Finite size effects in the $N = 1$ supersymmetric sine-Gordon models

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Study finite size effects in this family of models

- $N = 1$ **SSG models**
- **Light cone lattice regularization**
- **Nonlinear integral equations (NLIE)**
- **IR limit and the S-matrix**
- **UV limit of the model**
- **Check against PCFT**

Motivation

- Generalize the NLIEs for $N = 2$ SSG (relevant for string theory)

The $N = 1$ SSG model

The action

$$\mathcal{A} = \int d^2x \, \left\{ \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + i\bar{\psi}\gamma^\mu\partial_\mu\psi + \right.$$

$$\left. g\,\bar{\psi}\psi\cos\left(\frac{\beta}{2}\phi\right) + \frac{g^2}{\beta^2}\cos(\beta\phi) \right\},$$

- integrable
- has $N = 1$ SUSY
- perturbed $c = 3/2$ CFT
- $\Delta_{pert} = \frac{1}{2} + \frac{\beta^2}{32\pi}$
- $\Delta_{count.} = \frac{\beta^2}{8\pi}$

$c = 3/2$ CFT

Theory of a massless free boson and a Majoranna fermion

- The boson is compactified on a circle of radius R with periodic boundary conditions: $\phi(x+L) = \phi(x) + 2\pi m R$, $m \in \mathbb{Z}$ is the winding numer (topological charge)

- Free Majoranna fermion with periodic (R) or antiperiodic (NS) boundary conditions: $\psi(x+L) = \pm\psi(x)$

- NS sector: vertex operators

$$V_{n,m}^{(r_+,r_-)}(z,\bar{z}) = \bar{\psi}_{r_-}\psi_{r_+} : e^{\left[i\left(\frac{n}{R}+\frac{m}{2}R\right)\varphi(z)+\left(\frac{n}{R}-\frac{m}{2}R\right)\bar{\varphi}(\bar{z})\right]} :$$

$r_\pm \in \{0,1\}$, $\psi_0 = 1$, $\psi_1 = \psi$ of conformal dimensions

$$\Delta_{n,m}^{(r_+,r_-)\pm} = \frac{1}{2}\left(\frac{n}{R} \pm \frac{m}{2}R\right)^2 + \frac{r_\pm}{2}$$

- R sector: vertex operators

$$R_{n,m}(z,\bar{z}) = \sigma(z,\bar{z}) : e^{\left[i\left(\frac{n}{R}+\frac{m}{2}R\right)\varphi(z)+\left(\frac{n}{R}-\frac{m}{2}R\right)\bar{\varphi}(\bar{z})\right]} :$$

of conformal dimensions

$$\Delta_{n,m}^{(R)\pm} = \frac{1}{16} + \frac{1}{2}\left(\frac{n}{R} \pm \frac{m}{2}R\right)^2$$

- In this language if $R = \frac{\sqrt{16\pi}}{\beta}$

- Perturbing operator: $\frac{1}{\sqrt{2}}\left(V^{(1,1)}_{1,0} + V^{(1,1)}_{-1,0}\right)$

- Counterterm operator: $\frac{1}{2}\left(V^{(0,0)}_{2,0} + V^{(0,0)}_{-2,0}\right)$

- Convenient definition: $p = \frac{2}{R^2-1} = \frac{1}{\frac{8\pi}{\beta^2}-\frac{1}{2}}$.

Modular invariant theory

- guarantees locality

- Partition function (Di Francesco et. al. '88)

$$Z(R) = \frac{1}{|\eta|^2}\left\{(\chi_0\bar{\chi}_{1/2} + \chi_{1/2}\bar{\chi}_0)\sum_{n\in\mathbb{Z}+1/2, S\in 2\mathbb{Z}+1} + \right.$$

$$\left.(|\chi_0|^2 + |\chi_{1/2}|^2)\sum_{n\in\mathbb{Z}, S\in 2\mathbb{Z}} + |\chi_{1/16}|^2 \sum_{2n-S\in 2\mathbb{Z}+1}\right\} q^{\Delta^+(n,S)}\,\bar{q}^{\Delta^-(n,S)}$$

where $\Delta^{\pm}(n,S) = \frac{1}{2}\left(\frac{n}{R} \pm \frac{S}{2}R\right)^2$, and $q = e^{2i\pi\tau}$, τ is the modular paremeter.

The operator content:

NS sector: $V_{n,m}^{(r_+,r_-)}(z,\bar{z})$

$$n \in \mathbb{Z}+1/2, \quad m \in 2\mathbb{Z}+1, \quad \begin{cases} r_+ = 1, & r_- = 0 \\ r_+ = 0, & r_- = 1. \end{cases}$$

$$n \in \mathbb{Z}, \quad m \in 2\mathbb{Z}, \quad \begin{cases} r_+ = r_- = 0 \\ r_+ = r_- = 1. \end{cases}$$

R sector: $R_{n,m}(z,\bar{z})$

$$n \in \mathbb{Z}, \quad m \in 2\mathbb{Z}+1$$

$$n \in \mathbb{Z}+1/2, \quad m \in 2\mathbb{Z}$$

Spectrum

Repulsive regime: $p > 1$

- integrability $\rightarrow$ FST and S-matrix
- model has 3 degenerated vacua $\{0, 1/2, 1\}$
- elementary excitations are supersymmetric solitons (kinks)

$K^{\epsilon}_{ab}(\theta)$ of mass M where $a, b \in \{0, 1/2, 1\}$, $\quad |a - b| = 1/2$, $\epsilon \in \{\pm 1\}$ topological charge.

- S-matrix of kinks has tensor product form (Ahn '90)

$$S(\theta) = \underbrace{S_2(\theta)}_{\text{SUSY}} \otimes \underbrace{S_{SG}(\theta)}_{\text{bosonic degrees of freedom}}$$

.

$S_2(\theta)$ is the S-matrix of tricritical Ising perturbed by ϕ_{13}.

$S_{SG}(\theta)$ is a SG S-matrix at $\beta^2_{SG} = \frac{\beta^2}{1 + \frac{\beta^2}{16\pi}}$

Have two descriptions of theory

- **UV** $(L \to 0)$

- CFT data $c, \Delta, \bar{\Delta}$

- Perturbing term coupling $g = \kappa \; M^{2(1-\Delta)}$

- **IR** $(L \to \infty)$

- Soliton mass M

- S-matrix

- **Connect via Finite Size Effects**

- define the theory on a cylinder of circumference L

- scaling functions $E_i(L) = -\frac{\pi \;\; c_i(ML)}{6 \;\; L}$

- interpolation between the conformal and the massive spectrum

Lattice regularization and Bethe Ansatz

- SSG is suitable continuum limit of the 19-vertex model with alternating inhomogeneities (Reshetikhin, Saleur '93)

Description of the model

Let $V_i \simeq \mathbb{C}^{l_i+1}$ irred. spin $l_i/2$ representation,

$R_{ij}^{(l_i,l_j)}(\theta)$: $V_i \otimes V_j \rightarrow V_i \otimes V_j$ R-matrices satisfying the Yang-Baxter equations:

$$R_{12}^{(l_1,l_2)}(\theta) R_{13}^{(l_1,l_3)}(\theta+\theta') R_{23}^{(l_2,l_3)}(\theta') =$$

$$R_{23}^{(l_2,l_3)}(\theta') R_{13}^{(l_1,l_3)}(\theta+\theta') R_{12}^{(l_1,l_2)}(\theta)$$

Any $R_{12}^{(l_1,l_2)}(\theta)$ R-matrix can be built up from the simplest $R_{12}^{(1,1)}(\theta)$ R-matrix of the 6-vertex model by fusion

Monodromy matrices: $\Theta_i = (-1)^{i+1}\Theta \qquad N = \text{even}$

$$T^{(p)}(\theta, \{\theta_i\}) = R_{a1}^{(p,2)}(\theta - \Theta_1) \ldots R_{aN}^{(p,2)}(\theta - \Theta_N)$$

$$T^{(p)} : V_a \otimes V_H \rightarrow V_a \otimes V_H \qquad V_H = V_1 \otimes \cdots \otimes V_N$$

$$V_a \simeq \mathbb{C}^{p+1} \qquad V_i \simeq \mathbb{C}^3, \quad i \in \{1, \ldots, N\}$$

Transfer matrices:

$$T_p(\theta, \{\Theta_i\}) = \mathsf{Tr}_a T_p(\theta, \{\Theta_i\})$$

$$T_p : V_H \rightarrow V_H$$

Integrability

$$[T_p(\theta, \{\Theta_i\}), T_q(\theta', \{\Theta_i\})] = 0$$

Hamiltonian and momentum:

$$e^{i\frac{a}{2}(H \pm P)} \sim T_2\,(\pm\,\Theta\,, \{\Theta_i\})$$

Diagonalization $\longrightarrow$ Bethe Ansatz equations

$$\frac{\phi(\theta_j + i\pi)}{\phi(\theta_j - i\pi)} = -\frac{Q(\theta_j + i\pi)}{Q(\theta_j - i\pi)} \qquad j = 1, \ldots, M$$

$Q(\theta) = \prod_{j=1}^{M} \sinh \frac{\gamma}{2}(\theta - \theta_j)$ Baxter's Q

$$\phi(\theta) = \left(\sinh \frac{\gamma}{2}(\theta - \Theta) \cdot \sinh \frac{\gamma}{2}(\theta + \Theta)\right)^{N/2}$$

$\gamma = \frac{\pi}{p+2}$ anisotropy

Continuum limit

$$N \to \infty, \quad a \to 0, \qquad L = Na = \text{fixed},$$

$$\Theta = \ln\left(\frac{2N}{ML}\right) \to \infty$$

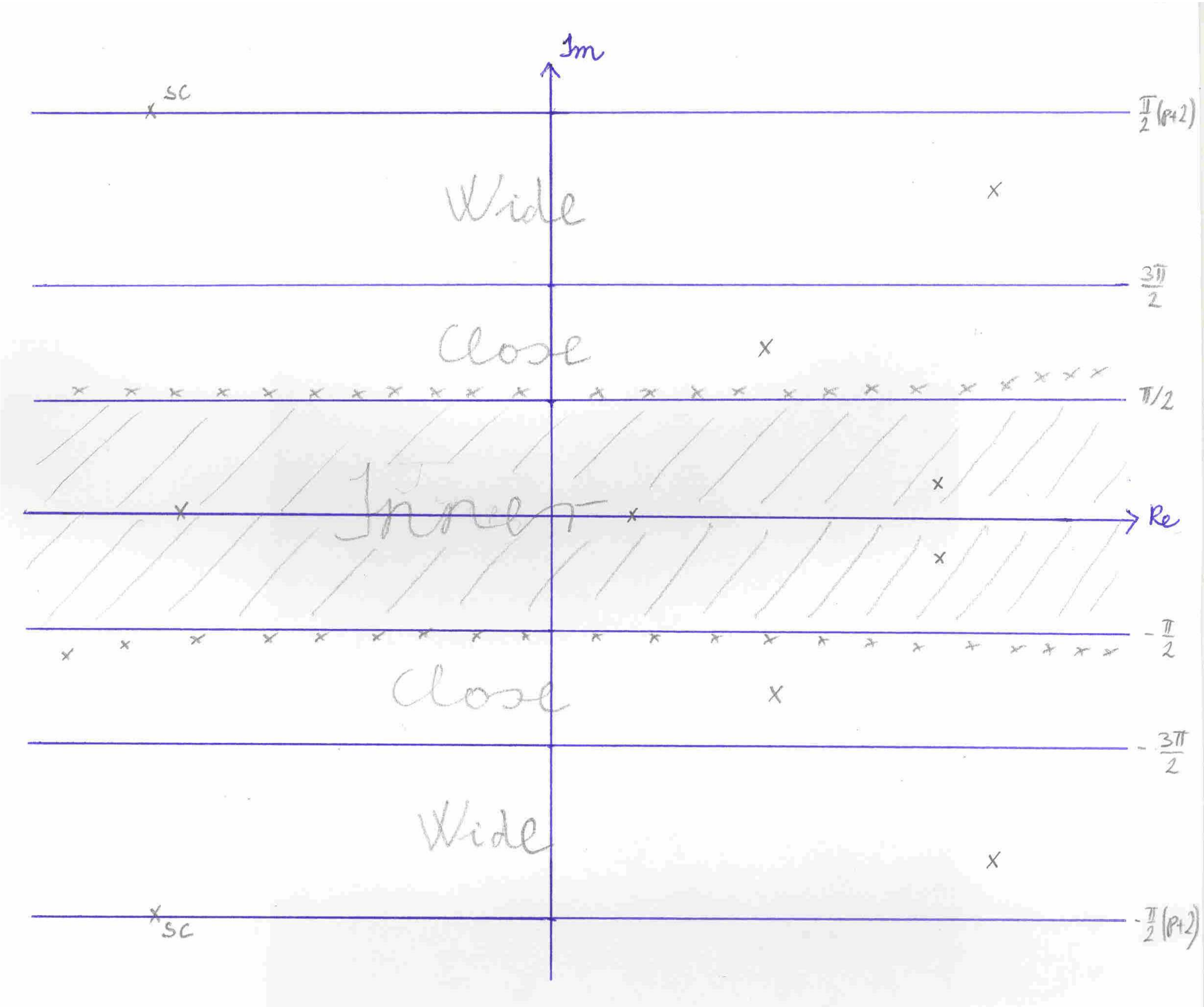
$a =$ lattice constant

$M =$ kink mass

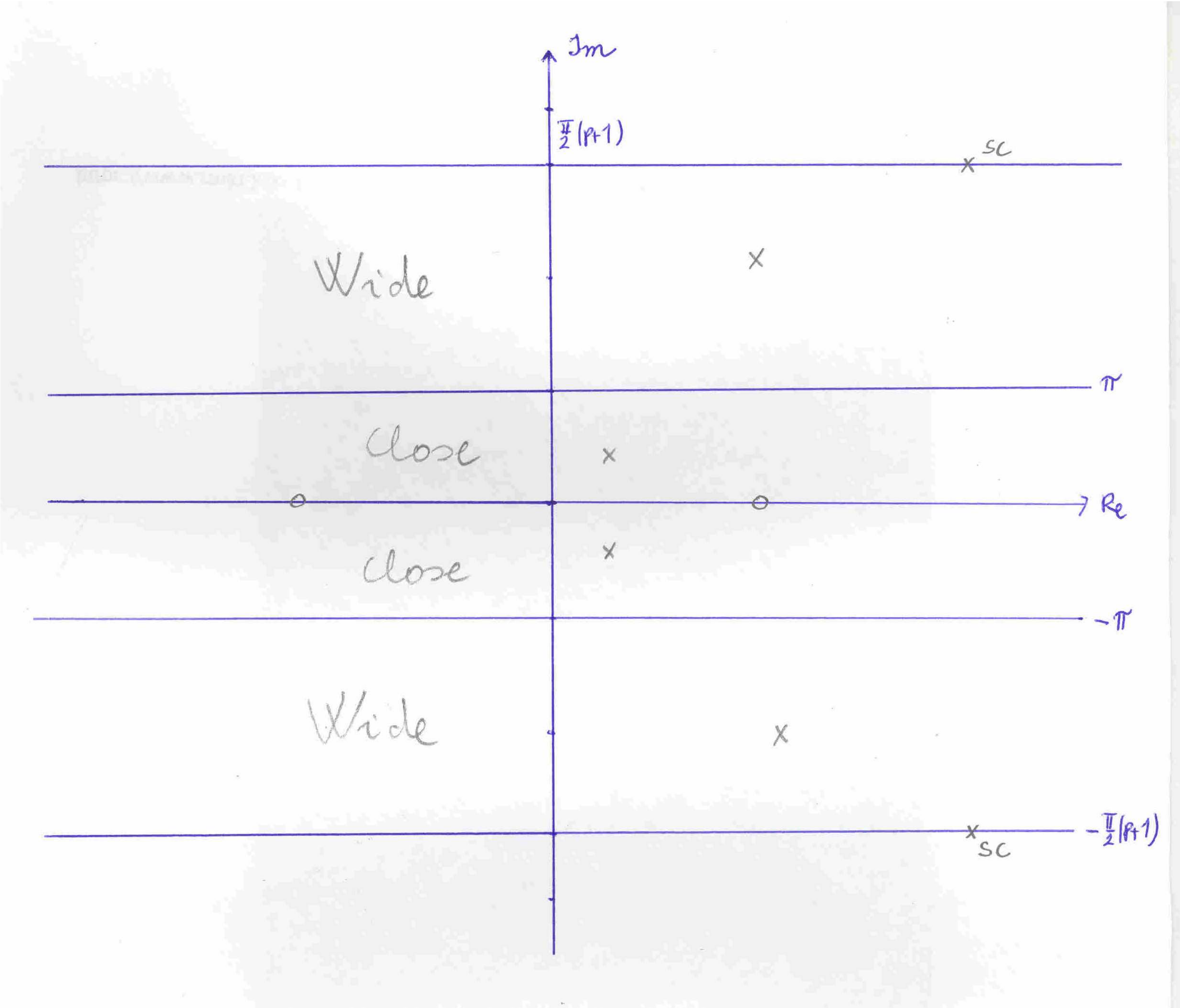
$L =$ volume

Classification of Bethe roots

- Ground state: sea of 2-strings



NLIE can be formuled by "effective roots"
roots with $\frac{\pi}{2} \geq |\mathrm{Im}\,\theta_j| \geq \frac{\pi}{2}(p+2)$ form the "effective plane"



Other important objects: real zeroes of $T_1(\theta)$ and $T_2(\theta)$. Real zeroes of $T_2(\theta)$ are called holes and lie on the real axis of the effective plane,

$$T_2(h_j) = 0, \quad h_j \in \mathbb{R}, j = 1, \dots, N_H$$

They correspond to holes in the 2-string distribution.

Real zeroes of $T_1(\theta)$ lie on the real axis of the original plane,

$$T_2(h_j^{(1)}) = 0, \quad h_j^{(1)} \in \mathbb{R}, j = 1, \ldots, N_1$$

They correspond to holes in the distribution of real roots.
Counting equations:

$$N_H = 2S + M_C + 2M_W$$

$$N_1 = S + M_C^{(2)} + M_W - M_R$$

Nonlinear integral equation (NLIE)

Introducing suitable auxiliary functions built from $T_1(\theta), T_2(\theta), Q(\theta)$ the Bethe Ansatz equations can be incorporated in NLIEs (Suzuki '98)

$$\log b(\theta) = i\,\delta_b\,\pi + il\,\sinh^{+\epsilon}(\theta) + ig_b^{+\epsilon}(\theta) + (G * \ln \mathcal{B})(\theta)$$

$$-(G^{+2\epsilon} * \ln \bar{\mathcal{B}})(\theta) + ig_1^{+\epsilon}(\theta) + (K^{-\frac{\pi}{2}+\epsilon} * \ln Y)(\theta)$$

$$\log y(\theta) = i\delta_y\pi + iD_y^{+\frac{\pi}{2}}(\theta) + (K^{+\frac{\pi}{2}-\epsilon} * \ln \mathcal{B})(\theta) + (K^{-\frac{\pi}{2}+\epsilon} * \ln \bar{\mathcal{B}})(\theta)$$

$$\mathcal{B}(\theta) = 1 + b(\theta), \qquad Y(\theta) = 1 + y(\theta), \qquad l = ML$$

Notations

$$f^{\pm\eta}(\theta) = f(\theta \pm i\eta), \quad (f * g)(\theta) = \int d\theta' f(\theta - \theta')g(\theta')$$

Kernel: derivative of the phase of the soliton-soliton scattering

$$G(\theta) = -i\,\frac{d}{d\theta}\,\ln S_{++}^{++}(\theta)|_{\beta^2 = 8\pi p/(p+1)}$$

$$K(\theta) = \frac{1}{2\pi\cosh(\theta)}$$

Sources:

$$g_b(\theta) = \sum_{j=1}^{N_H} \chi(\theta - h_j) - \sum_{j=1}^{M_C} \chi(\theta - c_j) - \sum_{j=1}^{M_W} \chi_{II}(\theta - w_j)$$

$$g_1(\theta) = \sum_{j=1}^{N_1} \chi_K(\theta - h_j^{(1)})$$

$$D_y(\theta) = \sum_{j=1}^{N_H} \chi_K(\theta - h_j) - \sum_{j=1}^{M_C} \chi_K(\theta - c_j) + \pi(S + M_W)$$

$\chi(\theta)$ = odd primitive of $2\pi\, G(\theta)$ and $\chi_K(\theta)$ = odd primitive of $2\pi\, K(\theta)$

Two constants: $\delta_b, \delta_y \in \{0, 1\}$

- $\delta_y = 0$ from the continuum limit of the 19-vertex model
- determination of source objects:

$$\frac{1}{i} \log b(\theta_j) = 2\pi I_j$$

$$\frac{1}{i} \log y(h_j^{(1)} - i\frac{\pi}{2}) = 2\pi I_j^{(1)}$$

$y(\theta) \to$ RSOS degrees of freedom

Energy and momentum

$$E = M\sum_{j=1}^{N_H}\cosh(h_j) - M\sum_{j=1}^{M_C}\cosh(c_j)$$

$$+i\,\frac{M}{2\pi}\int d\theta\,\sinh(\theta+i\epsilon)\ln\mathcal{B}(\theta) - i\,\frac{M}{2\pi}\int d\theta\,\sinh(\theta-i\epsilon)\ln\bar{\mathcal{B}}(\theta)$$

$$P = M\sum_{j=1}^{N_H}\sinh(h_j) - M\sum_{j=1}^{M_C}\sinh(c_j)$$

$$+i\,\frac{M}{2\pi}\int d\theta\,\cosh(\theta+i\epsilon)\ln\mathcal{B}(\theta) - i\,\frac{M}{2\pi}\int d\theta\,\cosh(\theta-i\epsilon)\ln\bar{\mathcal{B}}(\theta)$$

In the IR limit $(L\to\infty)$ $\ln\mathcal{B},\ln\bar{\mathcal{B}}\to 0$, so a hole excitation with h_j corresponds to a kink of rapidity h_j.

S-matrix from NLIE

Quantization of momenta of 2-kink states $\rightarrow$ S-matrix

$$e^{il\sinh(h_1)}\Lambda(h_1-h_2)=1$$

$\Lambda =$ "eigenvalue of the S-matrix"

Expectation:

$$e^{il\sinh(h_1)}\Lambda_2(h_1-h_2)\Lambda_{SG}(h_1-h_2)=1$$

$$\Lambda_2 \rightarrow \{\pm\lambda_1, \pm\lambda_2\}, \qquad \Lambda_{SG} \rightarrow \{S_{++}^{++}, S_+, S_-\}$$

Quantization of momenta from the NLIE:

$$\frac{1}{i}\log b(h_{1,2}) = 2\pi I_{1,2}$$

In $l \rightarrow \infty$ limit $\ln \mathcal{B}(\theta) \rightarrow 0$ and $y(\theta) \rightarrow y_\infty(\theta)$ finite limit

$$\log b(\theta) = i\ \delta_b\pi + i\ l\ \sinh(\theta) + i\ g_b(\theta) + (K^{-\frac{\pi}{2}} * \ln Y_\infty)(\theta) + \dots$$

$$\frac{1}{i}\log y(\theta) = \delta_y\pi + D_y(\theta + i\frac{\pi}{2})(\theta) + \dots$$

Eigenvalues from NLIE:

$$\Lambda_2(h_1-h_2) = e^{i\delta_b\pi + (K^{-\frac{\pi}{2}} * \ln Y_\infty)(h_1)}$$

$$\Lambda_{SG}(h_1-h_2) = -e^{i\, g_b(h_1)}.$$

2-hole solutions $N_H = 2$

$S = 1$ pure 2-hole $\to S_{++}^{++}$
$S = 0$ close pair $\to S_-$
$S = 0$ 1 self-conjugated root $\to S_+$

$\delta_y = 1$ is necessary to get all the 4 RSOS eigenvalues

$\delta_b \in \{0, 1\}, \delta_y \in \{0, 1\}$ choices give the 4 RSOS eigenvalues

Results obtained from NLIE agrees with Ahn's S-matrix ($S_2 \otimes S_{SG}$)

UV limit

In the UV limit the energy behaves like:

$$E_a(l) = -\frac{\pi}{6L}(c - 12(\Delta_a^+ + \Delta_a^-)) + \dots$$

Using standard techniques $(c, \Delta_a^\pm)$ can be calculated from NLIE:

$$\Delta^\pm = \frac{\delta_y}{16} + \frac{1}{2}\left(\frac{n_\pm}{R} \pm \frac{S}{2}R\right)^2 + J_\pm$$

$S =$ spin of the state

$\delta_y = 0$ case: NS sector

$J_\pm$ either $\mathbb{Z}$ or $\mathbb{Z} + 1/2$

Relations between $n_\pm$ and S are as follows

$n_\pm \in \mathbb{Z}$ if $S \in 2\mathbb{Z}$

$n_\pm \in \mathbb{Z} + 1/2$ if $S \in 2\mathbb{Z} + 1$

$\delta_y = 1$ case: R sector

$J_\pm \in \mathbb{Z}$

Relations between $n_\pm$ and S are as follows

$n_\pm \in \mathbb{Z} + 1/2$ if $S \in 2\mathbb{Z}$

$n_\pm \in \mathbb{Z}$ if $S \in 2\mathbb{Z} + 1$

This set of conformal weights correspond to the modular invariant partition function of Di Francesco et. al.

Conformal perturbation theory

R sector was proposed arbitrarily by setting $\delta_y = 1$, thus more checks are necessary

Charged 2-hole state: $(N_H = 2, S = 1, \delta_y = 1, \delta_b = 0)$

$\Delta^{\pm} = \frac{1}{16} + \frac{R^2}{8}$ in the UV limit

Using the massgap formula (Fateev,Baseilhac '98)

$$g = M^{1-\frac{\beta^2}{16\pi}} \cdot \kappa(\beta^2),$$

and standard methods of CPT the first nonzero correction of this state can be calculated:

$$\frac{6L}{\pi}E(L) = -(c - 12(\Delta^+ + \Delta^-)) - C_2\, l^{2y} + O(l^{4y})$$

$$y = 2(1 - \Delta_{pert}), \qquad l = ML$$

C_2 depends only on β.

Numerical comparison at $R^2 = 5/3$

$$\varepsilon(l) = \frac{6L}{\pi} E(l) + \frac{3}{2} - 12(\Delta^+ + \Delta^-)$$

l	$\varepsilon(l)$ (NLIE)	$\varepsilon(l)$ (PCFT)	l^{4y}
0.1	0.4411699(1)	0.42645819	0.025
0.05	0.24978747(1)	0.24493591	$8 \cdot 10^{-3}$
0.01	0.06795829(1)	0.067589069	$6 \cdot 10^{-4}$
0.005	0.03894153(2)	0.03881972	$2 \cdot 10^{-4}$
0.001	0.01072140(3)	0.010712145	10^{-5}
0.0005	0.00615559(2)	0.00615251	$5 \cdot 10^{-6}$
0.0001	0.0016979(1)	0.000169776	$4 \cdot 10^{-7}$

Conclusions

- Various checks support that NLIE provides correct description of FSE in SSG models
- Introduction of twist gives access to perturbed superminimal models
- Extension to attractive regime can give account of SUSY breathers and twisted versions could describe nonunitary perturbed superminimal models
- Proposal to $N = 2$ SUSY SG relevant for string theory