G)$.

Proof. We shall use the following characterizations:

(i) $H_1(G, \xi) = 0 \Leftrightarrow$ for one or any finite generating system S of G, every $g \in G'$ can be written $g = \pi(w)$ with $w \in F(S)$ and $\mu_\xi^S(w) \geq 0$ [thus $= 0$], ie $g = x_1 \cdots x_n$ with $x_i \in S \cup S^{-1}$ and $\xi(x_1 \cdots x_k) \geq 0$ for $k = 1, \cdots, n$.

(ii) $-\xi \in \Sigma(G) \Leftrightarrow$ every $g \in G'$ can be written $g = \prod_{i=1}^{n} g_i b_i g_i^{-1}$ with $g_i \in \langle A \rangle_+ = \pi(W_+(A))$ where A is a finite subset of $G_{\xi \geq 0}$ and $b_i \in B$, finite subset of G'.

(ii) $\Rightarrow$ *(i)*. We write $g \in G'$ as in (ii). We can assume that S contains $A \cup B$. Then this decomposition of g satisfies (i).

(i) $\Rightarrow$ *(ii)*. The group of "relations modulo Abelianization"

$$\pi^{-1}(G')/[F, F] = R[F, F]/[F, F] \subset F^{ab}$$

is finitely generated. Thus there exists a finite subset $V \subset \pi^{-1}(G')$ whose image generates $\pi^{-1}(G')/[F, F]$ as a monoid. Replacing each word $v \in V \subset F$ by a suitable cyclic permutation, since $\xi(v) = 0$ we can assume that $\mu_\xi^S(v) \geq 0$.

Lemma 1. *Any $g \in G'$ can be written $g = \pi(w) b_1 \cdots b_m$ with $w \in [F, F]$, $\mu_\xi^S(w) \geq 0$ and $b_i \in \pi(V)$.*

Proof of Lemma 1. By hypothesis, $g = \pi(w_0)$ with $\mu_\xi^S(w_0) \geq 0$. By the definition of V, there exists $v_1, \cdots, v_m \in V$ such that $w = w_0(v_1 \cdots v_m)^{-1} \in [F, F]$. Since $\mu_\xi^S(v_i) \geq 0$, we have $\mu_\xi^S(w) \geq 0$. Thus $g = \pi(w)\pi(v_1) \cdots \pi(v_m)$ with $w \in [F, F]$, $\mu_\xi^S(w) \geq 0$ and $b_i = \pi(v_i) \in \pi(V)$.

Lemma 2. *Any $w \in [F, F]$ such $\mu_\xi^S(w) \geq 0$ can be written $w = \prod w_i [x_i, y_i] w_i^{-1}$, with $x_i, y_i \in S \cup S^{-1}$ and $\mu_\xi^S(w_i) \geq 0$.*

Proof of Lemma 2. We make an induction on the length $|w|$. The result being obvious if $|w| = 0$, we assume that $|w| = m > 0$ and that the result is true up to length $m - 1$.

Let $y \in S \cup S^{-1}$ be the leftmost letter in w such that $\xi(y) \leq 0$. Since $w \in [F, F]$, y^{-1} occurs somewhere in w, thus for $x = y^{\varepsilon}$ with a suitable choice of $\varepsilon \in \{1, -1\}$, we can write $w = w_1 x w_2 x^{-1} w_3$ (reduced word), with w_1 a positive word in $S_+ = (S \cup S^{-1}) \cap G_{\xi \geq 0}$.

Also, $\xi(x) \leq 0$ or w_2 is a positive word in S_+. In both cases, $\mu_\xi^S(w_1 w_2) \geq 0$, thus

$$\mu_\xi^S(w_1 [x, w_2] w_1^{-1}) \geq 0.$$

And also $\mu_\xi^S(w_1 w_2 w_3) \geq 0$ We have

$$w_1 x w_2 x^{-1} w_3 = w_1 x w_2 x^{-1} w_2^{-1} w_1^{-1} w_1 w_2 w_3 = (w_1 [x, w_2] w_1^{-1}).(w_1 w_2 w_3),$$

where both factors are in $[F, F]$ and have nonnegative ξ-track. Since $|w_1 w_2 w_3| < |w|$, by the induction assumption, $w_1 w_2 w_3$ has the desired form.

Finally, writing $w_2 = x_1 \cdots x_n$, we have

$$w_1 [x, w_2] w_1^{-1} = \prod_{k=1}^{n} (w_1 x_1 \cdots x_{k-1}) [x, x_k] (w_1 x_1 \cdots x_{k-1})^{-1}.$$

Since $\mu_\xi^S(w_1 w_2) \geq 0$, $\mu_\xi^S(w_1 x_1 \cdots x_{k-1}) \geq 0$ for all k, which proves Lemma 2.

End of the proof of Proposition 5.3. Set $B = \pi(V \cup [S \cup S^{-1}, S \cup S^{-1}])$, which is a finite subset of G'. By the lemmas, every $g \in G'$ is of the form

$$g = \pi(\prod_{i=1}^{n} w_i [x_i, y_i] w_i^{-1}).b_1 \cdots b_m,$$

with $x_i, y_i \in S \cup S^{-1}$, $\mu_\xi^S(w_i) \geq 0$ and $b_i \in B$. We apply this to $g = xbx^{-1}$, $b \in B$, $x \in S \cup S^{-1}$. We obtain a finite number of words $w_{b,x}$ whose initial subwords are all in $G_{\xi \geq 0}$. We define A to be the image in G of these subwords.

Let M be the submonoid of G formed by elements of the form $\prod_{i=1}^{n} g_i b_i g^{-1}$ with $g_i \in \langle A \rangle_+$ and $b_i \in B$: it contains B, is invariant by product and is contained in G'. We want to prove that $M = G'$. It suffices to prove that if $g \in \langle A \rangle_+$, $b \in B$, $x \in S \cup S^{-1}$, then $x(gbg^{-1})x^{-1} \in M$. We can write $g = \pi(x_1 \cdots x_n)$, $x_i \in S \cup S^{-1}$, so that $\pi(x_1 \cdots x_k) \in \langle A \rangle_+$ for all k.

We use the equation

$$(hxyk)b(hxyv)^{-1} = h[x, y]h^{-1}.(hyxk)b(hyxk)^{-1}.h[y, x]h^{-1}.$$

to push x to the right until we obtain

$$x(gbg^{-1})x^{-1} = \Big(\prod_{i=1}^{n} g_i [x, x_i] g_i^{-1}\Big).gxbx^{-1}g^{-1}.\Big(\prod_{i=n}^{1} g_i [x_i, x] g_i^{-1}\Big)\ ,\ g_i = x_1 \cdots x_{i-1}.$$

Since $[x, x_i], xbx^{-1} \in M$ and M is invariant by product and by conjugation $m \mapsto hmh^{-1}$ with $h \in \langle A \rangle_+$, this finishes the proof of (ii), and thus of Proposition 5.3.

5.4 The case of geometric closed one-forms

Proposition. *If if $(\widetilde{X}, \widetilde{f}, G, \xi)$ is a geometric closed one-form with $X^{(1)}$ finite, the following are equivalent:*

(i) $H_1(G, \xi) = 0$.

(ii) Let h be the harmonic function which coincides with $\widetilde{f}$ on $\widetilde{X}^{(0)}$. For one or all $r \in \mathbb{R}$, $\widetilde{X}_{\widetilde{h} \geq r}$ is connected.

(iii) For one or all $r \in \mathbb{R}$, $\widetilde{X}_{\widetilde{f} \geq r}$ has a unique component on which ξ is unbounded.

(iv) There exist $r > 0$ such that the map $\widetilde{H}_0(X^{(1)}_{\widetilde{g} \geq r}) \to \widetilde{H}_0(X^{(1)}_{\widetilde{g} \geq r})$ zero.

If X is finite, one can replace $\widetilde{X}^{(1)}_{\widetilde{f} \geq r}$ by $\widetilde{X}_{\widetilde{f} \geq r}$ in (iii).

Proof. *(i)* $\Leftrightarrow$ *(ii)*. By the maximum principle, $\widetilde{X}_{h \geq r}$ is connected if and only if its intersection with $\widetilde{X}^{(1)}$ is connected. If $X^{(1)}$ is finite, this is true if and only if $H_1(G, \xi) = 0$.

(ii) $\Rightarrow$ *(iii)* This follows from the fact that $\widetilde{f} - \widetilde{g}$ is bounded by finiteness of $X^{(1)}$.

(iii)$\Rightarrow$ *(i)*

5.5 The valuation criterion of Brown

This section is inspired by [Br 1987].

Definitions. 1) An *HNN valuation with associated homomorphism ξ on G* is a function $v : G \to \mathbb{R} \cup \{+\infty\}$ with the following properties:

$$v(g) - v(g^{-1}) = \xi(g) \ , \ v(gh) \geq \min(v(g), \xi(g) + v(h)).$$

Remark. We changed $-\xi(g)$ of [Br 1987] to $\xi(g)$, in keeping with the fact that all our actions of G are on the left.

2) Let S be a finite system of generators for G. We define the *ξ-valuation*

$$\begin{aligned} v_\xi(g) &= \sup_{w \in F(s), \pi(w) = g} \mu_\xi^S(w) \\ &= \sup_{x_1 \cdots x_n = g, x_i \in S \cup S^{-1}} \min_{0 \leq i \leq n} \xi(x_1 \cdots x_i). \end{aligned}$$

One proves easily the

Properties. *(i) v_ξ takes values in $\mathbb{R}$, is an HNN valuation with associated homomorphism ξ, and satisfies $v_\xi(x) = \max(\xi(x), 0)$ for $x \in S \cup S^{-1}$.*

(ii) If v is an HNN valuation with associated homomorphism ξ, it satisfies

$$(\forall g_1, \cdots, g_n \in G) \ \ v(g_1 \cdots g_n) \geq \min_{1 \leq i \leq n} \xi(g_1 \cdots g_{i-1}) + v(g_i),$$

thus

$$(\forall x_1, \cdots, x_n \in S \cup S^{-1}) \ \ v(x_1 \cdots x_n) \geq \mu_\xi^S(x_1 \cdots x_n) + \min_{\xi(x_i) \geq 0} v(x_i),$$

and

$$(\forall g \in G) \ v(g) \geq v_\xi(g) + \min_{x \in S \cup S^{-1}, \xi(x) \geq 0} v(x) = v_\xi(g) - C.$$

Moreover,

$$v \text{ bounded below on } G_{\xi\geq 0} \Leftrightarrow v \text{ bounded below on } G'.$$

Proof of the last statement. It suffices to prove $\Leftarrow$. If $\xi(g) \geq 0$, by the commutativity of G/G', we can find $\gamma \in G'$ and a word $x_1 \cdots x_n$ such that $\gamma x_1 \cdots x_n = g$ and $\xi(\gamma x_1 \cdots x_i) \geq 0$ for $i = 1, \cdots, n$. Thus

$$v(g) \geq \min(v(\gamma), \xi(\gamma) + v(x_1 \cdots x_n)) \geq v(\gamma) + \mu_\xi^S(x_1 \cdots x_n) - C \geq v(\gamma) - C,$$

which is bounded below by hypothesis.

Proposition. *Assume that G is finitely generated. The following are equivalent:*

(i) $H_1(G, \xi) = 0$

(ii) The function v_ξ is bounded below on G', or on $G_{\xi\geq 0}$.

(iii) Every HNN valuation on G with associated homomorphism ξ is trivial, ie bounded below on G', or on $G_{\xi\geq 0}$.

Proof. $(i) \Leftrightarrow (ii)$. This follows from the characterizations (iv) and (v) in 7.2.

(i)$\Leftrightarrow$(iii). This follows from the fact that v_ξ is an HNN valuation with associated homomorphism ξ, and every such valuation v satisfies $v \geq v_\xi - C$.

5.6 Vanishing of $H_1(G, \xi)$ and actions on $\mathbb{R}$-trees

We first state a few definitions and facts about actions on $\mathbb{R}$-trees.

Abelian actions. (cf. Culler-Morgan 1984, Levitt 1994, Chiswell 2001) Let G be a group acting (on the left) by isometries on an $\mathbb{R}$-tree T, with *hyperbolic length function* $\ell(g) = \min_{x\in T} d(x, g.x)$, assumed to be nonzero (if G is finitely generated, this is equivalent to the nonexistence of a global fixed point). An element $g \in G$ is *elliptic* if $\ell(g) = 0$ ie g has a fixed point, and *hyperbolic*if $\ell(g) > 0$. The *characteristic set* $A_g = \{x \in T \mid d(x, g.x) = \ell(g)\}$ (the fixed point set if g is elliiptic, the translation axis if g is hyperbolic). The action is *Abelian* if it satisfies one of the equivalent following properties:

- $\ell(gh) \leq \ell(g) + \ell(h)$ for all $g, h \in G$; equivalently, $A_g \cap A_h \neq \emptyset$.
- There exists $\xi \in \mathrm{Hom}(G, \mathbb{R})$ such that $\ell = |\xi|$.
- The action fixes an end e of T; in that case, ξ can be recovered by $\xi(g) = \pm\ell(g)$, with the sign $+$ (when $\ell(g) > 0$ ie g hyperbolic) if g pushes away from e, and $-$ otherwise. Also, e contains one half of every translation axis A_g for g hyperbolic.

In particular, if G fixes a unique end, ξ is determined by the action. Otherwise, G stabilizes a unique line, which is equal to any translation axis A_g, g hyperbolic, and the two ends of this line give the two homomorphisms $\pm\xi$ whose absolute value is ℓ.

Minimal actions. An action of G on an $\mathbb{R}$-tree is *minimal* if it admits no invariant strict subtree. When $\ell \not\equiv 0$, there exists a unique minimal invariant subtree $T_{\min}$, which is the union of all translation axes A_g. If the action on T is Abelian, resp. has a unique fixed end, the same is true for the action on $T_{\min}$.

Geometric closed one-forms and $\mathbb{R}$-trees. We give two constructions of actions on $\mathbb{R}$-trees due to G. Levitt [Lev 1987] and Levitt 1994.

1) If $(\widetilde{X}, \widetilde{f}, G, \xi)$ is a geometric closed one-form, define on $\widetilde{X}$ the pseudodistance $d(x, y) =$ infimum of the total variation of ξ on a path from x to y in $\widetilde{X}$, and its metric quotient $T(\xi) = \widetilde{X}/{\sim}$, where $x \sim y$ if $d(x, y) = 0$. Using the fact that $\widetilde{X}$ is simply connected, one proves that $T(\xi)$ is an

$\mathbb{R}$-tree. An action on G on a tree is *geometric* if it is equivalent to such an action on $T(\xi)$ and is moreover Abelian.

2) If Y is any path-connected space and $g : Y \to \mathbb{R}$ a function, define the pseudodistance

$$\delta(y_1, y_2) = g(y_1) + g(y_2) - 2 \sup_\gamma \min_{t\in[0,1]} g(\gamma(t)),$$

the supremum being over all paths γ from y_1 to y_2 in Y. Note that $\delta(y_1, y_2) < \varepsilon$ if and only if there is a path from y_1 to y_2 in $Y_{g>\frac{g(y_1)+g(y_2)}{2}-\varepsilon}$. The metric quotient $T^+(g)$ is an $\mathbb{R}$-tree since it satisfies the 0-hyperbolicity condition, and if we still denote g the induced function on $T^+(g)$, the sublevels $T^+(g)_{\xi\leq r}$ are all path-connected by construction, thus $T^+(g)$ has a unique end $e \subset g^{-1}(\infty)$.

In the following Pproposition, the equivalence (i)-(ii) is due to [Bro 1987] and (ii)-(iii) to Levitt 1994. For the proofs, we follow Levitt 1994.

Proposition. *Let G be finitely generated and $\xi \in \operatorname{Hom}(G, \mathbb{R}) \setminus \{0\}$. The following are equivalent:*

(i) $H_1(G, \xi) \neq 0$.

(ii) There exists an action of G on an $\mathbb{R}$-tree T which has a unique fixed end, with associated homomorphism ξ.

(iii) There exists a geometric action of G on an $\mathbb{R}$-tree, whose associated homomorphism is $-\xi$.

Proof. (i) $\Rightarrow$ (ii). All sublevels $T^+(\xi)_{\xi\leq r}$ are path-connected: if π is the projection from $\widetilde{X}$ to $T^+(\xi)$ and γ is a path from x to y in $\widetilde{X}_{\xi\geq r}$, one can replace $\pi \circ \gamma$ by a constant on every maximal interval in $\xi^{-1}(]C, +\infty[)$, obtaining thus a path c from x to y in $T^+(\xi)_{\xi\leq r}$. Thus $T^+(\xi)$ has the fixed end $e = \xi^{-1}(-\infty)$, and the homomorphism associated to the action and e is clearly ξ (since the convention is that $\xi(g)$ measures how much g is pushed away from e).

Now we use the fact that $H_1(G, \xi) \neq 0$ to prove that there is no other fixed end, equivalently there is no end contained in $\xi^{-1}(+\infty)$. The hypothesis means that for every $x \in \widetilde{X}_{\xi>0}$ there exists $g \in \ker \xi$ such that x and $g.x$ are in different components of $\widetilde{X}_{\xi>0}$. Then $\pi(x)$ and $\pi(g.x)$ are also in different components of $T^+(\xi)_{\xi>0}$, otherwise one could find a path in $\widetilde{X}_{\xi>0}$ from x to y such that $\delta(y, g.x) = 0$. There would exists a path from y to $g.x$ in $\widetilde{X}_{\xi>0}$, thus a connecting path from x to $g.x$ in $\widetilde{X}_{\xi\geq 0}$, contradiction. Thus for every connected component C of $T^+(\xi)_{\xi>0}$, there exists $g \in \ker \xi$ such that $g.C \cap C = \emptyset$.

If e' is an end of $T^+(\xi)$ which is fixed by G and contained in $\xi^{-1}(+\infty)$, it is represented by a sequence (C_n) where C_n is a connected subset of $T^+(\xi)_{\xi>n}$, with $C_{n+1} \subset C_n$. The fact that e' is fixed by G implies that for every $g \in G$ one has $g.C_n \subset C_0$ for n large enough. In particular $g.C \cap C$ if C is the connected component of $T^+(\xi)_{\xi>0}$ containing C_0, and we get a contradiction.

(ii) $\Rightarrow$ (i). Replacing T by $T_{\min}$, we can assume that the action is minimal. Then for any $t_0 \in T$, the map $g \mapsto g.t_0$ extends affinely to $\varphi : \Gamma(G, S) \to T$ which is surjective. The fixed end is contained in $\xi^{-1}(-\infty)$ (equal to in fact), thus there is no fixed end in $\xi^{-1}(+\infty)$. Thus if $\xi(g) > 0$ and we orient the axis A_g by ξ, there exists $h \in G$ such that $h.A_g = A_{hgh^{-1}}$ does not contain a positive ray in A_g. This implies that A_g and $A_{hgh^{-1}}$ are not connected in $T_{\xi\geq a}$ for n large enough. Thus $T_{\xi\geq a}$ is disconnected for n large , and by the surjectivity of φ, so is $\Gamma(G, S)_{\xi\geq a}$, thus $H_1(G, \xi) \neq 0$.

(i) $\Rightarrow$ (iii) Let $\mathcal{P}$ be a presentation of G with a finite set of generators, to which we associate a geometric harmonic one-form $\xi : \widetilde{X} \to \mathbb{R}$. The hypothesis implies that $\widetilde{X}_{\xi\geq 0}$ is connected, thus

$T(\xi)_{\xi\geq 0}$ is connected: this implies that there is a unique end $e \subset \xi^{-1}(+\infty)$, which is thus G-invariant, thus the action is geometric. Finally, the associated homomorphism is $-\xi$ since we start from $+\infty$.

(iii)$\Rightarrow$ (i). By hypothesis, there exists a geometric closed one-form $(\widetilde{X}, G, \xi)$, such that $T(-\xi) = \widetilde{X}_{/\sim}$ is an $\mathbb{R}$-tree with a fixed end e and associated homomorphism $-\xi$, so that $e = \xi^{-1}(+\infty)$. Thus for every $g \in G_{\xi>0}$ and $h \in G$, A_g and $A_{hgh^{-1}}$ share a common ray R directed towards $+\infty$. If $t_0 \in R$, $g^n.t_0$ and $hg^nh^{-1}.t_0 \in R$ for every $n \in \mathbb{N}$, thus they are connected in $T_{\xi\geq 0}$ for n large enough. **One should deduce that g^n et hg^n are joined in $\Gamma(G,S)_{\xi\geq 0}$, which gives the result by (vii) in 5.2.**

5.7 The case of one-relator groups

A complete description of this case is given [Bro 1987]. We reprove here his results in a different way.

Let G be a group with one relator, such that $\mathrm{Hom}(G;\mathbb{R}) \neq 0$. If G has only one generator, $G \simeq \mathbb{Z}$. If G has at least three generators (maybe an infinite number), $H_1(G,\xi)$ never vanishes by 3.5. Thus we can assume that $G = \langle x, y | r\rangle$. By 2.5:

$$H_1(G,\xi) = 0 \Leftrightarrow \begin{cases} \dfrac{\partial r}{\partial x} \text{ is invertible in } \mathbb{Z}[G]_\xi \text{ if } \xi(y) \neq 0 \\ \dfrac{\partial r}{\partial y} \text{ is invertible in } \mathbb{Z}[G]_\xi \text{ if } \xi(x) \neq 0. \end{cases}$$

There are two cases:

1) If $r = s^n$, $n \geq 2$, is a nontrivial power, we have $\dfrac{\partial r}{\partial x} = (1 + s + \cdots + s^{n-1})\dfrac{\partial s}{\partial x}$, thus the image in $\mathbb{Z}[G/\ker\xi]$ is divisible by n since $\xi(s) = 0$, and so cannot be invertible in $\mathbb{Z}[G/\ker\xi]_{\overline{\xi}}$. A fortiori, $\dfrac{\partial r}{\partial x}$ is not invertible in $\mathbb{Z}[G]_\xi$ and similarly for $\dfrac{\partial r}{\partial y}$, thus $H_1(G,\xi)$ never vanishes.

2) Now assume that r is not a nontrivial power. By Howie 1982, G is right-orderable, thus $\mathbb{Z}[G]_\xi$ has only trivial invertibles: In particular,

$$\lambda \text{ is invertible in } \mathbb{Z}[G]_\xi \Leftrightarrow \xi \text{ has a unique minimum on } \mathrm{supp}(\lambda).$$

Write $r = x_1x_2\cdots s_n$, with $s_i \in \{x^{\pm 1}, y^{\pm 1}\}$; and note $g_i = x_1\cdots x_i$, $1 \leq i \leq n$, $g_0 = 1$.

Lemma. *The elements $1, g_1\cdots, g_{n-1}$ are distinct in G.*

Corollary. *If E is a subset of $[[0, n-1]]$ and $\lambda = \sum_{i\in E}\varepsilon_i g_i$, $\varepsilon_i = \pm 1$, λ is invertible if and only if $\xi|E$ has a unique minimum. This applies in particular to $\frac{\partial r}{\partial x}$ and $\frac{\partial r}{\partial y}$.*

Proof of the lemma (sketch). It suffices to prove that any strict finite subword of r does not represent 1 in G. This can be proved by induction on the length $|r|$ using the rewriting procedure of Magnus (cf. [Ly-Schu], IV.5).

We have

$$\frac{\partial r}{\partial x} = \sum_{i, s_i = x} g_{i-1} - \sum_{i, s_i = x^{-1}} g_i$$

$$\frac{\partial r}{\partial y} = \sum_{i, s_i = y} g_{i-1} - \sum_{i, s_i = y^{-1}} g_i.$$

Thus we get the

Proposition. *If $G = \langle x, y|r\rangle$, the following are equivalent:*

(i) $H_1(G, \xi) = 0$

(ii) if $\xi(x) \neq 0$: ξ *admits a unique minimum on* $\mathrm{supp}(\dfrac{\partial r}{\partial y}) = \{g_i | s_i = y^{-1} \text{ or } s_{i+1} = y\}$

(iii) if $\xi(y) \neq 0$: ξ *admits a unique minimum on* $\mathrm{supp}(\dfrac{\partial r}{\partial x}) = \{g_i | s_i = x^{-1} \text{ or } s_{i+1} = x\}$.

Remark. We stated the proposition without the restriction that r should not be a nontrivial power s^n. Indeed, in that case we have seen that $H_1(G, \xi)$ never vanishes. Since clearly ξ takes each value at least n times, (ii) and (iii) are also never satisfied.

It is easy to prove that this is equivalent to the result of [Br 1987], Theorem 5.2, which states that $\xi \in \Sigma(G)$ (ie $H_1(G, -\xi) = 0$) is equivalent to

(iii) If $\xi(x)$ *and* $\xi(y) \neq 0$, ξ *has a unique maximum on* $\{g_0, g_1, \cdots, g_{n-1}\}$. *If* $\xi(x)$ *or* $\xi(y) = 0$, ξ *has exactly two maxima on* $\{g_0, g_1, \cdots, g_{n-1}\}$, *and these are consecutive.*

Actually, [Br 1987] states this with $\{g_0, g_1 \cdots, g_{n-1}\}$ replaced by the inverse image of the extremal points of the convex hull of $\{\overline{g_0}, \overline{g_1}, \cdots, \overline{g_{n-1}}\}$: this clearly amounts to the same thing.

Corollary. *If G is finitely generated a one-relator group, the set*

$$\Sigma^1(G) = \{\xi \in H^1(G; \mathbb{R}) \setminus \{0\} \mid H_1(G, \xi) = 0\}$$

has a polyhedral rational structure (or: is rationally defined).

Examples. We have already seen the case of the Baumslag-Solitar group $B(1, 2)$. In general for $B(p, q) = \langle x, y \mid xy^p x^{-1} y^{-q}\rangle$ $(p, q \geq 2)$, we have

$$A = \{1, y^q\}\ ,\ B = \{x, xy \cdots, xy^{p-1}, 1, y, \cdots, y^{q-1}\}.$$

If $p \neq q$, $S(G) = \{\xi, -\xi\}$, with $\xi(x) = 1$, $\xi(y) = 0$. We have $H_1(G, \pm\xi) \neq 0$ if $p \geq 3$. If $p = 2$, $H_1(G, \xi) = 0$ if and only if $\xi(x) < 0$, as in the case of $B(1, 2)$.

If $p = q$, we have $\mathrm{Hom}(G; \mathbb{R}) \simeq \mathbb{R}^2$ via $\xi \mapsto (\xi(x), \xi(y))$. Then $H_1(G, \xi) = 0$ if and only if $\xi(y) \neq 0$.

5.8 The case of PL homeomorphisms of the interval

Let G be a finitely generated group of orientation-preserving and piecewise-linear homeomorphisms of $[0, 1]$. We have two special elements $\lambda, \rho \in H^1(G; \mathbb{R})$, given by $\lambda(g) = \log g'(0)$, $\rho(g) = \log g'(1)$. We assume that

- G is irreducible ie has no fixed points in $]0, 1[$. Equivalently, the closure of every interior orbit contains 0 and 1.
- λ and ρ are independent in the sense that $\lambda(\ker \rho) = \mathrm{im}(\lambda)$ and $\rho(\ker \lambda) = \mathrm{im}(\rho)$. Equivalently, $\ker \lambda$ and $\ker \rho$ generate G.

Theorem ([Bi-Ne-St], Theorem 8.1)

$\Sigma(G) = S(G) \setminus \{\lambda, \rho\}$.

Proof. 1) We prove that $[\lambda] \notin \Sigma(G)$, ie $H_1(G, \lambda) \neq 0$, the proof that $[\rho] \in S(G)$ being similar. For this, we follow [Bro 1987]. Define $v_-(g) = \log \varepsilon(g)$ where $\varepsilon(g)$ is the largest ε such that g is linear on $[0, \varepsilon]$. Then v_- is a HNN valuation with associated homomorphism λ, ie we have

$$v_-(g^{-1}) - v_-(g) = \lambda(g)\ ,\ v_-(gh) \geq \min(v_-(g), -\lambda(g) + v_-(h)).$$

Moreover, v_- has no minimum on $\ker\lambda$: if it had, there would exist $g_0 \in \ker\lambda$ such that every element of $\ker\lambda$ is the identity on $[0, \varepsilon(g_0)]$. Since $\ker\lambda$ is normal, $\varepsilon(g_0)$ would be fixed by every $g \in G$, thus $\varepsilon(g_0) = 0$ or 1 which is absurd. Thus v_- does not satisfy the conclusion of (iv) of Proposition 5.2, thus $H_1(G, \lambda) \neq 0$.

2) We prove that if $[\xi] \neq [\lambda], [\rho]$, $H_1(G, \xi) = 0$. For this, we follow [Bi-Ne-St]. By the independence of λ and ρ, ξ cannot vanish on $\ker\lambda$ and $\ker\rho$. We suppose that $\xi | \ker\rho \neq 0$, the other case being similar. Then there exists h such that $\xi(h) > 0$ and $\rho(\xi) = 0$, thus $\operatorname{supp}(h) \subset [0, b]$ with $b < 1$. Since $[\xi] \neq [\lambda]$, there exists g such that $\lambda(g) < 0 < \xi(g)$. Thus there exists $a > 0$ such that $g(x) < x$ on $]0, a[$ (thus $\operatorname{supp}(g) \supset [0, a]$), and using irreducibility we can replace h by a conjugate to achieve $a > b$.

Let S be a finite generating set of G, containing g and h, and let $\mathfrak{R} = [S^{\pm 1}, S^{\pm 1}]$. There exists $\varepsilon > 0$ such that $\operatorname{supp}(g) \subset [\varepsilon, 1 - \varepsilon]$ for all $r \in \mathfrak{R}$. Since $b < a$, we can choose k large enough so that $g^k(b) < \varepsilon$. Setting $w = g^k h g^{-k}$, one has $\operatorname{supp}(w) \subset [0, \varepsilon]$, which implies $r = w^N r w^{-N}$ for every $r \in \mathfrak{R}$ and every N **à compléter**.

6 Novikov homology and finiteness properties

In this section we state and prove a result relating the vanishing of Novikov homology with a property of finite domination. An immediate corollary is a relation with finiteness properties of type FP_m in group theory. In a slightly different formulation, all this is due to [Bi-Re 1988]. The special case of degree one is proved in [Bi-Ne-St] and [Sik 1987].

6.1 Statement of the main result

We consider the following objects:

- a group G equipped with a surjective homomorphism $\pi : G \to \mathbb{Z}^r$, for some $r \in \mathbb{N}^*$.
- the vector space H_π of all homomorphism $\xi : G \to \mathbb{R}$ vanishing on $\ker \pi$.
- a chain complex $C* = \bigoplus_{j\geq 0} C_j$ over $\mathbb{Z}[G]$, with an augmentation $\varepsilon = \partial_0 : C_0 \to \mathbb{Z}$ such that C_j is free and finitely generated if $0 \leq j \leq m$ and $\ker \varepsilon = \partial C_1$, ie $C_1 \to C_0 \to \mathbb{Z}$ is a partial resolution (ie ε induces $H_0(C) \approx \mathbb{Z}$).

We can now state the main result, which is very closely related to Theorem 6.1 in [Bi-Re], as well as its proof.

Theorem. *The following are equivalent:*

(i) $H_j(C_*, \xi) = 0$ *for every* $j \leq m$ *and* $[\xi] \in S(H_\pi)$.

(ii) C_* *is homotopy equivalent over* $\mathbb{Z}[\ker \pi]$ *to a complex which is free and finitely generated in degrees* $\leq m$.

6.2 Variants of the two equivalent properties

Identifying $\mathrm{Hom}(\mathbb{Z}^r, \mathbb{R}) = \mathbb{R}^r$ and denoting $\langle x, y\rangle$ the Euclidean product on $\mathbb{R}^r$, one has an isomorphism $\mathbb{R}^r \to H_\pi$ given by $v \mapsto \langle v, \pi\rangle$. Thus (i) translates to

(i') $H_j(C_*\langle v, \pi\rangle) = 0$ *for every* $j \leq m$ *and* $v \in S^{r-1}$.

We fix a basis B_j on each C_j, $j = 0, \cdots, m$, and define the support of chains as in 2.4. Thus for every $R > 0$ we can define $C_*(R)$ formed by the chains of support in $\pi^{-1}(B(R)))$, which is free and finite over $\mathbb{Z}[\ker \pi]$.

Moreover, every subcomplex of C_* which is finitely generated over $\mathbb{Z}[\ker \pi]$ is contained in some $C_*(R)$. And (ii) translates to

(ii') *There exists* $R > 0$ *and a* $\mathbb{Z}[\pi]$*-linear homotopy* $h : \mathrm{Id}_{C_*} \simeq \varphi$ *such that* $\varphi(C_{\leq m}) \subset C_*(R)$ *and* $h_{|C_*(R)} = 0$.

Indeed, if h exists, φ is a homotopy equivalence to the smallest subcomplex of C_* containing $C_{\leq m}(R)$, which is free and finitely generated in degrees $\leq m$. And if ψ is a homotopy equivalence to $\widetilde{C}_*$ which is free and finitely generated in degrees $\leq m$, then if χ is a homotopy inverse, $h = \chi \circ \psi - \mathrm{Id}_*$ satisfies (ii') since every submodule of C_j which is finitely generated over $\mathbb{Z}[\ker \pi]$ is contained in some $C_j(R)$

6.3 Proof of $(i') \Rightarrow (ii)$

We follow closely [BiRe]. By Proposition 2 in 4.1, the hypothesis means that for every $v \in S^{r-1}$ there exists a $\mathbb{Z}[G]$-linear homotopy $h_v : \mathrm{Id}_{C_*} \simeq \varphi_v$ of C_* such that for every $e \in B_0 \cup \cdots \cup B_m$,

$$k \in \pi(\mathrm{supp}(\varphi_v(e))) \Rightarrow \langle v, k\rangle > \max(\xi_{|\mathrm{supp}(e)}).$$

For a fixed e, this property is open in $v \in S^{r-1}$. Since $B_0 \cup \cdots \cup B_m$ is finite and S^{r-1} is compact, there exists finitely many $v_i \in S^{r-1}$, $i \in I$, such that

$$(*) \qquad (\forall v \in S^{r-1})\ (\forall e \in B_0 \cup \cdots \cup B_m)\ (\exists i \in I) \ \ \langle v, k\rangle > 1 \text{ on } \pi(\text{supp } \varphi_{v_i}(e)).$$

Lemma. (cf. [Bi-Re], 4.3 and 4.7) *Up to changing C_* by a finite number of elementary expansions, we can assume that we also have, for v, e, i as in $(*)$:*

$$(**) \qquad \langle v, k\rangle > 1 \text{ on } \pi(\text{supp } h_{v_i}(e)) \setminus \text{supp}(e)).$$

Proof of the lemma. Recall that an *elementary expansion* (in the sense of Whitehead) of C_*, associated to an element $u \in C_j$, is a new complex $\widetilde{C}_*$, obtained by adding two new free generators (on $\mathbb{Z}[G]$), $x \in \widetilde{C}_{j+1}$, $x' \in \widetilde{C}_{j+2}$, such that $\partial x = \partial u$ and such that $\partial x' = x - u$. Then the natural injection $C_* \subset \widetilde{C}_*$ is a homotopy equivalence, and property (resp. (ii)) holds simultaneously for C_* and for $\widetilde{C}_*$. We extend the definition of the support by supp $x =$ supp ∂u, supp $x' =$ supp u.

We construct the new complex $\widetilde{C}_*$ and the homotopies $\widetilde{h}_{v_i} : \text{Id}_{\widetilde{C}} \simeq \widetilde{\varphi}_{v_i}$, satisfying (i) and (ii), by induction on m. We can assume that $(C, (h_{v_i}), (\varphi_{v_i}))$ satisfies $(*)$ and $(**)$ for $j < m$. Then $\widetilde{C}_*$ is obtained from C_* by making all the elementary expansions associated to the elements $u_{e,i} = e - \varphi_{v_i} e - h_{v_i} \partial e$, $e \in B_m, i \in I$. This means that we add to the basis B_{m+1} the elements $x_{e,i}$, and to B_{m+2} the elements $x'_{e,i}$, with $\partial x_{e,i} = \partial u_{e,i}$, $\partial x'_{e,i} = x_{e,i} - u_{e,i}$. Note that $\partial u_{e,i} = \partial h_{v_i} e$, thus since supp $\partial h_{v_i} \subset$ supp $h_{v_i} e$, by induction we have

$$\langle v, k\rangle > 1 \text{ on } \pi(\partial u_{e,i} \setminus \text{supp}(e)).$$

Then we define $\widetilde{h}|C_j = h$ if $j \neq m$ and $\widetilde{h}_{v_i}(e) = x_{e,i}$, $\widetilde{\varphi}_{v_i} = \text{Id}_{\widetilde{C}} - \partial \widetilde{h}_{v_i}$. Then $\widetilde{h}_{v_i}|C_j = h_{v_i}$ if $j < m$ and $\widetilde{\varphi}_{v_i}|C_j = \varphi_{v_i}$ if $j \leq m$. Moreover,

$$\text{supp } \widetilde{h}_{v_i}(e) = \text{supp } x_{e,i} = \text{supp } \partial u_{e,i} = \text{supp } \partial h_{v_i} e \subset \text{supp } h_{v_i} e.$$

Thus $(C, (h_{v_i}), (\varphi_{v_i}))$ satisfies $(*)$ and $(**)$ for all $j \leq m$, which proves the Lemma.

Construction of h (cf. [Bi-Re], 5.6). For $c \in C_*$, define $||c|| = \max_{g \in \text{supp}(c)} ||\pi(g)||$ if $c \neq 0$, $||c|| = 0$ otherwise. By finiteness and $\mathbb{Z}[G]$-linearity there exists $K \in \mathbb{N}^*$ such that $||e|| \leq K$ for every $e \in B_0 \cup \cdots \cup B_m$ and

$$(\forall j \leq m, \forall c \in C_j, \forall i \in I) \qquad \pi(\text{supp}(h_{v_i} c)) \cup \pi(\text{supp}(\varphi_{v_i}\ c)) \subset V_K(\pi(\text{supp}(c))).$$

If $j = 0$, choose $c_0 \in C_0$ such that $\varepsilon(c_0) = 1$. We shall construct h which satisfies $\varphi(C_{\leq m}) \subset C_*(R)$ with $R = \max(||c_0||, \frac{1}{2}K^2)$.

We define $h|C_j$ and $\varphi|C_j = \text{Id} - (h\partial + \partial h)$ by induction on $j \leq m$ (for $j > m$, h is arbitrary). Let $t \in T$, $e \in B_0$, then $te - \varepsilon(e)c \in \ker \varepsilon = \partial C_1$. We set

- $h(te) = 0$ if $||te|| \leq R$
- $h(te) = c_1 \in C_1$ such that $\partial c_1 = te - \varepsilon(e)c_0$ if $||te|| > R$.

By construction, $h|C_0(R) = 0$. Moreover, $\varphi(te) = te$ in the first case, $\varphi(te) = te - \partial h(te) = \varepsilon(e)c_0$ in the second case, thus $||\varphi(te)|| \leq ||c_0|| \leq R$ in all cases.

Assume that $j \leq m$, $h|C_k$ has been constructed for $k < j$ with the property $\varphi(C_k) \subset C_*(R)$ and $\varphi|C_k(R) = 0$. For $t \in T$, $e \in B_j$, we set

- $h(te) = 0$ if $||te|| \leq R$
- $h(te) = u$ for some $u \in C_{j+1}$ which minimizes $||te - \partial u - h\partial(te)||$ if $||te|| > R$.

By construction, $h|C_j(R) = 0$. Moreover, in the first case, $\varphi(te) = te - h\partial(te) = te$ since $||\partial(te)|| \leq R$. In the second case $\varphi(te) = te - \partial u - h\partial(te) =: c$, so we have to prove that $||c|| \leq R$. Note first that $\partial c = \varphi\partial(te)$, thus $||\partial c|| \leq R$ by induction.

Let $a = ||c||$. We can assume that among all elements c' in $c + \partial C_{j+1}$ with $||c'|| = a$, c has the smallest number of points in $\pi(\mathrm{supp}(c)) \cap S(0,a)$.

Let $k_0 \in \pi(\mathrm{supp}(c))$ such that $||k_0|| = a$. Set $v = -\theta(k_0) \in S^{r-1}$. Then $\langle v, k_0\rangle = -a$, and if $k \in \pi(\mathrm{supp}(c)) \setminus \{k_0\}$ one has $\langle v, k\rangle > -a$. By (i)', there exists $i \in I$ such that for all $k \in \pi(\mathrm{supp}\ \varphi_{v_i}(e))$, $\langle v, k\rangle > 1$, equivalently $\langle k_0, k\rangle < -a$. We write $c = c' + c''$ where c' is the some of all terms whose π-support contains k_0. Thus $||k - k_0|| \leq K$ on $\pi(\mathrm{supp}(c)')$ and $k_0 \notin \pi(\mathrm{supp}(c)'')$. Set

$$\begin{aligned}\widetilde{c} &= c - \partial h_{v_i}(c') \\ &= \varphi_{v_i}(c') + h_{v_i}\partial c' + c''. \\ &= \varphi_{v_i}(c') + h_{v_i}\partial c - h_{v_i}\partial c'' + c''.\end{aligned}$$

Assuming that $a > R$, we shall get a contradiction by proving that

$$||\widetilde{c}|| \leq a \text{ and card } \pi(\mathrm{supp}\ \widetilde{c}) \cap S(0,a) < \text{card } \pi(\mathrm{supp}(c)) \cap S(0,a).$$

1) We prove that $\pi(\mathrm{supp}\ \widetilde{c})$ does not contain k_0. If $k \in \pi(\mathrm{supp}\ \widetilde{c})$, then $k \in \pi(\mathrm{supp}(\varphi_{v_i}(c'))) \cup \pi(\mathrm{supp}(h_{v_i}\partial c)) \cup \pi(\mathrm{supp}(h_{v_i}\partial c'')) \cup \pi(\mathrm{supp}(c''))$. We have four cases:

(i) If $k \in \pi(\mathrm{supp}\ \varphi_{v_i}(c'))$, then $\langle k_0, k - k_0\rangle < -a$, thus $k \neq k_0$.

(ii) If $k \in \pi(\mathrm{supp}\ h_{v_i}\partial c)$, then by $(**)$ one of the two following subcases occurs:
- $k \in \pi(\mathrm{supp}\ \partial c)$: then $||k|| \leq R < a$, thus $k \neq k_0$
- $\langle v, k\rangle > -a$, thus $k \neq k_0$.

(iii) If $k \in \pi(\mathrm{supp}\ h_{v_i}\partial c'')$, then either $k \in \mathrm{supp}\ \partial c'' \subset \mathrm{supp}(c)''$, or $\langle v, k\rangle \geq \sup\{\langle v, k'\rangle \mid k' \in \mathrm{supp}(c)''\} > -a$, and in both cases $k \neq k_0$.

(iv) If $k \in \pi(\mathrm{supp}(c''))$, $k \neq k_0$ by definition.

2) We prove that $\pi(\mathrm{supp}\ \widetilde{c} \setminus \mathrm{supp}(c)) \subset B(0,a)$. If $k \in \pi(\mathrm{supp}\ \widetilde{c} \setminus \mathrm{supp}(c))$, the first expression for $\widetilde{c}$ shows that $k \in \pi(\mathrm{supp}\ h_{v_i}c' \setminus \mathrm{supp}(c))$, thus $||k - k_0|| \leq K$ and $\langle v, k - k_0\rangle > 1$, ie $\langle k_0, k - k_0\rangle < -a$. Thus

$$\begin{aligned}||k||^2 = ||k_0 + (k - k_0)||^2 &= ||k_0||^2 + ||k - k_0||^2 + 2\langle k_0, k - k_0\rangle \\ &< a^2 + ||k - k_0||^2 - 2a \\ &\leq a^2 + K^2 - 2a.\end{aligned}$$

Since $a > \frac{1}{2}K^2$, $||k||^2 < a^2$, qed.

3) Finally, $\pi(\mathrm{supp}(\widetilde{c})) \subset \pi(\mathrm{supp}(c)) \cup B(0,a) \setminus \{k_0\}$, which proves the desired contradiction.

6.4 Proof of $(ii) \Rightarrow (i')$

Fix $v \in S^{r-1}$. Fix $g \in G$ such that $\langle v, \pi(g)\rangle > 0$. Choose $N \in \mathbb{N}^*$ such that

$$N\langle v, \pi(g)\rangle > R + \max_{e \in B} \max(\xi_{|\mathrm{supp}(e)})$$

and set $h_v(e) = g^N h(g^{-N}e)$ if e is a basis element. Then $\varphi_v(e) = g^{-N}\varphi(g^N e)$, thus if $\deg(e) \leq m$ one has

$$\varphi_v(e) = g^N \varphi(g^{-N}e) \in g^N C_*(R).$$

If $k \in \pi(\mathrm{supp}(\varphi_v(e)))$, we have

$$\langle v, k\rangle \geq N\pi(g) - R > \max(\xi_{|\mathrm{supp}(e)}),$$

ie (i') is satisfied, qed.

6.5 Relations with properties FP_m

Recall the definition (cf. [Bro 1982]): for $m \in \mathbb{N}^*$, a group G has *property* FP_m [for "finite projective"] if $\mathbb{Z}$ has a partial resolution of length m over $\mathbb{Z}[G]$ which is projective (or free) and finite. This is obviously implied by the existence of a CW-complex X which is a $K(G,1)$ with finite m-skeleton, thus

- If G is finitely generated, G satisfies FP_1. Actually, a simple nice argument shows that the converse is true.
- If G is finitely presented, G satisfies FP_2. the converse implication was an open question for thirty years, which was answered in the negative by [Be-Bra 1996]. We shall later interpret this result of Bestvina and Brady in terms of Novikov homology.

The theorem of the previous section has the

Corollary. *Assume that* $\pi : G \to \mathbb{Z}^r$ *is a surjective morphism and that* G *satisfies* FP_m. *The following are equivalent:*

(i) $H_j(G, \xi_v) = 0$ *for every* $[\xi] \in S(G)$ *vanishing on* $\ker\pi$.

(ii) $\ker\pi$ *satisfies* FP_m.

Proof of the corollary. By hypothesis, there is a free resolution $C_* \to \mathbb{Z}$ over $\mathbb{Z}[G]$, which is finitely generated in degrees $\leq m$. Then $(\forall[\xi \in S(G))\ H_j(G,\xi) = H_j(C_*,\xi)$ (cf. Section 2). Since all free resolutions are homotopy equivalent, (ii) means that C_* is homotopy equivalent over$\mathbb{Z}[\ker\pi]$ to a free $\mathbb{Z}[\ker\pi]$-complex which is finite up to degree m. By the Theorem, this is equivalent to (i).

Remark. The condition that G satisfies FP_m cannot be removed. Indeed, if $\ker\pi$ satisfies FP_m, then since $\mathbb{Z}^r \simeq G/\ker\pi$ satisfies FP_n for every n, G satisfies FP_m (cf. [Ser], p.87-88). On the other hand, if H is any group, then one has

$$H(H \times \mathbb{Z}, \mathrm{pr}_2) = 0.$$

Proof. If $C_* \to \mathbb{Z}$ is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[H]$, say with $C_0 = \mathbb{Z}[G]$, combining with the free resolution $\mathbb{Z}[\mathbb{Z}] \to \mathbb{Z}[Z] \to \mathbb{Z}$ we obtain a free resolution $\widetilde{C} \to \mathbb{Z}$ over $\mathbb{Z}[H \times \mathbb{Z}]$ (cf. [Ser]), with

$$\begin{aligned}
&\widetilde{C}_0 = \mathbb{Z}[H \times \mathbb{Z}]\\
&\widetilde{C}_n = (\mathbb{Z}[H \times \mathbb{Z}] \otimes_{\mathbb{Z}[H]} C_n) \oplus (\mathbb{Z}[H \times \mathbb{Z}] \otimes_{\mathbb{Z}[H]} C_{n-1}) \text{ if } n \geq 1\\
&\widetilde{D}_1 = \begin{pmatrix} D_1 \\ t-1 \end{pmatrix}, \ \widetilde{D}_n = \begin{pmatrix} D_n & 0 \\ (t-1)\mathrm{Id} & D_{n-1} \end{pmatrix},
\end{aligned}$$

where D_n is the (possibly infinite) matrix associated to ∂_n. We have to find matrices H_n such that $\mathrm{Id} + \widetilde{D}_{n+1}H_n + H_{n-1}\widetilde{D}_n$ has coefficients of the form $\sum_{k>0} \lambda_k t^k$, $\lambda_k \in \mathbb{Z}[H]$. It suffices to take

$$H_0 = \begin{pmatrix} 0 & 1 \end{pmatrix}$$
$$H_n = \begin{pmatrix} 0 & \mathrm{Id} \\ 0 & 0 \end{pmatrix} \text{ if } n \geq 1,$$

so that

$$1 + \partial H_0 = t$$
$$\mathrm{Id} + \widetilde{D}_{n+1}H_n + H_{n-1}\widetilde{D}_n = \begin{pmatrix} t\mathrm{Id} & 0 \\ 0 & t\mathrm{Id} \end{pmatrix} \text{ if } n \geq 1.$$

Question. What can we say of $\xi : G \to \mathbb{Z}$ such that $H(G, \xi) = 0$?

6.6 Topological version

Now we assume that X has a finite m-skeleton for some positive integer m, and that we are given a surjective morphism $p : \pi_1(X) \to \mathbb{Z}^r$. We realize p by a map $F : X \to T^r$, where $r = \mathrm{rk}(\xi)$ and $F_\sharp = p$. We impose $F(x_0) = [0]$ where x_0 is the basis point of X.

Let $\pi : \widehat{X}_\xi \to X$, the covering associated to $\ker v$, this F lifts to $\widehat{F} : \widehat{X}_\xi \to \mathbb{R}^r$. For $R > 0$, we define $\widehat{X}(R) \subset \widehat{X}_\xi$ as the smallest subcomplex containing $\widehat{F}^{-1}(B(R))$. Note that it is finite. Similarly, denote by $\widetilde{F} : \widetilde{X} \to \mathbb{R}$ the lift of $\widehat{F}$ to the universal covering, and by $\widetilde{X}(R) \subset \widetilde{X}$ the inverse image of $\widehat{X}(R)$.

Then the theorem in section 5 admits the following immediate interpretation.

Theorem. *The following are equivalent:*

(i) $H_j(X, \xi_v) = 0$ *for every* $j \leq m$ *and* $v \in \mathbb{R}^r \setminus \{0\}$.

(ii) $C_*(\widetilde{X})$ *is finitely dominated over* $\mathbb{Z}[\ker \pi]$ *up to degree* m.

Proof. (ii) implies that $C_*(\widetilde{X})$ is finitely dominated, up to degree m, over $\mathbb{Z}[\ker \pi]$. By the theorem in section 5, this implies (i).

Assume now that (i) holds. By the same theorem, for R large enough $\mathrm{Id}_{C_*(\widetilde{X})}$ is homotopic to a map $\varphi : C_*(\widetilde{X}) \to C_*(\widetilde{X}(R))_p$with values in $C_*(\widetilde{X}(R))$.

Remark. If $m \geq 2$, it is not true that (ii) is equivalent to $\widehat{X}$ being finitely dominated, up to degree m: it would imply that $\pi_1(\widehat{X})$ is finitely presented (in fact, by [Wa 1965] $\widehat{X}$ would be homotopy equivalent to a CW-complex with finite m-skeleton. The example of Bestvina and Brady shows that it is not always the case.

7 The Morse-Novikov complex

In this section we specialize to $X = M$ a closed and connected smooth manifold of dimension n, so that ξ is the cohomology class of one-form $\omega \in \Omega^1(M)$ which is closed but not exact.

7.1 Morse one-forms

A closed one-form ω is a *Morse form* if near every point in $\mathrm{Sing}(\omega) = \omega^{-1}(0)$, there exists a Morse function f such that $\omega = df$. Thus there are coordinates $(x_1, \cdots, x_n)$ such that $\omega = d(-\sum_{j=1}^{i} x_j^2 + \sum_{j=i+1}^{n} x_j^2)$, where $i = \mathrm{ind}(x)$ is the *index* of x. If $\mathrm{ind}(x) = 0$ or n, x is called a *center*.

As in the case of functions, Morse one-forms are dense among closed forms in a given cohomology class in the $C{+}\infty$ topology. The fact that M admits a Morse function with exactly one local maximum and one local minimum has the following analogue.

Proposition (cf Arnoux-Levitt 1986]). *If $\xi \neq 0$, there exists a Morse form representing ξ which has no centers.*

7.2 The complex

Let ω be a closed nonexact Morse form on a closed manifold M. Denote $[\omega] = \xi \in H^1(M;\mathbb{R}) = \mathrm{Hom}(\pi_1(M), \mathbb{R})$. Let $\pi : \widetilde{M} \to M$ be the universal covering, with $\mathrm{Aut}(\widetilde{M}|M)$ identified with $\pi_1(M)$ acting on the left. Thus $\pi^*\omega = df$ for some Morse function f on $\widetilde{M}$.

For each $x \in \mathrm{Sing}(\omega)$, choose a lift $\widetilde{x} \in \widetilde{M}$, defining thus a bijection from $\mathrm{Sing}(\omega) \subset M$ to $\widetilde{\mathrm{Sing}}(\omega) \subset \widetilde{M}$, and $\mathrm{crit}(f)$

Let X be a vectorfield X on M which is quasigradient with respect to ω. More precisely, we assume that $\omega(X) > 0$ on $M \setminus \mathrm{Sing}(\omega)$, and that near every $x \in \mathrm{Sing}(M)$ there is a coordinate chart in which

$$\omega = \sum \pm x_i dx_i \ , \ X = \sum \pm x_i \frac{\partial}{\partial x_i}.$$

Thus for each $x \in \mathrm{Sing}(\omega)$ one has the stable manifold $W^s(z)$, injective immersion of $\mathbb{R}^{\mathrm{ind}(x)}$, and the unstable manifold $W^u(z)$, injective immersion of $\mathbb{R}^{n-\mathrm{ind}(x)}$. We choose an orientation of each stable manifold, and thus a coorientation of each unstable manifold.

This lifts to orientations and coorientations of the stable and unstable manifolds of the lift $\widetilde{X}$. Note that in $\widetilde{M}$ these are true submanifolds. If $z, w \in \mathrm{Crit}(f)$, define $\mathcal{M}(z,w) = W^z(z) \cap W^u(w)$. Generically, $W^s(z)$ and $W^u(w)$ are transverse, thus $\mathcal{M}(z,w)$ is an oriented submanifold of dimension $\mathrm{ind}(z) - \mathrm{ind}(w)$, which admits a natural $\mathbb{R}$-action. Thus $\mathcal{M}(z,w)$ is empty unless $\mathrm{ind}(z) > \mathrm{ind}(w)$. The quotient $\overline{\mathcal{M}}(z,w)$ is an oriented manifold of dimension $\mathrm{ind}(z) - \mathrm{ind}(w) - 1$: it corresponds to unparametrized trajectories of $-\widetilde{X}$ descending from z to w.

Theorem. *For X generic, one has the following properties:*

(i) *$W^s(z)$ and $W^u(w)$ are transverse if* $\mathrm{ind}(z) > \mathrm{ind}(w)$.

(ii) *$\overline{\mathcal{M}}(z,w)$ is compact (ie finite) if* $\mathrm{ind}(z) - \mathrm{ind}(w) = 1$*; it is thus a compact oriented zero-dimensional manifold, and one can define $m(z,w) \in \mathbb{Z}$, its algebraic number of points.*

(iii) *One can define a $\mathbb{Z}[\pi_1(M)]_\xi$-complex $C_*(\omega, X)$ which is free over $\widetilde{\mathrm{Sing}}(\omega)$, graduated by the index, and such that*

$$\partial\widetilde{x} = \sum_{g \in \pi_1(M)} m(\widetilde{x}, g\widetilde{y}) g\widetilde{y}.$$

(iv) The homology of this complex is canonically isomorphic to $H(M, \xi)$.

(v) The simple homotopy type of $C_*(\omega, X)$ *is the same as* $C_*(\widetilde{M})(-\xi)$*,where* $C_*(\widetilde{M})$ *is the singular complex of* $\widetilde{M}$*, or any free* $\mathbb{Z}[\pi_1(M)]$*-complex having the same homotopy type, eg a cellular chain complex.*

7.3 Theorems of Pazhitnov and Latour

8 Abelian Novikov homology (i): Euclidean property, Morse inequalities

In this section we assume that $G = \mathbb{Z}^m$ for some $m \in \mathbb{N}^*$. The group ring $\mathbb{Z}[\mathbb{Z}^m]$ can be identified with the ring of m-variable Laurent polynomials $\mathbb{Z}[t_1^{\pm 1}, \cdots, t_m^{\pm 1}]$. For $n = (n_1, \cdots, n_m) \in \mathbb{Z}^m$, we denote $t^n = t_1^{n_1} \cdots t_m^{n_m}$.

Let $\xi \in \mathrm{Hom}(\mathbb{Z}^m; \mathbb{R}) \setminus \{0\}$, ie $\xi(n_1, \cdots, n_m) = \xi(t_1^{n_1} \cdots t_m^{n_m}) = \sum_{i=1}^m \alpha_i n_i$ with the α_i not all zero. We investigate the properties of

$$\mathbb{Z}[\mathbb{Z}^m]_\xi = \{\sum_{i=1}^{+\infty} a_i t^{n_i} \mid a_i \in \mathbb{Z} \ , \ n_i \in \mathbb{Z}^m \ , \ \lim_{i \to \infty} \xi(n_i) = +\infty\}.$$

Note first that, since $\mathbb{Z}^m$ is orderable, the invertibles of $\mathbb{Z}[\mathbb{Z}^m]$ are trivial, and the invertibles of $\mathbb{Z}[\mathbb{Z}^m]_\xi$ are trivial.

8.1 The generic case: $\mathbb{Z}[\mathbb{Z}^m]_\xi$ is Euclidean

We assume here that $\xi \in \mathrm{Inj}(\mathbb{Z}^m, \mathbb{R})$, the set of injective homomorphisms. In other words, the α_i are linearly independent over $\mathbb{Q}$. Then every $\lambda \in \mathbb{Z}[\mathbb{Z}^m]_\xi \setminus \{0\}$ is ξ-simple, thus n_ξ is defined on all of $\mathbb{Z}[\mathbb{Z}^m]_\xi$.

Proposition (cf Pazhitnov ?) *Assume that $\xi \in \mathrm{Hom}(\mathbb{Z}^m, \mathbb{R})$ is injective. Then n_ξ is a Euclidean norm on $\mathbb{Z}[\mathbb{Z}^m]_\xi$, ie: if $\lambda, \mu \in \mathbb{Z}[\mathbb{Z}^m]_\xi$ with $\mu \neq 0$, there exists $\gamma \in \mathbb{Z}[\mathbb{Z}^m]_\xi$ such that $n_\xi(\lambda - \mu\gamma) < n_\xi(\mu)$. Thus $\mathbb{Z}[\mathbb{Z}^m]_\xi$ is a principal ideal domain.*

Proof. We have $m_\xi(\lambda) = a_0 g_0$, $m_\xi(\mu) = b_0 h_0$ with $a_0 = \pm n_\xi(\lambda)$, $g_0 = \gamma_\xi(\lambda)$, $b_0 = \pm n_\xi(\mu)$, $h_0 = \gamma_\xi(\mu)$. Divide a_0 by b_0 in $\mathbb{Z}$: $a_0 = b_0 q_0 + r_0$, with $0 \leq r_0 < n_\xi(\mu)$. If $r_0 \neq 0$ ie $n_\xi(\mu)$ does not divide $n_\xi(\lambda)$, then $\gamma = q_0 g_0 h_0^{-1}$ has the desired property. The same is true if $\lambda = \mu q_0$.

If $r_0 = 0$ and $\lambda \neq \mu q_0 g_0 h_0^{-1}$, we define

$$\lambda_1 = \lambda - \mu q_0 g_0 h_0^{-1} = t_\xi(\lambda) - q_0 g_0 t_\xi(\mu) h^{-1} \ , \ g_1 = \gamma_\xi(\lambda_1).$$

Applying the same construction with (λ, μ) replaced by (λ_1, μ), and iterating the process as long as possible, we define $(\lambda_n, g_n, \gamma_n, q_n)$ so that $\lambda_0 = \lambda, \gamma_0 = 1$ and

$$\begin{cases} \lambda_n = \lambda_{n-1} - \mu q_{n-1} g_{n-1} h_0^{-1} = t_\xi(\lambda_{n-1}) - q_{n-1} g_{n-1} t_\xi(\mu) h_0^{-1} \\ g_n = \gamma_\xi(\lambda_n) \\ \gamma_n = \gamma_{n-1} + q_n g_n h_0^{-1} = \displaystyle\sum_{i=0}^{n} q_i g_i h_0^{-1}. \end{cases}$$

We have thus $\lambda - \mu\gamma_{n-1} = \lambda_n$.

There are two possibilities:

- The process stops because we reach some n for which $n_\xi(\mu)$ does not divide $n_\xi(\lambda_n)$ or $\lambda_n = \mu q_n \gamma_n$. Then we can take $\gamma = \gamma_n$.
- Or it goes on indefinitely. Since $g_n = \gamma_\xi(\lambda_n)$ and $\lambda_n = t_\xi(\lambda_{n-1}) - q_{n-1} g_{n-1} t_\xi(\mu) h_0^{-1}$, we have

$$g_n \in E_n := \bigcup_{i=0}^{n} \mathrm{supp}(\lambda) \mathrm{supp}((t_\xi(\mu) h_0^{-1})^i),$$

where $B^i = \{b_1 \cdots b_i \mid b_j \in B\}$. Also, $\xi(g_n)$ is increasing. Since $\mathrm{supp}((t_\xi(\mu)h_0^{-1}) \subset]0, +\infty[$, $\bigcup_{n=0}^{\infty} E_n$ has only a finite number of elements in $\{\xi \leq C\}$, thus $\xi(g_n) \to +\infty$. Thus $\sum_{n=0}^{+\infty} q_n g_n h_0^{-1}$ converges to some $q \in \mathbb{Z}[\mathbb{Z}^m]_\xi$, and by construction one has $\lambda = \mu q$. This proves Proposition 1.

An immediate consequence of the proof is that the gcd in $\mathbb{Z}[\mathbb{Z}^m]_\xi$ of two elements in $\mathbb{Z}[\mathbb{Z}^m]$ is often in $\mathbb{Z}[\mathbb{Z}^m]$.

Proposition 8.2 *Let $\xi \in \mathrm{Inj}(\mathbb{Z}^m, \mathbb{R})$, and let λ, μ be nonzero elements of $\mathbb{Z}[\mathbb{Z}^m]$. Then either μ divides λ in $\mathbb{Z}[\mathbb{Z}^m]_\xi$, or the Euclidean division holds in $\mathbb{Z}[\mathbb{Z}^m]$, ie there exists $\gamma \in \mathbb{Z}[\mathbb{Z}^m]$ such that $n_\xi(\lambda - \gamma\mu) < n_\xi(\mu)$.*

Novikov completion of an ideal. Let I be an ideal in $\mathbb{Z}[\mathbb{Z}^m]$. Its *Novikov completion* I_ξ is the ideal it generates over $\mathbb{Z}[\mathbb{Z}^m]_\xi$.

Proposition 8.3 *If $\xi \in \mathrm{Inj}(\mathbb{Z}^m, \mathbb{R})$, the Novikov completion I_ξ of an ideal I in $\mathbb{Z}[\mathbb{Z}^m]_\xi$ is generated by an element of I.*

Proof. The ring $\mathbb{Z}[\mathbb{Z}^m]$ is Noetherian, thus I has a finite generating system $(\lambda_1, \cdots, \lambda_n)$. We prove the result by induction on n, it is clear that it suffices to prove the case $n = 2$: this follows from Proposition 2, since the Euclidean algorithm takes place inside $\mathbb{Z}[\mathbb{Z}^m]$ until we reach a stage when we have two elements in $\mathbb{Z}[\mathbb{Z}^m]$ one of which divides the other in $\mathbb{Z}[\mathbb{Z}^m]_\xi$, in which case this element is the desired generator.

8.3 Morse inequalities

Recall first the structure of homology groups over a principal ideal ring R. If $C_* = (C_i)$ is a complex over R which is free of finite rank in each dimension, the R-module $H_i(C)$ is finitely generated, thus it is isomorphic to $F_i \oplus T_i$ where T_i is the torsion submodule and $F_i \simeq H_i(C)/T_i$ is a free module of finite rank. This rank is called the i-eth Betti number of C_*, and denoted by $b_i(C, R)$ or $b_i(C)$.

Moreover, one has an essentially unique decomposition into cyclic modules $T_i \simeq \bigoplus_{j=1}^{t_i} R/(e_{i,j})$, where $e_{i,j} \in R$ (elementary divisor) is not a unit and divides $e_{i,j+1}$. The number $t = t_i(C, R)$ is the minimal number of generators of T_i.

This applies in particular when $R = \mathbb{Z}[G/\ker \xi]_{\overline{\xi}}$, the ring of Abelian Novikov homology. The numbers b_i, t_i will then be denoted by $b_i^{ab}(C_*, \xi), t_i^{ab}(C_*, \xi)$.

Proposition. *Let C_* be a $\mathbb{Z}[G]$-complex, which is free of finite rank up to degree k, with $\mathrm{rk}\, C_i = c_i$. One has the strong Morse inequalities*

$$(\forall i \leq k) \qquad \sum_{j \leq i} (-1)^{i-j} c_{i-j} \geq \sum_{j \leq i} (-1)^{i-j} (b_j^{ab}(C_*, \xi) + t_j^{ab}(C_*, \xi) + t_{j-1}(C_*, \xi)).$$

Summing form 0 to i, one obtains the Morse inequalities

$$(\forall i \leq k) \qquad c_i \geq b_i^{ab}(C_*, \xi) + t_i^{ab}(C_*, \xi) + t_{i-1}^{ab}(C_*, \xi)).$$

9 Abelian Novikov homology (ii): structure of the vanishing locus

9.1 Principal ideals of $\mathbb{Z}[\mathbb{Z}^m]_\xi$

It follows from general results on completions that $\mathbb{Z}[\mathbb{Z}^m]_\xi$ is factorial and Noetherian. However, when ξ is not injective, it is no longer a principal ideal domain, since $m_\xi(I)$ is an arbitrary ideal of $\mathbb{Z}[\ker\xi] \simeq \mathbb{Z}[t_1^{\pm1}, \cdots, t_k^{\pm1}]$, with $k \geq 1$.

Proposition. *Let I be an ideal of $\mathbb{Z}[\mathbb{Z}^m]$. Its Novikov completion I_ξ is principal if and only if $m_\xi(I)$ is principal. More precisely, let $\lambda \in I$. Then λ generates I_ξ if and only if $m_\xi(\lambda)$ generates $m_\xi(I)$.*

Proof. "only if": obvious.

"if": it suffices to prove that λ divides $\mu \in I$ in $\mathbb{Z}[\mathbb{Z}^n]_\xi$. The Euclidean division algorithm of μ by λ goes on indefinitely since $m_\xi(\lambda)$ always divides $m_\xi(\mu)$, and goes on indefinitely, qed.

9.2 Divisibility in $\mathbb{Z}[\mathbb{Z}^m]_\xi$ of elements in $\mathbb{Z}[\mathbb{Z}^m]$

Proposition. *Let λ, μ be two elements of $\mathbb{Z}[\mathbb{Z}^m]$, such that μ divides λ in $\mathbb{Z}[\mathbb{Z}^m]_\xi$. Then there exist $s \in S_\xi$ such that μ divides λs in $\mathbb{Z}[\mathbb{Z}^m]$.*

Proof. The image of the multiplication by μ over $\mathbb{Z}[\mathbb{Z}^m]_\xi$ contains $\lambda \in \mathbb{Z}[\mathbb{Z}^m]$. Since $\mathbb{Z}[\mathbb{Z}^m]_\xi$ is faithfully flat over $S_\xi^{-1}\mathbb{Z}[\mathbb{Z}^m]$, it is already true over $S_\xi^{-1}\mathbb{Z}[\mathbb{Z}^m]$, ie there exists $s \in S_\xi$ and $\nu \in \mathbb{Z}[\mathbb{Z}^m]$ such that $\mu\nu = \lambda s$.

9.3 Finiteness of the number of quotients

Let $\lambda, \mu \in \mathbb{Z}[\mathbb{Z}^m]$. We investigate the number of possible quotients in $\mathbb{Z}[\mathbb{Z}^m]$, ie γ such that $n_\xi(\lambda - \mu\gamma) < n_\xi(\lambda)$ for some ξ (not necessarily injective).

Lemma. *There exists $N = N(\lambda, \mu)$ with the following property. Assume that μ is ξ-simple and that there exists $\gamma \in \mathbb{Z}[\mathbb{Z}^m]$ such that*

$$\mathrm{supp}(\lambda - \mu\gamma) \cap \Big(\bigcup_{i=0}^{N} \mathrm{supp}(\lambda)\mathrm{supp}(t_\xi(\mu)\gamma_\xi(\mu)^{-1})^i \Big) = \emptyset.$$

Then μ divides λ in $\mathbb{Z}[\mathbb{Z}^m]_\xi$.

Proof. If $v_\xi(\lambda - \mu\gamma) > v_\xi(\mu)$, $m_\xi(\mu)$ divides $m_\xi(\lambda)$. As in 8.1, define

$$E_n := \bigcup_{i=0}^{n} \mathrm{supp}(\lambda)\mathrm{supp}((t_\xi(\mu)\gamma_\xi(\mu)^{-1})^i) \ , \ E_\infty = \bigcup_{n=0}^{\infty} E_n.$$

In the field $\mathbb{Q}[\mathbb{Z}^m]_\xi$, we have $\lambda\mu^{-1} = \sum_{k \in E_\infty} a_k t^k$, $a_k \in \mathbb{Q}$. We want to prove that the a_k are integers. Let $S = \mathrm{supp}(t_\xi \mu)\gamma_\xi(\mu)^{-1} \subset \xi^{-1}(]0, +\infty[)$, so that $\mu = \pm\gamma_\xi(\mu)(n_\xi(\mu) + \sum_{\ell \in S} m_\ell t^\ell)$. The equation $\lambda = \mu \sum_k a_k t^k$ gives

$$(*) \qquad \pm n_\xi(\mu) a_k = -\sum_{\ell \in S} m_\ell a_{k-\ell}.$$

The hypothesis implies that $a_k \in \mathbb{Z}$ for all k in E_N. We can order $E_\infty = \{k_n \mid n \in \mathbb{N}\}$ so that $\{k_0, \cdots, k_n\} \subset E_n$ for all n. Set $x_n = a_{k_n} \in \mathbb{Q}$, the formula $(*)$ becomes a recurrence formula

$$x_n = b_1 x_{n-1} + \cdots + b_r x_{n-r}\ ,\ b_i \in \mathbb{Q}.$$

Moreover, $x_n \in \mathbb{Z}$ if $n \leq N$, and $r, x_0, \cdots, x_{r-1}, b_1, \cdots, b_r$ depend only on (λ, μ). To finish the proof, we need to show that $x_n \in \mathbb{Z}$ for all $n \in \mathbb{N}$ if N is large enough.

Let $F_N \subset \mathbb{Z}^r$ be the subgroup generated by $(x_{n-1}, \cdots, x_{n-r})$, $r \leq n \leq N$. Then $b_1 y_1 + \cdots + b_r y_r \in \mathbb{Z}$ for every $(y_1, \cdots, y_r) \in F_N$. Since F_n is a non-decreasing sequence of subgroups of $\mathbb{Z}^r$, there exists

$$N = N(r, x_0, \cdots, x_{r-1}, b_1, \cdots, b_r) = N(\lambda, \mu) \in \mathbb{N}$$

such that $F_n = F_N$ for $n \geq N$. Thus if $N = N(\lambda, \mu)$, $x_n \in \mathbb{Z}$ for all $n \in \mathbb{N}$. Thus $a_k \in \mathbb{Z}$ for all k, qed.

Proposition. *Let λ, μ be nonzero elements of $\mathbb{Z}[\mathbb{Z}^m]$. Then there exists $A(\lambda, \mu) \subset \mathbb{Z}[\mathbb{Z}^m]$ finite with the following property.*

Assume that μ is ξ-simple (for instance ξ injective) and does not divide λ in $\mathbb{Z}[\mathbb{Z}^m]_\xi$, and the Euclidean division process of λ by μ terminates in a finite number of steps, ie there exists $\gamma \in \mathbb{Z}[\mathbb{Z}^m]$ such that $0 < n_\xi(\lambda - \mu\gamma) < n_\xi(\mu)$. Then it terminates in at most $N(\lambda, \mu)$ steps, and the quotient of λ by μ belongs to $A(\lambda, \mu)$.

Proof. By induction on n, the set $A_n(\lambda, \mu)$ of quotients of λ by μ which are obtained after at most n steps is finite, thus $A(\lambda, \mu) := A_{N(\lambda,\mu)}(\lambda, \mu)$ is finite. By the lemma,

$$(\forall \gamma \in \mathbb{Z}[\mathbb{Z}^m]) \qquad \mathrm{supp}(\lambda - \mu\gamma) \cap \Big(\bigcup_{i=0}^{N} \mathrm{supp}(\lambda)(\mathrm{supp}(t_\xi(\mu)\gamma_\xi(\mu)^{-1})^i \Big) \neq \emptyset,$$

which means that the process of division by μ stops after at most N steps. The quotient γ belongs to A, and satisfies $n_\xi(\lambda - \mu\gamma) < n_\xi(\lambda)$. This proves the proposition.

9.4 Finiteness of the number of g.c.d.'s

Let I be an ideal of $\mathbb{Z}[\mathbb{Z}^m]$.

Proposition. *There exists a finite set $A \subset I$ with the following property. For every nonzero $\xi \in \mathrm{Hom}(\mathbb{Z}^m, \mathbb{R})$ such that I_ξ is principal, A contains a containing a generator of I_ξ. This is in particular true for every ξ which is injective.*

Proof. The ideal I has a finite generating family $(\lambda_1, \cdots, \lambda_k)$. When ξ varies in $\mathrm{Inj}(\mathbb{Z}^m, \mathbb{R})$, the k-tuple $(\gamma_\xi(\lambda_1) \cdots, \gamma_\xi(\lambda_k))$ takes a finite number of values. We define a *chamber* $C \subset \mathrm{Inj}(\mathbb{Z}^m, \mathbb{R})$ to be the inverse image of one of these values. On each chamber C, the integers $m_\xi(\lambda_i)$, $\gamma_\xi(\lambda_i)$ and $n_\xi(\lambda_i)$ are determined. We define $n(C) = \min(n_\xi(\lambda_1), \cdots, n_\xi(\lambda_k))$.

Since there is only a finite number of chambers, it suffices to find for each C a finite set $A(C) \subset I$ which contains a generator of I_ξ for every $\xi \in C$ such that I_ξ is principal. We prove this by induction on $n(C)$.

• If $n(C) = 1$, for some i we have $m_\xi(\gamma_\xi(\lambda_i)^{-1}\lambda_i) = 1$ for every $\xi \in C$. Then $\alpha(C) = \gamma_\xi(\lambda_i)^{-1}\lambda_i$ generates $\mathbb{Z}[\mathbb{Z}^m]_\xi$ for all $\xi \in C$, so that $A(C) = \{\alpha(C)\}$ has the desired property.

• Assume the existence of $A(C')$ is known for every C' with $n(C') < n(C) = n$. Adding a generator, we can assume that $n_\xi(\lambda_k) = d(C)$. We divide $\lambda_1, \cdots, \lambda_{k-1}$ by λ_k in $\mathbb{Z}[\mathbb{Z}^m]_\xi$. If λ_k

divides all the λ_i in $\mathbb{Z}[\mathbb{Z}^m]_\xi$, we can take $A(C) = \{\lambda_k\}$. If not, by Proposition 8.4, there exists γ in the finite set $\bigcup_{i<k} A(\lambda_i, \lambda_k)$ such that $n_\xi(\lambda_i - \lambda_k\gamma) < n_\xi(\lambda_k)$. Replacing λ_i by $\lambda_i - \lambda_k\gamma$, we obtain finitely many k-uplets $(\lambda_1^{(j)}, \cdots, \lambda_k^{(j)})$ such that $n_\xi(\lambda_i^{(j)}) < n_\xi(\widetilde{\lambda}_k^{(j)})$. Denote $C^{(j)}$ the associated chambers. Then $\min(n_\xi(\lambda_1^{(j)}), \cdots, n_\xi(\lambda_k^{(j)})) < n$, thus the inductive hypothesis gives finitely many sets $A(C^{(j)})$, their union is the desired set $A(C)$.

Corollary. *Define $N(I) = \{\xi \in \mathbb{R}^m \setminus \{0\} \mid I_\xi = \mathbb{Z}[\mathbb{Z}^m]_\xi\}$. Then $N(I)$ is a finite union of finite intersections of open integer half-spaces. Or (in the language of [Bi-Ne-St]) it is* rationally defined.

Proof. Let $A = \{a_1, \cdots, a_r\}$ be as in the proposition. Then $N(I) = \mathbb{Z}[\mathbb{Z}^m]_\xi$ if and only if some a_i belongs to S_ξ ie $m_\xi(a_i) = \pm 1$. Thus

$$N(I) = \bigcup_{i,g \mid a_i(g) = \pm 1} \quad \bigcap_{h \in \operatorname{supp}(a_i) \setminus \{g\}} \{\xi \mid \xi(hg^{-1}) > 0\},$$

which proves the corollary.

9.5 Structure of the set of homomorphisms with vanishing homology

Proposition. *Let (C, ∂) be a differential module over $\mathbb{Z}[\mathbb{Z}^r]$, which is free of finite rank. Define*

$$N(C) = \{\xi \in (\mathbb{R}^m)^* \setminus \{0\} \mid H(C_*, \xi) = 0\}.$$

Then $N(C)$ has a rational polyhedral structure (or: is rationally defined), ie it is the union of a finite number of cones which are each the intersection of a finite number of rational (or integer) open half-spaces.

Proof. Since $\mathbb{Z}[\mathbb{Z}^r]$ is Noetherian, we have a presentation $(\mathbb{Z}[\mathbb{Z}^r])^m \to (\mathbb{Z}[\mathbb{Z}^r])^n \to H(C)$, where the first morphism is given by right multiplication with a matrix $A \in \mathrm{M}_{m,n}(\mathbb{Z}[\mathbb{Z}^r])$. Then the same matrix gives a presentation $(\mathbb{Z}[\mathbb{Z}^r]_\xi)^m \to (\mathbb{Z}[\mathbb{Z}^r]_\xi)^n \to H(C_*, \xi)$.

Thus $H(C_*, \xi)$ vanishes if and only if A is left invertible, ie the ideal $I_\xi \subset \mathbb{Z}[\mathbb{Z}^r]_\xi$ generated by the n-minors of A is equal to $\mathbb{Z}[\mathbb{Z}^r]_\xi$. By the corollary in 2.5, this is a rational polyhedral condition on ξ.

9.6 Faithful flatness of $\mathbb{Z}[G]_\xi$ over $\Sigma_\xi^{-1}\mathbb{Z}[G]$

Proposition [Lat 1994, Proposition 1.14] *Assume that G is Abelian. Then $\mathbb{Z}[G]_\xi$ is faithfully flat over $\Sigma_\xi^{-1}\mathbb{Z}[G] = S_\xi^{-1}\mathbb{Z}[G]$.*

Proof. We can assume that G is finitely generated. We can then write $G_{\xi \geq 0} = \bigcup_{p \in \mathbb{N}} C_p$, where $C_p \subset C_{p+1}$ is the semi-group generated by a suitable finite subset F_p. Then $\mathbb{Z}[F_p]$ is a Noetherian ring and $\mathbb{Z}[F_p]_{\xi > 0}$ is an ideal. Thus the completion

$$\widehat{\mathbb{Z}[F_p]} := \varprojlim \mathbb{Z}[F_p] / (\mathbb{Z}[F_p]_{\xi > 0})^n$$

is faithfully flat over the localization $(1 + \mathbb{Z}[F_p]_{\xi > 0})^{-1}\mathbb{Z}[F_p]$ (cf. [Bou 1985], Prop. 9 p.248 and Prop. 12 p.251).

Denote $\mathbb{Z}[\widehat{G_{\geq 0}}]$ the completion with respect to the ideal $\mathbb{Z}[G_{>0}]$, then $\mathbb{Z}[\widehat{G_{\geq 0}}] = \bigcup_{p \in \mathbb{N}} \widehat{\mathbb{Z}[F_p]}$, and thus it is faithfully flat over

$$\bigcup_{p \in \mathbb{N}} (1 + \mathbb{Z}[F_p^+])^{-1}\mathbb{Z}[F_p] = (1 + \mathbb{Z}[G_{\xi < 0}])^{-1}\mathbb{Z}[G_{\xi \geq 0}] = S_\xi^{-1}\mathbb{Z}[G_{\xi \geq 0}].$$

Since $\mathbb{Z}[G]_\xi = G\widehat{\mathbb{Z}[G_{\geq 0}]}$ and $S_\xi^{-1}\mathbb{Z}[G] = GS_\xi^{-1}\mathbb{Z}[G_{\geq 0}]$, this implies the proposition.

Other proof. We show the linear extension property. Consider a linear system $Ax = b$, $A \in \mathrm{M}_{p,q}(S_\xi^{-1}\mathbb{Z}[G])$, $x \in \mathrm{M}_{q,1}(\mathbb{Z}[G]_\xi)$, $b \in \mathrm{M}_{p,1}(S_\xi^{-1}\mathbb{Z}[G])$. We want to show

(i) if it has a solution in $\mathrm{M}_{q,1}(\mathbb{Z}[G]_\xi)$, it has a solution in $\mathrm{M}_{q,1}(S_\xi^{-1}\mathbb{Z}[G])$,

(ii) every solution $x \in \mathrm{M}_{q,1}(\mathbb{Z}[G]_\xi)$ is of the form $x = x_0 + \sum_{i=1}^{r} \lambda_i x_i$, where $x_0, x_1, \cdots, x_r \in S_\xi^{-1}\mathbb{Z}[G]$, $\lambda_1, \cdots, \lambda_r \in \mathbb{Z}[G]_\xi$ and $Ax_0 = b$, $Ax_i = 0$ if $i > 0$.

Writing $A = S^{-1}\widetilde{A}$, $b = S_2\widetilde{b}$ with $S_i \in S_\xi$, $\widetilde{A} \in \mathrm{M}_{p,q}(\mathbb{Z}[G])$, $\widetilde{b} \in \mathrm{M}_{p,1}(\mathbb{Z}[G])$, the equation becomes $(S_2\widetilde{A})x = S_1 S_2 b$, thus we can assume that A and b have coefficients in $\mathbb{Z}[G]$.

10 Abelian Novikov homology (iii): relation with twisted Alexander polynomials

10.1 Alexander ideals and Alexander polynomials for a finitely generated group

In this section and the next we follow mostly [Mc-Mullen 2002], with some ideas coming from [Kaw] and [Tu 2002].

Let G be a finitely generated group. We denote

$$G^{ab} := G/G' \ , \ \mathrm{ab}(G) = G^{ab}/\mathrm{Torsion} = G/\sqrt{G'},$$

where for $A \subset G$ we define $\sqrt{A} := \{x \in G \mid (\exists n \in \mathbb{N}^*)\ x^n \in A\}$. Note that $\sqrt{G}'$ is a characteristic subgroup since G' is one.

We have $\mathrm{ab}(G) \approx \mathbb{Z}^n$ with $n = b_1(G)$, thus the ring $\mathbb{Z}[\mathrm{ab}(G)]$ is factorial. Using a multiplicative notation for G, and choosing $t_1, \cdots, t_n$ generators of ab(G, we have

$$\mathbb{Z}[\mathrm{ab}(G)] = \mathbb{Z}[t_1^{\pm 1}, \cdots, t_n^{\pm 1}].$$

The image of $g \in G$ in $\mathrm{ab}(G)$ will be denoted $\mathrm{ab}(g)$. We denote

$$m(G) = \langle x - 1 \mid x \in \mathrm{ab}(G) \rangle = \langle t_1 - 1, \cdots, t_n - 1 \rangle,$$

the Abelianized augmentation ideal. More generally, if H is a subgroup of G, we define $m_G(H)$ as the left ideal of $\mathbb{Z}[\mathrm{ab}(G)]$ generated by $\mathrm{ab}(x) - 1$, $x \in H$.

Definition: Alexander module. Let H be a normal subgroup of G such that G/H is free Abelian. The *Alexander module* of (G, H) is $A_G(H) = H/H'$, viewed as a $\mathbb{Z}[G/H]$-module.

Topological interpretation. If $\pi_1 X = G$ and $\widehat{X} \to X$ is the covering such that $\pi_1 \widehat{X} = H$, one has$A_G(H) = H_1(\widehat{X})$.

Special case. If $H = \sqrt{G'}$, $A_G(\sqrt{G'}) = \sqrt{G'}/(\sqrt{G'})'$ is the *Alexander module of* G.

Definition: Alexander ideal, Alexander polynomial. The *Alexander ideal* of (G, H) is the 0-eth Fitting ideal, or *order ideal*, of the Alexander module:

$$I_G(H) := \mathrm{Fitt}_0(A_G(H)) = \mathrm{Fitt}_0(H/H').$$

Recall (cf. [Ei], pp. 492 sqq.) that if R is a commutative ring and M is a finitely generated R-module and $R^{(I)} \xrightarrow{u} R^p \to M$ is a presentation of M, with I possibly infinite and u represented by the right multiplication with $A \in M_{I,p}(R)$, the i-eth Fitting ideal $\mathrm{Fitt}_i(M)$ is the ideal generated by the $(p-i)$-minors of the matrix A. It is easy to show that this is independent of the presentation.

Special case. If $H = \sqrt{G'}$, $I_G(\sqrt{G'}) =: I(G)$ is the *Alexander ideal of* G.

10.2 Matrix computations

Let $\langle x_1, \cdots, x_p \mid r_i, i \in I \rangle$ be a presentation of G,

Proposition. *The Alexander module of* (G, H) *is the first Fitting ideal of* $H_1(\widehat{X}, \pi^{-1}(x_0))$.

Alexander polynomial of (G,H) *is the greatest common divisor of* $\mathrm{Fitt}_0(H/H')$ *(where* H/H' *is viewed as a* $\mathbb{Z}[G/H]$*)-module).*

Here, $R = \mathbb{Z}[G/H]$ and $A_H(G) = H_1(\widehat{X}, \pi^{-1}(\{x_0\})) = C_1(\widehat{X})/\mathrm{im}(\widehat{\partial}_2)$. If $\langle x_1, \cdots, x_p \mid r_i, i \in I\rangle$ is a presentation of G, the chain module $C_1(\widehat{X})$ is isomorphic to R^p, $C_2(\widehat{X})$ to $R^{(I)}$, and (identifying $G/H = F$) $\widehat{\partial}_2$ is the right multiplication by $A = \Big(\varphi(\dfrac{\partial r_i}{\partial x_j})\Big)_{i,j}$.

Definition: Alexander polynomial. The *Alexander polynomial* of (G,H) is the greatest common divisor of the elements of $I_H(G)$. Recall that $\mathbb{Z}[G/H] \approx \mathbb{Z}[t_1^{\pm 1}, \cdots, t_n^{\pm 1}]$ is a unique factorization domain. The Alexander polynomial is an element

$$\Delta_G(H) \in \mathbb{Z}[G/H]$$

well-defined up to multiplication by a unit in $\mathbb{Z}[G/H]$, i.e. a monomial $\pm\overline{g}$, or $\pm t_1^{i_1} \cdots t_r^{i_r}$ using Laurent polynomials.

Special case. Let ξ be a nonzero homomorphism from G to $\mathbb{R}$. Then im ξ is a free Abelian group, and we obtain $\Delta_\xi(G) \in \mathbb{Z}[\mathrm{im}\ \xi] \subset \mathbb{Z}[\mathbb{R}]$, thus we can write $\Delta_\xi(G)$ as a finite sum $\sum_{\alpha\in\mathbb{R}} a_\alpha t^\alpha$. We can normalize $\Delta_\xi(G)$ by asking that the lowest degree α for which $a_\alpha \neq 0$ be 0 [this is not always a good idea, for instance if $G = \pi_1(M^3)$ Poincaré duality gives a natural $\Delta_\xi(G)$ which is symmetric i.e. such that $a_{-\alpha} = a_\alpha$].

In this special case, we say that $\Delta_\xi(G)$ is *unitary* (resp. *infra-unitary*) if its highest (resp. lowest) coefficient is ± 1, and bi-unitary if it is unitary and infra-unitary.

10.2 A characterization of the Alexander polynomial

This other definition is given by the following

Proposition. *The Alexander polynomial of* (G,H) *is the greatest common divisor of* $\mathrm{Fitt}_0(H/H')$ *(where* H/H' *is viewed as a* $\mathbb{Z}[G/H]$*)-module).*

Proof. We simplify the proof of [Kaw] p. 92. We recall the short exact sequence

$$0 \to H/H' \to A_G(H) \to m(G/H) \to 0.$$

The module $m(G/H) \approx m(\mathbb{Z}^r) = \langle t_1 - 1, \cdots, t_r - 1\rangle \subset \mathbb{Z}[t_1^{\pm 1}, \cdots, t_r^{\pm 1}]$ is torsion-free, thus $\mathrm{Fitt}_0(m(\mathbb{Z}^r)) = 0$. The Proposition follows then from the following

Lemma. *Assume that* $0 \to M_1 \to M \to M_2 \to 0$ *is a short exact sequence of finitely generated* R*-modules, with* $\mathrm{Fitt}_0(M_2) = 0$. *Then*

$$\mathrm{Fitt}_0(M) = 0$$
$$\mathrm{Fitt}_d(M) = \mathrm{Fitt}_{d-1}(M_1) \text{ if } d \geq 1.$$

Proof of the lemma. Fix $p \in R$ irreducible (chosen in a set P of representatives modulo the units). Let p^{a_1}, p^a be the p-primary factors of $\mathrm{Fitt}_{d-1}(M_1)$ and $\mathrm{Fitt}_d(M)$. It suffices to prove that they are equal.

The sequence of modules localized at p ($M_p = R_p \otimes_R M$, where $R_p = R[M \setminus (p)]^{-1}$),

$$0 \to (M_1)_p \to M_p \to (M_2)_p \to 0,$$

is exact since R_p is flat over R (cf. [Bou], [Ei] p.66, [Mat]). Since R_p is a principal ideal domain, M_1, M and M_2 have presentations given by square matrices P_1, P, P_2, with $P = \begin{pmatrix} P_1 & 0 \\ A & P_2 \end{pmatrix}$. Moreover, $\mathrm{Fitt}_{d-1}((M_1)_p) = p^{a_1}$, $\mathrm{Fitt}_d(M_p) = p^a$, and $\mathrm{Fitt}_0((M_2)_p) = \cdots = \mathrm{Fitt}_r((M_2)_p) = 0$. Thus we are reduced to the case where $M = M_p$, ie M is local and principal.

By hypothesis, $\mathrm{Fitt}_0(M_2) = \cdots = \mathrm{Fitt}_{r-1}(M_2) = 0$ ie every k-minor of P_2 vanishes if $p_2 - k \leq r - 1$. Consider a nonvanishing $(p_1 + p_2 - d)$-minor of P. If $d \leq r - 1$, the matrix has to contain P_1, thus the minor belongs to $(\mathrm{Fitt}_d(M_1)_p)$

The localization preserves the torsion-freeness, thus

then $\Delta_0((M_1)_p) = p^{a_1}$, $\Delta_0(M_p) = p^a$ and $\Delta_0((M_2)_p) = p^{a_2}$ (modulo units).

we have $\Delta_0(M)) = \prod_{p \in P} (\Delta_0(M_p))$

11 The case of three-dimensional manifolds

In this section we consider a closed three-manifold M with $H^1(M;\mathbb{R}) \neq 0$.

11.1 A special chain complex equivalent to $C_*(\widetilde{M})$

For $p \in \mathbb{N}$ (resp. $p \in \mathbb{N}^*$), denote

$$D_p = D^2 \setminus \bigcup_{i=1}^{p} \mathrm{Int}(D_i^2)\ ,\ \widetilde{D}_p = B \setminus \bigcup_{i=1}^{p-1} \mathrm{Int}(D_i^2)$$

where B is a Möbius band. The product $H_p = D_p \times [0,1]$ (resp. $\widetilde{H}_p = \widetilde{D}_p \times [0,1]$) is a handlebody (resp. a twisted handlebody) of genus p.

By[Tu 1979], if M is orientable,it can be written as the union of two handlebodies (H_p), glued by a diffeomorphism φ of ∂H_p which is the identity except on the upper part $D_p \times \{1\}$. If M is not orientable, we have a similar statement with twisted handlebodies. In both cases we write $M = M_\varphi$.

Orientable case. Choose base points $P_0 \in \partial D^2$ and $P_i \in \partial D_i^2$, and fix arcs c_i in D_p from P_0 to P_i, non-intersecting and ordered anticlockwise from P_0 to $P_1, \cdots, P_p$. Let γ_i be ∂D_i^2 positively oriented, viewed as a loop in P_i. Let x_i be the loop $c_i\gamma_i c_i^{-1}$ in P_0. The group $\pi_1(D_p, P_0)$ is free on generators $x_1, \cdots, x_p$, and up to isotopy, φ is determined by the classes $s_i = \varphi(c_i)c_i^{-1} \in \langle x_1, \cdots, x_p\rangle$.

Then $\pi_1(M_\varphi)$ has the presentation $\langle x_1, \cdots, x_p \mid s_1, \cdots, s_p\rangle$, with the property

$$\prod_{i=1}^{p} s_i x_i s_i^{-1} = \prod_{i=1}^{p} x_i.$$

Conversely, every group with such a presentation is the fundamental group of an orientable 3-manifold M_φ, with φ determined by the s_i. Such a presentation is called an *Artin presentation* by [Win].

Setting $r_1 = s_1, r_i = x_1 \cdots x_{i-1} s_i (x_1 \cdots x_{i-1})^{-1}$, one obtains another form of presentation:

$$(*) \qquad \prod_{i=1}^{p} [r_i, x_i] = 1.$$

Thus fundamental groups of orientable 3-manifolds are the groups admitting a presentation $\langle x_1, \cdots, x_p \mid r_1, \cdots, r_p\rangle$ satisfying $(*)$. We shall call it an *Artin presentation of the second form.*

Another way to obtain such a presentation is to consider the natural Morse function associated to the decomposition $M_\varphi = H_p^+ \cup H_p^-$. It admits a gradientlike vectorfield for which the associated chain complex is

$$0 \to \mathbb{Z}[\pi_1 M] \xrightarrow{\partial_3} \mathbb{Z}[\pi_1 M]^p \xrightarrow{\partial_2} \mathbb{Z}[\pi_1 M]^p \xrightarrow{\partial_1} \mathbb{Z}[\pi_1 M] \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

with $\partial_1(e_i) = x_i - 1$, $\partial_2(f_i) = \dfrac{\partial r_i}{\partial x_j}$, $\partial_3(1) = \sum_{i=1}^{p}(x_i^{-1} - 1)f_i$. The main interest of this is the following property: *if we represent ∂_i by the multiplication on the right by a matrix D_i, we have $D_3 = D_1^*$ where $(a_{i,j})^* = (\overline{a}_{j,i})$ with $\overline{\sum n_g g} = \sum n_g \alpha(g^{-1})$, the canonical involution on $\mathbb{Z}[\pi_1(M)]$ and $\alpha(g) = \pm 1$ is the orientation homomorphism ($\alpha(x_i) = 1$.*

Nonorientable case. We define c_i, γ_i, x_i similarly except that γ_p is the soul of B and $P_p \in \gamma_p$. Then $\pi_1(M_\varphi)$ has the presentation $\langle x_1, \cdots, x_p \mid s_1, \cdots, s_p\rangle$, with the property

$$\prod_{i=1}^{s-1} s_i x_i s_i^{-1} s_p x_p s = \prod_{i=1}^{p} x_i.$$

Setting $r_1 = s_1, r_i = x_1 \cdots x_{i-1} s_i (x_1 \cdots x_{i-1})^{-1}$, we obtain

$$\prod_{i=1}^{p-1} [r_i, x_i] r_p x_p r_p x_p^{-1} = 1.$$

11.2 Vanishing of $H_1(M, \xi)$ in the non aspherical case

Suppose that M is not aspherical, ie $H_2(\widetilde{M}) = \pi_2(M) \neq 0$. Then either M is a connected sum $M_1 \sharp M_2$ with $\pi_1(M_1)$ and $\pi_1(M_2)$ non trivial, or M is an S^2-bundle over S^1 with a homotopy sphere, ie M is an S^2-bundle over S^1 by Perelman. In the first case, $\pi_1(M)$ is a nontrivial free product, thus $H_1(M, \xi) = H_1(\pi_1(M), \xi)$ never vanishes. In the second case, $\pi_1(M) = \mathbb{Z}$, thus $H_1(M, \xi)$ always vanishes.

11.3 Vanishing of $H_1(M, \xi)$ in the aspherical case

Here we assume that M is aspherical, thus the complex $(*)$ is exact, and so is its dual

$$0 \to \mathbb{Z}[\pi_1 M] \xrightarrow{\partial_3^*} \mathbb{Z}[\pi_1 M]^p \xrightarrow{\partial_2^*} \mathbb{Z}[\pi_1 M]^p \xrightarrow{\partial_1^*} \mathbb{Z}[\pi_1 M] \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$

where $\partial_i = \partial_{3-i}^*$, defined by the matrix D_{3-i}^*.

Proposition. *Let M a closed orientable 3-manifold. We fix an Artin presentation of the second form $\pi_1(M) = \langle x_1, \cdots, x_p \mid r_1, \cdots, r_p\rangle$, to which is associated a resolution with matrices $D_1, D_2, D_3 = D_1^*$. Let $\xi \in H^1(M, \mathbb{R})$ be nonzero, and let $i \in [[1, p]]$ be such that $\xi(x_i) \neq 0$. Denote $A \in \mathrm{M}_{p-1,p-1}(\mathbb{Z}[\pi_1(M)]$ the matrix obtained from D_2 by deleting the i-eth line and the i-eth column. The following are equivalent:*

(i) $H_1(M, \xi) = 0$

(ii) A is left invertible in $\mathrm{M}_p(\mathbb{Z}[\pi_1 M]_\xi)$

(iii) A is invertible in $\mathrm{M}_p(\mathbb{Z}[\pi_1 M]_\xi)$

(iv) A is invertible in $\mathrm{M}_p(\mathbb{Z}[\pi_1 M]_{-\xi})$

(v) $H_1(M, -\xi) = 0$.

Proof. Up to reordering we can assume that $i = p$. We write D_1, D_2, D_3 as block matrices (L being a line and C, C' columns):

$$D_3 = D_1^* = (C^* \mid x_p^* - 1) \ , \ D_2 = \begin{pmatrix} A & C' \\ L & a \end{pmatrix} \ , \ D_1 = \begin{pmatrix} C \\ x_p - 1 \end{pmatrix}.$$

The identities $D_2 D_1 = 0$, $D_3 D_2 = 0$ (corresponding to $\partial_1 \circ \partial_2 = 0$, $\partial_2 \circ \partial_3 = 0$) give

$$AC + C'(x_p - 1) = 0 \ , \ LC + a(x_p - 1) = 0$$
$$C^* A + (x_p^* - 1)L = 0 \ , \ C^* C' + (x_p^* - 1)a = 0.$$

Since $x_p - 1$ and $x_p^* - 1$ are invertible in $\mathbb{Z}[\pi_1 M]_\xi$, this implies that, considered as a matrix with coefficients in $\mathbb{Z}[\pi_1(M)]$, we have

$$D_2 = \begin{pmatrix} A & AC(1-x_p)^{-1} \\ (1-x_p^*)^{-1}C^*A & a \end{pmatrix}.$$

By Proposition 2.8, (i) is equivalent to the left invertibility of $\begin{pmatrix} A \\ (1-x_p^*)^{-1}C^*A \end{pmatrix}$. This is equivalent to the left invertibility of A, thus (i) $\Leftrightarrow$ (ii). Since $\mathbb{Z}[\pi_1 M]_\xi$ is stably finite, this is equivalent to (iii).

Then (iii) is equivalent to the invertibility of A^* in $\mathrm{M}_p(\mathbb{Z}[\pi_1 M]_{-\xi})$. Since (D_1^*, D_2^*, D_3^*) has the same properties as (D_1, D_2, D_3), this is equivalent to (iv). finally, (v) $\Leftrightarrow$ (iv) in the same way as (i) $\Leftrightarrow$ (ii).

11.4 Thurston norm

Theorem [Thu]. *Let M be a closed oriented three-manifold.*

(i) There exists a unique pseudo-norm $||\xi||_T$ defined on $H^1(M,\mathbb{R})$, ie $||\xi||_T \in \mathbb{R}_+$, $||\xi+\eta||_T \leq ||\xi||_T + ||\eta||_T$, $||\xi(\lambda\xi)||_T = |\lambda|||\xi||_T$, with the following property: if $\xi \in H^1(M,\mathbb{Z})$ is primitive (ie $\mathrm{im}(\xi) = \mathbb{Z}$), one has

$$||\xi||_T = \max\left\{ \sum_{i=1}^k \max(0, -\chi(S_i)) \;\middle|\; \begin{array}{l} S = \coprod_{i=1}^k S_i \text{ is a smooth surface, with connected} \\ \text{components } S_i \text{ and such that } [S] = PD(\xi). \end{array} \right\}$$

Here, $PD(\xi)$ is the Poincaré dual of ξ, so that ($[S] = PD(\xi) \Leftrightarrow S$ is a regular value of a map $f : M \to S^1$ whose associated homomorphism $\pi_1(M) \to \mathbb{Z}$ is ξ), and $\chi(S)$ is the Euler characteristic.

(ii) There exists a finite number of classes $\alpha_1, \cdots, \alpha_k \in H_1(M;\mathbb{Z})$ such that

$$||\xi||_T = \max_{1\leq i\leq k} |\xi(g_i)|.$$

Equivalently, the unit ball $B_T(M) = \{\xi \in H^1(M;\mathbb{R}) \mid ||\xi||_T \leq 1\}$ is rational polyhedron (convex and symmetric).

(iii) $\xi^{-1}(0)$ is generated by $PD(S)$ with $\chi(S) \geq 0$, ie S is a sphere or a torus. In particular, ξ is a norm if and only if $\pi_1(M)$ is atoroidal, in particular if M is hyperbolic.

Definition. This pseudonorm $||\xi||_T$ is called the *Thurston norm.*

Remarks. 1) Actually, Thurston defines his norm on $H_2(M,\mathbb{R})$, which is canonically isomorphic to $H^1(M;\mathbb{R})$.

2) Property (ii) holds in fact for any pseudo-norm on $\mathbb{R}^n$ which is integral-valued on $\mathbb{Z}^n$.

3) Thurston also shows that any given polyhedron $P \subset \mathbb{R}^n$ which is convex, symmetric and rational, is the unit ball $B_T(M)$ of on some $H^1(M;\mathbb{R})$. Equivalently, every norm on $\mathbb{R}^n$ wihich is integer-valued on $\mathbb{Z}^n$, is the norm $||.||_T$ on some $H^1(M;\mathbb{R}) \simeq \mathbb{R}^n$.

4) There does not seem to exist an analogue of this in the non-orientable case.

11.5 Vanishing of $H_1(M,\xi)$ and non-singular closed one-forms

Let M be a closed three-manifold. Denote

$$\mathcal{N}(M) = \{\xi \in H^1(M;\mathbb{R}) \mid \xi \text{ is represented by a closed non-singular one-form}\}.$$

It is an open subset of $H^1(M;\mathbb{R}) \setminus \{0\}$, with $-\mathcal{N}(M) = \mathcal{N}(M)$. The two main facts about M are:

- ([Stallings]) If ξ is rational, $\xi \in \mathcal{N} \Rightarrow \ker\xi$ is finitely generated. In fact, if $\mathrm{im}(\xi) \subset \mathbb{Z}$, this is equivalent to the existence of a fibration $p : M \to S^1$ whose associated homomorphism $\pi_1(M) \to \mathbb{Z}$ is ξ, and Stallings states that p exists if and only if $\ker\xi$ is finitely generated.
- ([Thu]) If M is oriented, $\mathcal{N}(M)$ is the union of the cones on some open faces of $\partial B_T(M)$, which are called fibered. Moreover, one can realize any couple $(P,\mathcal{F})$ where $P \subset \mathbb{R}^n$ is a convex, symmetric and rational polyhedron, and $\mathcal{F}$ a set of open faces of ∂P.

Corollary. *If M is orientable, $\mathcal{N}(M)$ is rationally defined.*

Theorem. *Let M be an orientable closed-three manifold, and $\xi \in H_1(M;\mathbb{R}) \setminus \{0\}$. The following are equivalent:*

(i) $H_1(M,\xi) = 0$

(ii) $\xi \in \mathcal{N}(M)$.

Equivalently, $\Sigma(\pi_1(M)) = \mathcal{N}(M)$.

Proof. By section 7, $(ii) \Rightarrow (i)$. Note that we do not need 11.2, since $-\mathcal{N}(M) = \mathcal{N}(M)$, and not the orientability of M either.

In the other direction: if ξ is rational, $(ii) \Rightarrow (i)$ follows from 4.x and the criterion of Stallings, the orientability of M being unnecessary. If ξ is irrational, it belongs to the cone $\mathbb{R}_+^*.F$ of some open face F of $\partial B_T(M)$, and since property (i) is open, every $\eta \in \mathbb{R}_+^*.F$ close enough satisfies it, thus belongs to $\mathcal{N}(M)$. Thus F is fibered, which implies that $\xi \in \mathcal{N}(M)$.

Questions. 1) Do we still have $(i) \Rightarrow$ if M is non orientable ? This would follow if we knew that $\mathcal{N}(M)$ is rationally defined.

2) Can one prove directly that $\Sigma(\pi_1(M))$ is rationally defined ?

11.6 The case of a manifold fibered over S^1

Let M be a closed three-manifold equipped with a fibration $p : M \to S^1$. We shall assume that M is orientable, so that the fiber is a closed orientable surface Σ_k, with fundamental group presented by $\langle x_1, y_1, \cdots, x_k, y_k \mid \prod_{i=1}^k [x_i, y_i]\rangle$. We assume that $k \geq 1$, so that M is aspherical. The fibration is characterized by the homotopic monodromy $\varphi \in \mathrm{Aut}(\pi_1(\Sigma_k))$, defined up to interior automorphism. Moreover, one can normalize φ so that it lifts to $\Phi \in \mathrm{Aut}(F_{2k})$, $\widetilde{\varphi}(x_i) = X_i$, $\widetilde{\varphi}(y_i) = Y_i$ and $\prod_{i=1}^k [X_i, Y_i] = \prod_{i=1}^k [x_i, y_i]$.

Then $\pi_1(M)$ has a presentation with $2k+1$ generators

$$(s_1, \cdots, s_{2k+1}) = (x_1, y_1, \cdots, x_k, y_k, t)$$

and $2k+1$ relations

$$(r_1, \cdots, r_{2k+1}) = \big(tx_1t^{-1}X_1^{-1}, ty_1t^{-1}Y_1^{-1}, \cdots, tx_kt^{-1}X_k^{-1}, ty_kt^{-1}Y_k^{-1}, \prod_{i=1}^k [x_i, y_i]\big).$$

The associated matrix $D_2 = \left(\frac{\partial r_i}{\partial s_j}\right) \in \mathrm{M}_{2k+1}(\mathbb{Z}[\pi_1(M)])$ is

$$D_2 = \begin{pmatrix} & & & & 1-X_1 \\ & & & & 1-Y_1 \\ & t\mathrm{Id}_{2k} - D\Phi & & & \cdots \\ & & & & 1-X_k \\ & & & & 1-Y_k \\ 1-y_1 \mid x_1 - 1 \mid \cdots \mid c_1\cdots c_{k-1}(1-y_k) \mid c_1 \cdots c_{k-1}(x_k-1) & & & & 0 \end{pmatrix},$$

with $c_i = [x_i, y_i]$, and $D\Phi \in \mathrm{M}_{2k}(\mathbb{Z}[\pi_1(M)])$ is given by

$$(D\Phi)_{s_i,s_j} = \frac{\partial}{\partial s_j}\Phi(s_i)\ ,\ 1 \le i,j \le 2k.$$

Denote $A = \Phi^{ab} \in \mathrm{GL}_{2k}(\mathbb{Z})$ the effect of Φ on homology. Via $(\xi(s_1),\cdots,\xi(s_{2k+1}))$, we can identify $H^1(M;\mathbb{R})$ with $\mathrm{coker}({}^t\!A - \mathrm{Id}_{2k}) \times \mathbb{R}$. Let $\xi \in H^1(M;\mathbb{R})$ such that $\xi(t)(= \xi(s_{2k+1})) \neq 0$. By 11.3,

$$H_1(M,\xi) = 0 \Leftrightarrow t\mathrm{Id}_{2k} - D\Phi \text{ is invertible in } \mathrm{M}_{2k}(\mathbb{Z}[\pi_1(M)]_\xi).$$

Examples. 1) The case where $k = 1$ is very simple: either $A = \mathrm{Id}_2$ and $M = T^3$, or A is similar to $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, and there is a fibration $p : M \to T^2$ which induces an isomorphism on H^1, or $H^1(M;\mathbb{R}) \simeq \mathbb{R}$. In all cases, $H_1(M,\xi)$ always wanishes, and $\mathcal{N}(M) = H^1(M;\mathbb{R}) \setminus \{0\}$, without using 11.5.

2) Let $||D\Phi||_\xi = \max\{\xi(g) \mid g \in \mathrm{supp}(\Phi)\}$. If $||D\Phi||_\xi < |\xi(t)|$, $H_1(M,\xi) = 0$. Indeed, if $\xi(t) > ||D\Phi||_\xi$, $\mathrm{Id}_{2k} - t^{-1}D\Phi \in S_{-\xi}$, thus $H_1(M,-\xi) = 0$, thus $H_1(M,\xi) = 0$. And if $\xi(t) < -||D\Phi||_\xi$, $\mathrm{Id}_{2k} - t^{-1}D\Phi \in S_\xi$, thus $H_1(M,\xi) = 0$.

12 Residual units

12.1 Definitions and conjectures

Definitions. Let G be a residually finite group, and $n \in \mathbb{N}^*$. If R is a ring, $R[G]$ is the group ring with coefficients in R, and $\mathrm{M}_n(R)$ is the ring of (n,n)-matrices with coefficients in R.

- An element of $\mathrm{M}_n(\mathbb{Z}[G])$ is *a residual unit* if its image in $\mathrm{M}_n(\mathbb{Z}[G/H])$ is a unit for every subgroup $H \lhd_{f.i.} G$ (normal with of finite index).
- $\mathrm{M}_n(\mathbb{Z}[G])$ has *finitely detectable units* if every residual unit is a unit.

Recall that the ring $\mathbb{Z}[G]$ is *stably finite*, meaning that if $A \in \mathrm{M}_n(\mathbb{Z}[G])$, A is a unit $\Leftrightarrow$ A has a right inverse $\Leftrightarrow$ A has a left inverse.

The Novikov case. Let ξ be an element of $\mathcal{N}(G) = (\mathrm{Hom}(G,\mathbb{R}) \setminus \{0\})_{/\mathbb{R}_+^*}$. Recall that the Novikov ring $\mathbb{Z}[G]_\xi$ is the ring of series $\sum_{i=0}^{\infty} n_i g_i$ such that $n_i \in \mathbb{Z}$, $g_i \in G$ and $\xi(g_i) \to +\infty$. If H is a normal subgroup of G, we denote $\overline{\xi}_H \in \mathcal{N}(G/(H \cap \ker \xi))$ the image of ξ.

Recall that the Novikov ring is also stably finite. We adapt the previous definitions to the Novikov case:

- An element of $\mathrm{M}_n(\mathbb{Z}[G]_\xi)$ is a residual unit if its image in $\mathrm{M}_n(\mathbb{Z}[G/(H \cap \ker \xi)]_{\overline{\xi}_H})$ is a unit for every subgroup $H \lhd_{f.i.} G$.
- $\mathrm{M}_n(\mathbb{Z}[G]_\xi)$ has finitely detectable units if every residual unit is a unit.

Conjectures. Let G be a "nice" group. For instance:

- virtually (residually finite and right-orderable); by [Boyer-Rolfsen-Wiest] and Perelman, this is the case if $G = \pi_1(M)$ where M is a closed 3-dimensional manifold with $b_1(M) > 0$. By Agol, it suffices that if M be hyperbolic.
- (stronger) virtually (residuallly torsion-free nilpotent); by [Koberda 2013] this is the case if $G = \pi_1(M)$ where M is a closed 3-dimensional geometric manifold which is not Sol, in particular if M is hyperbolic. Other examples: free groups (Magnus), surface groups (Baumslag), right-angled Artin groups (Duchamp-Krob).

Conjecture 1. *If G is "nice", $\mathrm{M}_n(\mathbb{Z}[\pi_1(M)]$ has finitely detectable units for every $n \in \mathbb{N}^*$.*

Conjecture 2. *If G is "nice", $\mathrm{M}_n(\mathbb{Z}[\pi_1(M)]_\xi)$ has finitely detectable units for every $n \in \mathbb{N}^*$.*

Remarks. 1) One motivation for Conjecture 2 is that, by Sections 10 and 11, it implies that a class $\xi \in H^1(M,\mathbb{R}) \setminus \{0\})$ is represented by a nonsingular closed one-form if and only if that every twisted Alexander polynomial $\Delta^H_{M,u}$ is unitary.

A related application is the following (cf. Section 4):

Proposition. *Let G be a group satisfying Conjecture 2, and let ξ be a nonzero morphism from G to $\mathbb{Z}$. Then the following are equivalent:*

- $\ker \xi$ *is finitely generated*
- *for every subgroup $H <_{f.i.} G$, the Abelianization of* $\ker(\xi_{|H})$ *is finitely generated.*

Corollary. *If $\lambda \in \mathbb{Z}[G]$ is a residual unit, the module $\mathbb{Z}[G]/\mathbb{Z}[G]\lambda$ has no nontrivial finite quotient.*

It is convenient to generalize these notions.

12.2 Residually full left ideals

Definitions. Let G be a group. A left ideal $I \subset \mathbb{Z}[G]$ is *residually full* if I surjects onto $\mathbb{Z}[G/H]$ for every $H \lhd_{f.i.} G$. Similarly, if ξ is a nonzero morphism from G to $\mathbb{R}$, a left ideal of $\mathbb{Z}[G]_\xi$ is residually full if it surjects onto $\mathbb{Z}[G/H \cap \ker \xi]_{\overline{\xi}}$ for every $H \lhd_{f.i.} G$.

Similarly, a left submodule $M \subset \mathbb{Z}[G]^n$ is residually full if its image in every quotient $\mathbb{Z}[G/H]^n$, $H \subset G$ normal subgroup of finite index, is equal to $\mathbb{Z}[G/H]^n$. An a left submodule $M \subset \mathbb{Z}[G]_\xi$ is residually full if its image is full in every quotient $\mathbb{Z}[G/H \cap \ker \xi]^n_{\overline{\xi}}$, $H \subset G$ normal subgroup of finite index.

The group ring $\mathbb{Z}[G]$ (resp. the Novikov ring $\mathbb{Z}[G]_\xi$) has *finitely detectable full left ideals* if every left ideal $I \subset \mathbb{Z}[G]$ (resp. $I \subset \mathbb{Z}[G]_\xi$) which is residually full is equal to $\mathbb{Z}[G]$. (resp. to $\mathbb{Z}[G]_\xi$).

Conjecture 1'. *If G is "nice", $\mathrm{M}_n(\mathbb{Z}[\pi_1(M)]$ has finitely detectable full ideals.*

Conjecture 2. *If G is "nice", $\mathrm{M}_n(\mathbb{Z}[\pi_1(M)]_\xi)$ has finitely detectable full left ideals.*

These conjecture imply Conjectures 1 and 2, thanks to the

Proposition. *Assume that $\mathbb{Z}[G]$ (resp. $\mathbb{Z}[G]_\xi$) has finitely detectable full left ideals.*

(i) Let M be a residually full left submodule of $\mathbb{Z}[G]^n$ (resp. $\mathbb{Z}[G]^n_\xi$). Then $M = (\mathbb{Z}[G])^n$ (resp. $\mathbb{Z}[G]^n_\xi$).

(ii) The ring $\mathrm{M}_n(\mathbb{Z}[G])$ (resp. $\mathrm{M}_n(\mathbb{Z}[G]_\xi)$ has finitely detectable units for every n.

Proof. It suffices to treat the case of $\mathbb{Z}[G]$, the case of $\mathbb{Z}[G]_\xi$ being completely analogous.

(i) By induction on n, the result being the hypothesis if $n = 1$. Assume that $n > 1$ and the result is true for $n - 1$. Consider the set I of $\lambda \in \mathbb{Z}[G]$ such that there exists $\lambda_1, \cdots, \lambda_{n-1} \in \mathbb{Z}[G]$ with $(\lambda_1, \cdots, \lambda_{n-1}, \lambda) \in M$. It is a left ideal, and its image is full in every quotient $\mathbb{Z}([G/H])$ with G/H finite. Thus $I = \mathbb{Z}[G]$, ie M contains an element $x = (\lambda_1, \cdots, \lambda_{n-1}, 1)$.

One has a direct sum decomposition $\mathbb{Z}[G]^n = \mathbb{Z}[G]^{n-1} \oplus \mathbb{Z}[G]x$. Substracting $\mu_n x$ from every element $(\mu_1, \cdots, \mu_n) \in M$, ones sees that $M = (M \cap (\mathbb{Z}[G]^{n-1}) \oplus \mathbb{Z}[G]x$. It suffices to prove that $M \cap \mathbb{Z}[G]^{n-1} = \mathbb{Z}[G]^{n-1}$. Clearly, $M \cap \mathbb{Z}[G])^{n-1}$ is a left submodule which has a full image in every quotient $(\mathbb{Z}[G/H])^{n-1}$, $H \subset G$ normal subgroup of finite index. By the induction hypothesis, $M \cap \mathbb{Z}[G]^{n-1} = \mathbb{Z}[G]^{n-1}$, qed.

(ii) Let $A \in \mathrm{M}_n(\mathbb{Z}[G])$ be a residual unit. The right multiplication by A is a left-linear map u from $\mathbb{Z}[G]^n$ to itself. By hypothesis, $\mathrm{im}(u)$ has full image in every quotient $(\mathbb{Z}[G/H])^n$, $H \subset G$ normal subgroup of finite index. By (i), $\mathrm{im}(u) = \mathbb{Z}[G]^n$, ie A is left invertible, thus invertible, qed.

12.3 Main results

Proposition. *1) Let G be virtually polycyclic (in particular, finitely generated and virtually nilpotent). Then $\mathbb{Z}[G]$ has finitely detectable full left ideals. Thus $\mathrm{M}_n(\mathbb{Z}[G])$ has finitely detectable units for any n.*

2) Under the same hypothesis, $\mathbb{Z}[G]_\xi$ has finitely detectable units. Thus $\mathrm{M}_n(\mathbb{Z}[G]_\xi)$ has finitely detectable units for any n.

3) Let G be residually torsion-free nilpotent. Then $\mathbb{Z}[G]$ has finitely detectable units.

12.4 Comparison with finite quotients

Proposition. *Let $I \subset \mathbb{Z}[G]$ be a left ideal. The following are equivalent:*

1) I is residually full, ie I surjects onto $\overline{\lambda} \in \mathbb{Z}[G/H]$ for every $H \triangleleft_{f.i.} G$

2) I surjects onto every finite quotient modules $\mathbb{Z}[G]/J$, J a left ideal.

Proof. Assume that 1) holds. If $\mathbb{Z}[G]/J$ is a finite quotient, the normal subgroup

$$H := \{g \in G \mid \text{acts by the identity on } \mathbb{Z}[G]/J\}$$

has finite index in G, and we have a factorization

$$\begin{array}{ccc} \mathbb{Z}[G] & \rightarrow & \mathbb{Z}[G]/J \\ \Big\downarrow & \nearrow & \\ \mathbb{Z}[G/H]. & & \end{array}$$

Since I surjects onto $\mathbb{Z}[G/H]$, it also surjects onto $\mathbb{Z}[G]/J$. Thus 2) holds.

Conversely, assume that 2) holds. Then for every $H \triangleleft_{f.i.} G$, the image $\overline{\lambda} \in \mathbb{Z}[G/H]$ becomes a unit in $\mathbb{F}_p[G/H]$ for every prime p. Taking an associated matrix $M \in \mathrm{M}_{[G:H]}(\mathbb{Z})$, the image of M in $\mathrm{M}_{[G:H]}(\mathbb{F}_p)$ is invertible for every prime p, thus M is invertible, thus $\overline{\lambda}$ is a unit.

Corollary. *If $I \subset \mathbb{Z}[G]$ is residually full, $\mathbb{Z}[G]/I$ has no nontrivial finite quotient.*

12.5 From a finite index subgroup to the group

Proposition. *1) If $\Gamma < G$ has finite index and $\mathbb{Z}[\Gamma]$ has finitely detectable full left ideals, so has $\mathbb{Z}[G]$.*

Samme statement, with the Novikov rings $\mathbb{Z}[G]_\xi$ and$\mathbb{Z}[\Gamma]_{|\xi\Gamma}$.

We can assume that Γ is normal. Let $I \subset \mathbb{Z}[G]$ be a residually full left ideal. Then, as a left $\mathbb{Z}[\Gamma]$-module, $\mathbb{Z}[G]$ isisomorphic to $\mathbb{Z}[\Gamma]^m$, $m = [G : H]$, so that $\mathbb{Z}[G]$ embeds in $\mathrm{M}_m(\mathbb{Z}[\Gamma])$. Thus I is identified with a submodule $M \subset \mathbb{Z}[\Gamma]^m$.

Let $H < \Gamma$ be a normal subgroup of finite index. Then it contains H_1 which is normal in G and of finite index, and the isomorphism of $\mathbb{Z}[\Gamma]$-modules $\mathbb{Z}[G] \to \mathbb{Z}[\Gamma]^m$ induces an isomorphism of $\mathbb{Z}[\Gamma/H_1]$-modules $\mathbb{Z}[G/H_1] \to \mathbb{Z}[\Gamma/H_1]^m$. By hypothesis, the image of I in $\mathbb{Z}[G + H_1]$ is full, thus the image of M in $\mathbb{Z}[\Gamma/H_1]^m$ is full, and a fortiori the image in $\mathbb{Z}[\Gamma/H]^m$ is full. Since $\mathbb{Z}[\Gamma]$ has finitely detectable full left ideals, $M = \mathbb{Z}[\Gamma]^m$, thus $I = \mathbb{Z}[G]$, which proves the Proposition.

2) The proof is essentially the same.

12.6 Proof of the main results

Recall the Proposition 12.3.

Proposition. *1) Let G be virtually polycyclic (in particular, finitely generated and virtually nilpotent). Then $\mathbb{Z}[G]$ has finitely detectable full left ideals. Thus $\mathrm{M}_n(\mathbb{Z}[G])$ has finitely detectable units for any n.*

2) Under the same hypothesis, $\mathbb{Z}[G]_\xi$ has finitely detectable units. Thus $\mathrm{M}_n(\mathbb{Z}[G]_\xi)$ has finitely detectable units for any n.

3) Let G be residually torsion-free nilpotent. Then $\mathbb{Z}[G]$ has finitely detectable units.

Proof. 1) Let $I \subset \mathbb{Z}[G]$ be a residually full left ideal. By contradiction, assume that $M = \mathbb{Z}[G]/I$ is nonzero.

By 12.4, M has no nontrivial finite quotient. Since $I \neq \mathbb{Z}[G]$, there exists a maximal left ideal $I_1 \supset I$ (without the axiom of choice, since $\mathbb{Z}[G]$ is Noetherian).

Thus the $\mathbb{Z}[G]$-module $M_1 = \mathbb{Z}[G]/I_1$ is simple (nonzero and with no nontrivial submodule). By [Philip Hall 1959] (quoted in [Passman 1976, p.544]), since $\mathbb{Z}$ is a principal ideal domain with an infinite number of primes (up to units), and G is virtually polycyclic, M_1 is finite, a contradiction since M_1 is a quotient of M.

2) By 12.5, we can assume that G is polycyclic and torsion free, thus left-orderable. We fix a left order on G. Let $I \subset \mathbb{Z}[G]$ be a residually full left ideal. Let $I_0 \subset \ker\xi$ be the ideal generated by $G\lambda_0 \cap \ker\xi$ where λ_0 ranges over the ξ-minimal parts of the elements of I.

Then $\ker\xi$ is polycyclic and I_0 is a left ideal of $\mathbb{Z}[\ker\xi]$. Let us prove that I_0 is full. If $H_0 \lhd_{f.i.} \ker\xi$, there exists $H \lhd_{f.i.} G$ such that $H_0 = H \cap \ker\xi$ (it suffices to add to H_0 the lifts of a basis of im ξ over $\mathbb{Z}$).

Since I is full, I surjects onto $\mathbb{Z}[G/H \cap \ker\xi]_{\overline{\xi}} = \mathbb{Z}[G/H_0]_{\overline{\xi}}$.

A left ideal $I \subset \mathbb{Z}[G]$ is *residually full* if I surjects onto $\mathbb{Z}[G/H]$ for every $H \lhd_{f.i.} G$. Similarly, if ξ is a nonzero morphism from G to $\mathbb{R}$, a left ideal of $\mathbb{Z}[G]_\xi$ is residually full if it surjects onto $\mathbb{Z}[G/H \cap \ker\xi]_{\overline{\xi}}$ for every $H \lhd_{f.i.} G$.

Proof that $\mathbb{Z}[G]_\xi$ has finitely detectable units

Let $\lambda \in \mathbb{Z}[G]_\xi$ be a residual unit. We want to prove that its ξ-minimal part $\lambda_0 \in \mathbb{Z}[\ker\xi]$ is invertible. We can assume that $\ker\xi = \mathbb{Z}^s \times \{0\}$, $s = r - \mathrm{rk}(\xi)$. If $H = k\mathbb{Z}^r$, $H \cap \ker\xi = k\mathbb{Z}^s \times \{0\}$ and $\mathbb{Z}^r/(H \cap \ker\xi) = (\mathbb{Z}/k\mathbb{Z})^s \oplus \mathbb{Z}^{r-s}$ (canonical isomorphism).

There exists a group homomorphism $f : \mathbb{Z}^s \to \mathbb{Z}$ which is injective on $\mathrm{supp}(\lambda_0)$. Then the invertibility of λ_0 is equivalent to that of $f(\lambda_0) \in \mathbb{Z}[\mathbb{Z}]$. Moreover, one can extend f to a homomorphism $f : \mathbb{Z}^r \to \mathbb{Z} \times \mathbb{Z}^{r-s}$ by sending all the other generators to themselves. This gives a ring homomorphism $f : \mathbb{Z}[\mathbb{Z}^r]_\xi \to \mathbb{Z}[\mathbb{Z} \times \mathbb{Z}^{r-s}]_{\overline{\xi}}$. The hypothesis implies that $f(\lambda)$ is again residually invertible, thus it suffices to treat the case $s = 1$.

Write $\lambda_0 = \sum_{i=0}^{d} a_i t^i = P(t)$. The image of λ in $\mathbb{Z}[\mathbb{Z}/k\mathbb{Z}][\mathbb{Z}^{r-s}]_{\overline{\xi}}$ is invertible, and so is its image in $\mathbb{Z}[\omega_k][\mathbb{Z}^{r-s}]_{\overline{\xi}}$. For k prime $> d$, $P(\omega_k) \neq 0$, thus it is the $\overline{\xi}$-minimal part of this image. Thus $P(\omega_k)$ is invertible in $\mathbb{Z}[\omega_k]$ for every prime $k > d$. By the proof of (i) (it suffices to have the invertibility for a sequence $k_i \to +\infty$), $P(t) = \pm t^r$, qed.

3) Let $\lambda \in \mathbb{Z}[G]$ be a residual unit. By hypothesis, there exists a normal subgroup $H \lhd G$ such that G/H is torsion-free nilpotent and $\mathrm{supp}(\lambda)$ injects in G/H. Also, G/H is finitely generated, thus satisfies 1).

The image $\overline{\lambda}$ of λ in $\mathbb{Z}[G/H]$ is a residual unit. By 1), $\overline{\lambda}$ is a unit. Since G/H is torsion-free nilpotent thus left orderable, $\overline{\lambda} \in \pm G/H$. Finally, since $\mathrm{supp}(\lambda)$ injects in G/H, $\lambda \in \pm G$, qed.

12.7 Proof that $\mathbb{Z}[\mathbb{Z}^r]_\xi$ has finitely detectable units

Let $\lambda \in \mathbb{Z}[\mathbb{Z}^r]_\xi$ be residually invertible. We want to prove that its ξ-minimal part $\lambda_0 \in \mathbb{Z}[\ker\xi]$ is invertible. We can assume that $\ker\xi = \mathbb{Z}^s \times \{0\}$, $s = r - \mathrm{rk}(\xi)$. If $H = k\mathbb{Z}^r$, $H \cap \ker\xi = k\mathbb{Z}^s \times \{0\}$ and $\mathbb{Z}^r/(H \cap \ker\xi) = (\mathbb{Z}/k\mathbb{Z})^s \oplus \mathbb{Z}^{r-s}$ (canonical isomorphism).

There exists a group homomorphism $f : \mathbb{Z}^s \to \mathbb{Z}$ which is injective on $\mathrm{supp}(\lambda_0)$. Then the invertibility of λ_0 is equivalent to that of $f(\lambda_0) \in \mathbb{Z}[\mathbb{Z}]$. Moreover, one can extend f to a homomorphism $f : \mathbb{Z}^r \to \mathbb{Z} \times \mathbb{Z}^{r-s}$ by sending all the other generators to themselves. This gives a ring homomorphism $f : \mathbb{Z}[\mathbb{Z}^r]_\xi \to \mathbb{Z}[\mathbb{Z} \times \mathbb{Z}^{r-s}]_{\overline{\xi}}$. The hypothesis implies that $f(\lambda)$ is again residually invertible, thus it suffices to treat the case $s = 1$.

Write $\lambda_0 = \sum_{i=0}^{d} a_i t^i = P(t)$. The image of λ in $\mathbb{Z}[\mathbb{Z}/k\mathbb{Z}][\mathbb{Z}^{r-s}]_{\overline{\xi}}$ is invertible, and so is its image in $\mathbb{Z}[\omega_k][\mathbb{Z}^{r-s}]_{\overline{\xi}}$. For k prime $> d$, $P(\omega_k) \neq 0$, thus it is the $\overline{\xi}$-minimal part of this image. Thus $P(\omega_k)$ is invertible in $\mathbb{Z}[\omega_k]$ for every prime $k > d$. By the proof of (i) (it suffices to have the invertibility for a sequence $k_i \to +\infty$), $P(t) = \pm t^r$, qed.

(iii) By hypothesis, we have a short exact sequence $\mathbb{Z}^r \to G \to Q$ where Q is a finite group. Using a section $\sigma : Q \to G$, we have an isomorphism of $\mathbb{Z}[\mathbb{Z}^r]$-modules $\mathbb{Z}[G] \approx (\mathbb{Z}[\mathbb{Z}^r])^{|Q|}$, $|Q| = \mathrm{card}(Q)$. Thus $\mathbb{Z}[G]$ can be represented as a subring of $\mathrm{M}_{|Q|}(\mathbb{Z}[\mathbb{Z}^r])$, and more generally $\mathrm{M}_n(\mathbb{Z}[G])$ can be represented as a subring of $\mathrm{M}_{n|Q|}(\mathbb{Z}[\mathbb{Z}^r])$.

Let $\lambda \in \mathrm{M}_n(\mathbb{Z}[G])$ be residually invertible. We have

$$\lambda = (\lambda_{i,j}) \ , \ \lambda_{i,j} = \sum_{q \in Q} a_{ijq} \sigma(q) \ , \ a_{ijq} \in \mathbb{Z}[\mathbb{Z}^r].$$

The associated matrix $M_\lambda \in \mathrm{M}_{n|Q|}(\mathbb{Z}[\mathbb{Z}^r])$ satisfies

$$(M_\lambda)_{i,q;j,q'} = a^{\sigma(q)}_{i,j,q^{-1}q'} c(q, q^{-1}q')$$

where $c : Q \times Q \to \mathbb{Z}^r$ is the 2-cocycle associated to σ and $a^g = gag^{-1}$. In particular, $(M_\lambda)_{i,1;j,q} = a_{i,j,q}$. This matrix M_λ is residually invertible, thus by (ii) it has an inverse $N = (b_{i,q;j,q'})$, we have $\sum_{k,q} a_{i,k,q} b_{k,q;j,1} = \delta_{i,j}$. Thus, setting $\mu = (\mu_{i,j})$ with $\mu_{i,j} = \sum_{q \in Q} \sigma(q)^{-1} b_{i,q;j,1}$, we have $\lambda\mu = \mathrm{Id}_n$, thus λ is invertible.

The proof for $\mathrm{M}_n(\mathbb{Z}[G]_\xi)$ is entirely similar.

Remark. More generally, the proof of (iii) shows that if $\mathrm{M}_n(\mathbb{Z}[G])$ (resp. $\mathrm{M}_n(\mathbb{Z}[G]_\xi)$) has finitely detectable inversibles, the same is true for $\mathrm{M}_n(\mathbb{Z}[\Gamma])$ (resp. $\mathrm{M}_n(\mathbb{Z}[\Gamma]_\xi)$) if Γ contains G as a finite index subgroup. Since the finite detectability of invertibles obviously passes to a subgroup, the same is true for Γ commensurable with G.

Bibliography

[Ag] I. Agol, *The virtual Haken conjecture*, Doc. Math. 18 (2013), 1045-1087.

[Ar-L] P. Arnoux, G. Levitt, *Sur l'unique ergodicité des un-formes fermées singulières*, Invent. Math. 84 (1986), 141–156.

[Be-Br] M. Bestvina, N. Brady, *Morse theory and finiteness properties of groups*, Invent. Math. 129 (1997), 445–470.

[Bi-Ge] R. Bieri, R. Geoghegan, *Sigma invariants of direct products of groups*, arXiv 0808.0013

[Bi-GeKo] R. Bieri, R. Geogeghan, D.H. Kochloukova, *The Sigma invariants of Thompson's group F*, Groups, Geom. and Dynam. 4 (2010), 263-273.

[Bi-Ne-St] R. Bieri, W.D. Neumann, R. Strebel, R., *A geometric invariant of discrete groups*, Invent. Math. 90 (1987), 451–477.

[Bi-Re] R. Bieri, B. Renz, *Valuations on free resolutions and higher geometric invariants of groups*, Comment. Math. Helv. 63 (1988), 464–497.

[Bot] R. Bott, *Morse theory, old and new*, Bull. Amer. Math. Soc. (N.S.) 7 (1982), 331–358.

[Bou] N. Bourbaki, *Algèbre commutative, Chapitres 1 à 4*, Masson, 1985.

[Boy-Ro-Wie] S. Boyer, D. Rolfsen, B. Wiest, *Orderable 3-manifold groups*, Ann. Inst. Fourier (Grenoble) 55 (2005), 243–288.

[Br 1982] K.S. Brown, *Cohomology of groups*, Springer Grad. Texts in Math., 1982.

[Br 1987] K.S. Brown,*Trees, valuations and the Bieri-Neumann-Strebel invariant*, Invent. Math. 90 (1987), 479-504.

[Ca] D. Calegari, *Foliations and the geometry of* 3*-manifolds*, Oxford Mathematical Monographs, 2007.

[Ch] L. Chiswell, *Introduction to* Λ*-trees*, World Scientific, 2001.

[Cohen] M.M. Cohen, *A course in simple-homotopy theory*, Grad. Texts in Math. 10, Springer, 1973.

[Cohn] P.M. Cohn, *Free rings and their relations*, Acad. Press, 1985.

[Cu-Mo] M. Culler, J.W. Morgan., *Group actions on* $\mathbb{R}$*-trees*, Proc. London Math. Soc. 55 (1987), 571–604.

[Da 2000] M. Damian, *Formes fermées non singulières et propriétés de finitude des groupes*, Ann. Sci. Ec. Norm. Sup. (4) 33 (2000), 301–320.

[Da 2005] M. Damian, *On the higher homotopy groups of a finite CW-complex*, Topology and its Applications 149 (2005), 273–284.

[De 2008a] T. Delzant, *Trees, valuations and the Green-Lazarsfeld set*, Geom. and Funct. Anal. 18 (2008), 1236-1250.

[De 2008b] T. Delzant, personal communication, June 2008.

[De 2010] T. Delzant, *L'invariant de Bieri-Neumann-Strebel des groupes fondamentaux des variétés kählériennes*, Math. Ann. 34 (2010), 119-125.

[Du] N. Dunfield, *Alexander and Thurston norms of fibered* 3*-manifolds* Pacific J. Math. 200 (2001), 43–58.

[Ei] D. Eisenbud, *Commutative algebra. With a view towards algebraic geometry*, Grad. Texts in Math. 150, Springer, 1995.

[Em] I. Emmanouil, *Idempotent matrices over complex group algebras*, Springer Universitext, 2006.

[Fa] M. Farber, *Topology of closed one-forms*, Mathematical Surveys and Monographs, vol. 108, Amer. Math. Soc., 2004.

[Fa-Schü] M. Farber, D. Schütz, *Moving homotopy classes to infinity*, Forum Math. 19 (2007), 281-296.

[Fo] R.C. Forman, *Combinatorial Novikov-Morse theory*, Internat. J. Math. 13 (2002), 333–368.

[Fr-Ki] S. Friedl, T. Kim, *The Thurston norm, fibered manifolds and twisted Alexander polynomials*, Top. 45 (200), 929-953.

[Fr-Vi 2008a] S. Friedel, S. Vidussi, *Symplectic* $S^1 \times N^3$*, subgroup separability, and vanishing Thurston norm*, J. Amer. Math. Soc. 21 (2008), 597–610.

[Fr-Vi 2008b] S. Friedel, S. Vidussi, *Twisted Alexander polynomials and symplectic structures*, Amer. J Math 130 (2008), 455–484.

[Fr-Vi 2011] S. Friedel, S. Vidussi, *Twisted Alexander polynomials detect fibered* 3*-manifolds*, Annals of Math. 173 (2011), 1587–1643.

[G] D. Gabai, *Foliations and the topology of three-manifolds I*, J. Diff.Geom. 18 (1983), 445-503.

[Hag-Wis] F. Haglund, D. Wise, *Special cube complexes*, Geom. and Funct. Anal. 17 (2008), 1551-1620.

[Har] S.L. Harvey, *Higher-order polynomial invariants of 3-manifolds giving lower bounds for the Thurs-ton norm*, Topology 44 (2005), no. 5, 895–945.

[Hi-Ko] J. A. Hillman, D.H Kochloukova, *Finiteness conditions and* PD_r *groups covers of* PD_n*-complexes*, Math. Z. 256 (2007), 45-56.

[Ho] J. Howie, *On locally indicable groups*, Math. Z. 180 (1982), 445–461.

[Hu] M. Hutchings, *Reidemeister torsion in generalized Morse theory*, Forum Math. 14 (2002), 209–244.

[Hu-Le] Hutchings, M., Lee, Y.J., *Circle-valued Morse theory, Reidemeister torsion, and Seiberg-Witten invariants of* 3*-manifolds*, Topology 38 (1999), 861–888.

[Kap] Kaplansky, I., *Fields and rings*, The University of Chicago Press, 1969.

[Kaw] A. Kawauchi, *A survey of knot theory*, Birkhäuser, 1996.

[Ko 1999] D.H Kochloukova, *The* Σ^m*-conjecture for a class of metabelian groups*, in *Groups St. Andrews 1997 in Bath II*, London Math. Soc. Lect. Notes 261 (1999), 492–502.

[Ko 2006] D.H Kochloukova, *Some Novikov rings that are von Neumann finite*, Comment. Math. Helv. 81 (2006), 931–943.

[Ku] C. Kutluhan, C.H.Taubes, *Seiberg-Witten Floer homology and symplectic forms on* $S^1 \times M^3$, Geom. Topol. 13 (2009), 493–525

[Lam] T.Y. Lam, *Lectures on modules and rings*, Grad. Texts in Math. 189, Springer, 1999.

[Lat] Latour, F., *Existence de un-formes fermées non singulières dans une classe de cohomologie de de Rham*, Publ. IHES 80 (1994), 135–194.

[Lev 1987] G. Levitt, *Un-formes fermées singulières et groupe fondamental*, Invent. Math. 88 (1987), 635–667.

[Lev 1993] G. Levitt, *Constructing free actions on* $\mathbb{R}$*-trees*, Duke Math. J. 69 (1993), 615-633.

[Lev 1994] G. Levitt, $\mathbb{R}$*-trees and the Bieri-Neumann Strebel invariant*, Publ. Mat. 38 (1994), 195-202.

[Ly-Schu] R.C.Lyndon, P.E. Schupp, *Combinatorial group theory*, Springer, 1977.

[Mas] W.S. Massey, *Homology and cohomology theories*, Marcel Dekker, New York 1978.

[Mat] H. Matsumura, *Commutative ring theory*, Cambridge Studies in Adv. Math., Cambridge Univ. Press, 1986.

[Mc-M] C. Mc-Mullen, *The Alexander polynomial of a 3-manifold and the Thurston norm on cohomology*, Ann. Sci. Ec. Norm. Sup. 35 (2002), 153–171.

[Me] G. Meigniez, *Bouts d'un groupe opérant sur la droite. II. Applications à la topologie des feuilletages*, Tohoku Math. J. (2) 43 (1991), 473–500.

[Mi 1965] J.W. Milnor *Lectures on the h-cobordism theorem. Notes by L. Siebenmann and J. Sondow*, Math. Notes 1, Princeton Univ. Press, 1965.

[Mi 1966] J.W. Milnor, *Whitehead torsion*, Bull. Amer. Math. Soc. 72 (1966), 385-426. Collected papers 2, *The fundamental group*, 151–219.

[Mi 1968] J.W. Milnor, *Infinite cyclic coverings*, in *Conference on the topology of manifolds (East Lansing 1967)*, Prindle, Weber and Schmidt 1968, 115-133. Collected Papers 2, *The fundamental group*, 71–89.

[No 1981] S.P. Novikov, *Multivalued functions and functionals. An analogue of the Morse theory*, Soviet Math. Doklady 24 (1981), 222–226.

[No 1982] S.P. Novikov, *The Hamiltonian formalism and a multivalued analogue of the Morse theory*, Russian Math. Surveys 37 (1982), 3–49, 248

[Pas 1971] D.S. Passman, *Idempotents in group rings*, Proc. Amer. Math. Soc. 28 (1971), 371–374.

[Pas 1977] D.S. Passman, *The algebraic structure of group rings*, Pure and Applied Mathematics, Wiley, 1977.

[Pazh 1989] A. Pazhitnov, *On the sharpness of inequalities of Novikov type for manifolds with a free abelian fundamental group*, Math. Sbornik 180 (1989), 1486–1523, 1584. Translation in Math. USSR-Sbornik 68 (1991), 351–389.

[Pazh 1995] A. Pazhitnov, *On the Novikov complex for rational Morse forms*, Ann. Fac. Sci. Toulouse Math. (6) 4 (1995), 297–338.

[Pazh 1999] A. Pazhitnov, *The simple homotopy type of the Novikov complex, and the Lefschetz* ζ*-function of the gradient flow*, Russian Math. Surveys 54 (1999), 119–169.

[Sch 2002a] D. Schütz, *One-parameter fixed-point theory and gradient flows of closed one-forms*, K-Theory 25 (2002), 59–97.

[Sch 2002b] D. Schütz, *Controlled connectivity of closed one-forms*, Algebraic and Geometric Topology 2 (2002), 171–217.

[Sch 2006] D. Schütz, *Finite domination, Novikov homology, and nonsingular closed one-forms*, Math Z 25 (2006), 623–654.

[Seh] S.K. Sehgal, *Group rings*, in *Handbook of Algebra, vol. 3*, Elsevier, 2003, pp. 457–541.

[Ser] J.-P. Serre, *Cohomologie des groupes discrets*, in *Prospects in mathematics*, Ann. of Math. Studies 70 (1971), 77–169.

[Sie] L.C. Siebenmann, *A total Whitehead obstruction to fibering over the circle*, Comment. Math. Helv. 45 (1970), 1-48.

[Sik] J.-C. Sikorav, J.-C., *Homologie de Novikov associée à une classe de cohomologie de degré un*, in: Thèse d'Etat, Université Paris-Sud (Orsay), 1987.

[Sm] S. Smale, *Morse inequalities for dynamical systems*, Bull. Amer. Math. Soc. 66 (1960), 43–49.

[St] J. Stallings, *On fibering certain* 3*-manifolds*, in *Topology of* 3*-manifolds and related topics (Proc. The Univ. of Georgia Tech. Institute, 1961)*, p.95-100, Prentice-Hall.

[Tho] R. Thom, R, *Sur une partition en cellules associée à une fonction sur une variété*, Comptes Rendus Acad. Sci. Paris 228 (1949), 973–975.

[Thu] W.P. Thurston, *A norm for the homology of* 3*-manifolds*, in Mem. Amer. Math. Soc. 59 (1986), no. 339, i–vi and 99–130.

[Ti] D. Tischler, *On fibering certain foliated manifolds over* S^1, Topology 9 (1971), 153-154.

[Tu 1979] V.G. Turaev *Fundamental groups of three-dimensional manifolds and Poincaré duality. Topology (Moscow, 1979)*, Trudy Mat. Inst. Steklov. 154 (1983), 231–238.

[Tu 2002] V.G. Turaev *Torsions of* 3*-dimensional manifolds*, Birkhäuser Progress in Mathematics 208, 2002.

[Vi] S. Vidussi, *Norms on the cohomology of a 3-manifold and SW theory*, Pacific J. Math. 208 (2003), 169–186.

[Wa 1965] C.T.C. Wall, *Finiteness conditions for CW-complexes*, Annals of Math. 81 (1965), 56–69.

[Wa 1966] C.T.C. Wall, *Finiteness conditions for CW-complexes, II*, Proc. Roy. Soc. London Ser. A, 295 (1966), 129–139.

[Win] H.E. Winkelnkemper, *Artin presentations. I. Gauge theory,* $3+1$ *TQFTs and the braid groups*, J. Knot Theory Ramifications 11 (2002), 223–275.

[Wit] E. Witten, *Morse theory and supersymmetry*, J. Diff. Geom. 17 (1982), 661–692.