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Universality of rare fluctuations in turbulence and critical phenomena

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A statistical treatment of three-dimensional turbulent flow continues to pose a challenge to theorists[1,2]. One suggestion invokes an analogy with equilibrium phase transitions[3]. Here we approach this idea experimentally, presenting evidence of a strong analogy between the statistical behaviour of a confined turbulent flow and that of a model of the critical behaviour of a ferromagnet. Both systems experience large fluctuations limited only by the system size. We find that the power consumption measured in turbulent-flow experiments and the magnetization at the critical point of the ferromagnet have probability distributions of the same functional form, irrespective of Reynolds number on the one hand and system size on the other. The distributions both have non-gaussian tails that characterize the large-amplitude fluctuations. In this region, the scaled distributions for the two systems collapse onto a single universal curve over at least four orders of magnitude. This suggests a basic similarity in the finite-size corrections to the fluctuation statistics in the limit of infinite system size (for the magnetic system) or infinite Reynolds number (for turbulent flow).

The power consumption of turbulent flow is an important practical quantity in fluid mechanics, which is related, for example, to the drag experienced by a moving body. In particular, large amplitude fluctuations in this quantity, although rare, can have serious practical consequences. We have previously described an experiment[4] that measures the temporal fluctuations in the power consumption of the turbulent flow produced in an enclosed air gap between two counter-rotating disks[5]. The two important constraints of this system are that the flow is spatially confined in a cylindrical vessel, and that the integral Reynolds number is fixed at a constant value. The Reynolds number Re characterizes the complexity of the fluid motion; it is defined as the ratio of the nonlinear to dissipative forces. For this experiment, it takes the value $\mathrm{Re} = L^2\Omega/\nu$, where L and Ω are the radius and rotation rate of the disks, respectively, and ν is the fluid's kinematic viscosity. Re is fixed by controlling the electric motors that drive the disks, such that their rotation rate Ω is constant. This means that the rate of energy consumption of the flow $P(t)$, measured as the power delivered by the driving motors, fluctuates in time. The probability density function $Q_P(P)$ for the fluctuations in $P(t)$ has been measured[4] for different Reynolds numbers. The mean value, $\bar{P}$, and the r.m.s. half-width, σ_P, of the distribution depend on Re, but the functional form of the distribution does not. A general method of confirming that different normalized probability distributions $f_n(x)$ (where n is 1, 2, 3 …) have the same functional form is to plot $\sigma_n f_n(x)$ versus $(x - \bar{x}_n)/\sigma_n$, which brings the curves for different n into coincidence. The test is applied to the turbulence data in Fig. 1a, where $\sigma_P Q_P(P)$ is plotted against $(P - \bar{P})/\sigma_P$ for different Re. Two important results can be deduced from Fig. 1a: first, the sets of measurements do indeed obey the same statistics, and second, the statistics are non-gaussian. In the figure, the rescaled curves are plotted on a logarithmic scale to emphasize the tail of the distribution.

The magnetic system under consideration is the spin-wave approximation to the two-dimensional XY model, which has proved to be a convenient system for calculating critical behaviour in the finite-size regime[6]. It is defined by the hamiltonian $H = -J\Sigma_{\langle i,j\rangle}\cos(\theta_i - \theta_j)$, where J is the ferromagnetic coupling constant and θ_i the angle of orientation of the classical spin vector $\mathbf{S}_i$, constrained to lie in a plane. The summation is over nearest neighbours and the spins are set on a square lattice with periodic boundary conditions. The spin-wave approximation, which is valid at low temperature, is obtained by expanding the cosine interaction to the quadratic term. Magnetic ordering is described by the instantaneous scalar order parameter:

$$M = \frac{1}{N}\sqrt{\left(\sum_{i=1,N}\mathbf{S}_i\right)^2}$$

which has a probability distribution $Q_M(M)$, with mean value $\langle M\rangle$ and standard deviation σ_M. In Fig. 1b, we show the Monte Carlo results of ref. 6. As in the turbulence case, we observe universal statistics, this time as a function of N and T.

The central empirical result of this work is presented in Fig. 1c, where the probability density functions of the turbulence experiment and the critical system are overlaid on the same graph. These two apparently disconnected quantities obey essentially the same, universal, non-gaussian statistics. The high end of the distribution has a guassian shape. Near the centre, small systematic discrepancies are found, both within and between the two sets of curves. In the magnetic case, these are $O(1/N)$ corrections to universality[7], and in the turbulence case, they vary systematically with Re. The main region of interest, however, is the distinctive exponential tail towards the low end of the distribution, where the data coincide over four orders of magnitude, for a span of eight standard deviations. This result emphasizes the fact that in practical applications it is unsafe to assume that such fluctuations of a turbulent system are gaussian. For example, it can be deduced from the curve that the probability of a rare fluctuation of greater than six standard deviations from the mean is still 2×10^{-4}, as opposed to 1×10^{-9} in the gaussian case.

To see how the overlap of the two distributions can occur, we first observe that any possible analogue between the two systems must occur at a critical point; a singular point in the thermodynamic phase diagram, which is characterized by fluctuations that are correlated over a divergent length scale[8]. In a thermodynamic system, the driving force for the fluctuations is on the atomic scale. All trace of fluctuations on larger scales disappears everywhere except at the critical point, where they are maintained and grow right up to the macroscopic scale. In the turbulent fluid, the driving force is on the macroscopic scale, and the fluctuations are driven downward in scale to a dissipation length η, below which the system is uniform. Therefore, as in the critical system, each one of these length scales is important for establishing both the mean value of the dissipated power and the fluctuations about the mean.

Away from a critical point, fluctuations in a global quantity such as M can generally be treated as a small perturbation about the extremely precise mean value that defines equilibrium[9]. The standard deviation of the distribution, σ_M, is then proportional to $1/\sqrt{N}$. This is a direct result of the statistical independence of the internal degrees of freedom and is a cornerstone of statistical thermodynamics. As N becomes large, the distribution becomes vanishingly narrow and is represented by a gaussian function, to an excellent approximation. In contrast, at a critical point the system is scale-free or 'self-similar'[10], meaning that on moving from one length scale to another, the form and intensity of the fluctuations remains unchanged and the scale invariance is only interrupted on arriving at the characteristic size L of the system[11]. The fluctuations over different length scales all contribute to the fluctuations in M, with the result that $Q_M(M)$ is much broader; σ_M no longer varies as $1/\sqrt{N}$ and the distribution becomes non-gaussian.

The two-dimensional XY model provides an excellent example for studying critical behaviour as it has a continuous line of critical points over a finite range of temperature[12,13]. At any one of these points, a renormalization-group treatment converges onto the spin-wave approximation[14], which therefore describes the critical behaviour perfectly at all temperatures. The scale independence can be illustrated within the spin wave approximation, where one can easily calculate $\langle M \rangle$ and σ_M (refs 6, 15): $\langle M \rangle = (1/2N)^{k_B T/8\pi J}$, $\sigma_M = aT\langle M \rangle$, $a \approx 0.04$. In the thermodynamic limit, $N \to \infty$, the order parameter $\langle M \rangle$ is strictly zero[16]. However, the corrections to the thermodynamic limit disappear more slowly than the $\sim 1/\sqrt{N}$ variation discussed above[11,17,18] and the ratio $\sigma_M/\langle M \rangle$ is independent of system size. Further, the self-similar nature of the fluctuations around this mean value is expected to reflect itself in the kind of scale independence for Q_M observed numerically in Fig. 1b (refs 19, 20). We have recently proved this explicitly[7], showing that $Q_M(M)$ is indeed a universal function, independent of both N and T.

We now argue that there is a connection between system size and Reynolds number that leads to an analogy, at a stochastic level, between the magnetic model and the turbulent fluid. The fluid motion is described by the three-dimensional Navier–Stokes equation and turbulence has its origin in the nonlinear term, whose importance compared to the smoothing viscous forces is given by the Reynolds number. In our experiment, motion is created at large scales where energy is fed into the flow and transferred by the nonlinearity to smaller scales where the viscous forces become progressively more important. At a crossover length scale η, they dominate; energy dissipation exceeds energy transfer and the cascade terminates. Turbulence can thus be viewed as a super-

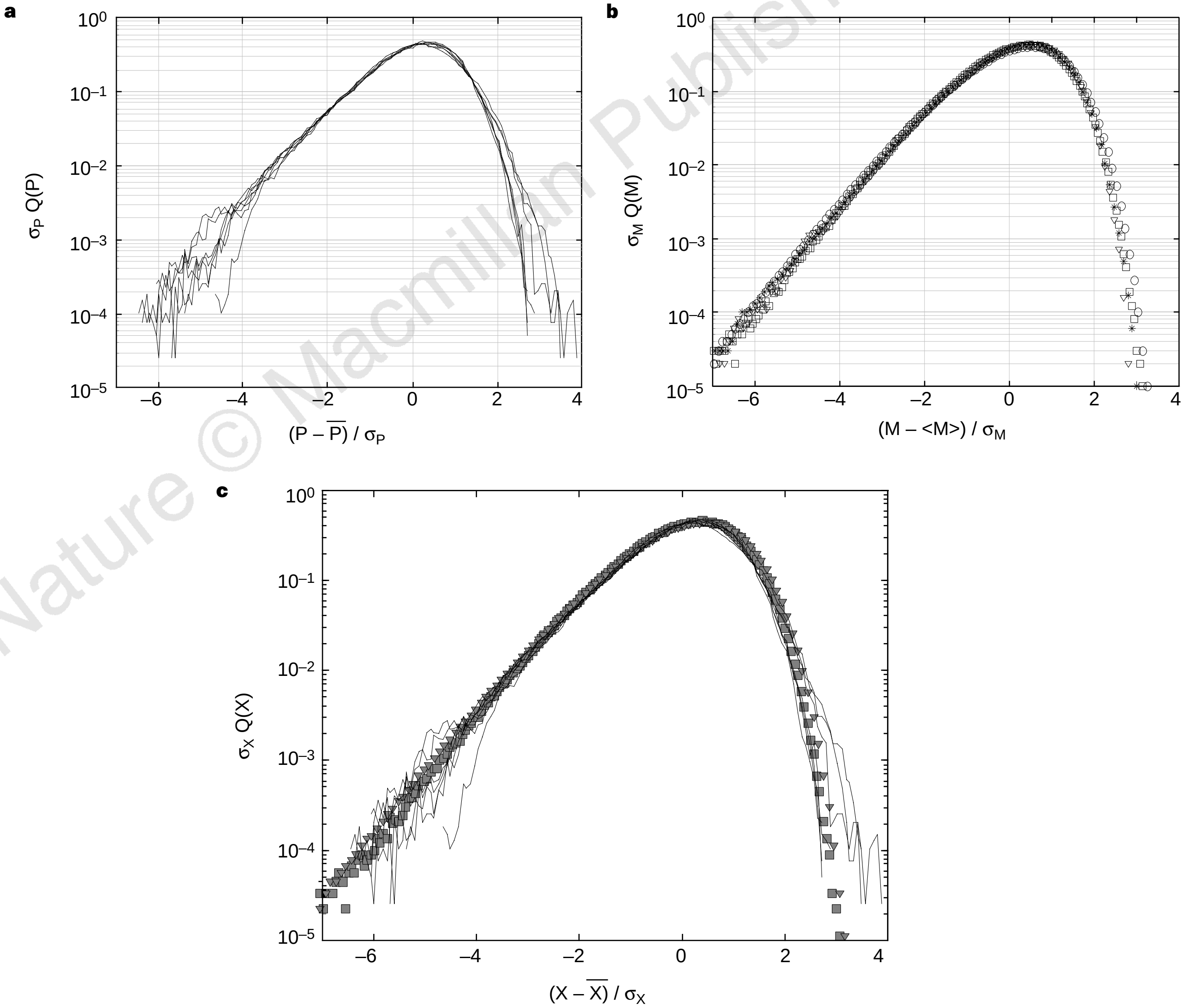

Figure 1 Universality of global fluctuations. Probability density function of power fluctuations, $\sigma_P Q_P$, in the turbulent closed flow produced in the gap between two counter-rotating disks (from ref. 4). Data are shown for rotation rates Ω of 25, 30, 35, 40 and 45 Hz. Using the reduced variable $(P - \bar{P})/\sigma_P$, the data measured at different Reynolds number, $Re \propto \Omega$, collapse onto a single curve. **b**, Probability density for the instantaneous magnetization from Monte Carlo simulation of the spin wave approximation of the 2D-XY model at $k_B T/J = 0.5$, for $N = 100$ spins (circles), $N = 1{,}024$ spins (stars), $N = 10{,}000$ spins (triangles) and $k_B T/J = 1.0$ for $N = 1{,}024$ spins (squares)[6]. The data collapse onto a single universal curve for the reduced variable $(M - \langle M \rangle)/\sigma_M$. **c**, Data from **a** and **b** superimposed; where X is the power P or magnetization M, and $\bar{X} = \bar{P}$ or $\langle M \rangle$.

position of moving eddies on a wide range of scales. The smallest one is the dissipation length η. In a closed flow, the largest one is equal to the system size L. We emphasize the difference with open flows where eddies escape through the open boundaries and the power consumption has gaussian fluctuations[4]. The effective number of degrees of freedom contributing to the turbulence motion can be estimated as $N = (L/\eta)^3$ (ref. 21). Using Kolmogorov mean-field arguments, N can be related to the Reynolds number through $L/\eta = \mathrm{Re}^{4/3}$ (ref. 1). Hence any system of finite Reynolds number can be considered as containing a finite number of degrees of freedom. Given that many scales are important, the simplest scenario is a self-similar one. Indeed, both Kolmogorov's original hypothesis about the energy cascade and subsequent corrections to include intermittency effects assume that the energy transfer per unit mass on scale l, ω_l, has a scale-invariant behaviour (for example, $\langle \epsilon_l^p \rangle \propto l^{\pi(p)}$), as confirmed by experiment[1,22]. In a closed turbulent flow, the scale invariance holds up to the system size. The experimental evidence presented here suggests that this is sufficient to provide a complete analogy with the critical magnetic system and the same kind of universality for $Q_P(P)$ as for $Q_M(M)$.

A consequence of this interpretation is that the fluctuations in the dissipated power can be thought of as a finite-size correction to the result at infinite Reynolds number. The $\mathrm{Re} = \infty$ case is by definition inaccessible for any experiment studying turbulence in a closed flow, as there is a well defined upper and lower length scale between which fluctuations are important. For very small values of Re, one might expect to observe transient effects that depend on the details and dimensions of the experiment, but once some threshold has been exceeded, $Q_P(P)$ should be independent of Re, however large it may be.

As many length scales are important at a critical point, the microscopic details often get washed away and the critical behaviour of apparently radically differing systems can be the same. Only large scale details, such as the symmetry of the hamiltonian and the dimension, are important, and systems with the same critical behaviour are grouped together in 'universality classes'[8]. The probability distribution function for all systems falling in the same universality class should therefore have the same universal form. However, there is no *a priori* reason to expect that the same should be true in going from one universality class to another. It is therefore surprising to find that these two functions are so similar. We do not believe that we have miraculously fallen onto the correct universality class for the turbulent fluid experiment. Rather, it is likely that, for certain universality classes, the departure from gaussian behaviour at a critical point is described to an excellent approximation by the spin-wave limit of the two-dimensional XY model, with the fine details that characterize and separate the universality classes being concentrated in the central part of the distribution function, or otherwise being hidden by experimental errors.

Our results indicate that the universality observed in the turbulence experiment can be explained in terms of a self-similar structure of fluctuations, just as in a finite critical system. This analogy provides an important new experimental application of finite size scaling approaches to a critical point[20] and it suggests a new range of experiments to characterize turbulent flow. Finally, it provides a systematic method of predicting the probability of rare fluctuations in a confined turbulent system. ☐

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Observation of 'third sound' in superfluid ^{3}He

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Waves on the surface of a fluid provide a powerful tool for studying the fluid itself and the surrounding physical environment. For example, the wave speed is determined by the force per unit mass at the surface, and by the depth of the fluid[1]: the decreasing speed of ocean waves as they approach the shore reveals the changing depth of the sea and the strength of gravity. Other examples include propagating waves in neutron-star oceans[2] and on the surface of levitating liquid drops[3]. Although gravity is a common restoring force, others exist, including the electrostatic force which causes a thin liquid film to adhere to a solid. Usually surface waves cannot occur on such thin films because viscosity inhibits their motion. However, in the special case of thin films of superfluid ^{4}He, surface waves do exist and are called 'third sound'. Here we report the detection of similar surface waves in thin films of superfluid ^{3}He. We describe studies of the speed of these waves, the properties of the surface force, and the film's superfluid density.

Superfluid ^{3}He can be described by a 'two fluid' model[4] where the liquid is considered to be two interpenetrating fluids: a normal fluid component and a superfluid component. Each component has a mass density fraction, ρ_n/ρ and ρ_s/ρ respectively, and is governed by a different equation of motion. In particular, the superfluid component flows without viscosity while the normal component experiences viscous drag. This 'two fluid' nature allows several different types of acoustic phenomena to exist. For example, oscillations with both components moving in-phase are called first sound, while out-of-phase oscillations are called second sound.

Third sound[5,6] is the name given to a surface wave travelling on a thin superfluid film. Here, the superfluid component oscillates parallel to the substrate while the normal-fluid component is held stationary by viscosity, as shown schematically in Fig. 1a. A typical wave amplitude is 0.1 nm. In the simplest case, the speed of