# Some Functions Computable with a Fused-mac

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## Abstract

*The fused multiply accumulate instruction (fused-mac) that is available on some current processors such as the Power PC or the Itanium eases some calculations. We give examples of some floating-point functions (such as* $\mathrm{ulp}(x)$ *or* $\mathrm{Nextafter}(x, y)$*), or some useful tests, that are easily computable using a fused-mac. Then, we show that, with rounding to the nearest, the error of a fused-mac instruction is exactly representable as the sum of two floating-point numbers. We give an algorithm that computes that error.*

## 1 Introduction

The fused multiply accumulate instruction (fused-mac) is available on some current processors such as the IBM Power PC or the Intel/HP Itanium. That instruction computes an expression $\pm ax \pm b$ with one final rounding error only. This allows one to perform correctly rounded division using Newton-Raphson division [17, 7, 16] (the main idea behind that is that if $q$ approximates $x/y$ with enough accuracy, then the remainder $x - yq$ will be exactly computed with a fused-mac, allowing to correct the quotient estimation). Also, this makes evaluation of scalar products and polynomials faster and, generally, more accurate than with conventional (addition and multiplication) floating-point operations. This is important, since scalar products appear everywhere in linear algebra, and since polynomials are very often used for approximating functions.

It is well known [9, 2, 3] that (assuming rounding to nearest) the error of a floating point addition or the remainder of a square root is exactly representable as a floating-point number of the same format. This is also true (for any rounding mode) for the error of a multiplication or the remainder of a division. A natural question arises: is there a similar property for the fused-mac operation?

Also, expert floating-point programming sometimes requires the evaluation of functions such as $\mathrm{Nextafter}(x, y)$, or the successor of a given floating-point number, or (for error estimation), $\mathrm{ulp}(x)$. We may also, for some calculations, need to know if the last bit of the significand of a number is a zero [4]. These various functions can always be computed at a low level, using masks and integer arithmetic: this results in software that is not portable, and sometimes quite slow, since the corresponding calculations are not performed in the floating-point pipeline. With conventional arithmetic, designing portable software for these functions is feasible [5] but might be costly. We aim at showing that the availability of a fused-mac instruction facilitates portable yet efficient implementation of such functions.

## 2 Definitions and notations

Define $\mathbb{M}_n$ as the set of exponent-unbounded, $n$-bit significand, binary floating-point (FP) numbers (with $n \geq 1$), that is: $\mathbb{M}_n = \left\{M \times 2^E, 2^{n-1} \leq M \leq 2^n - 1, M, E \in \mathbb{Z}\right\} \cup \{0\}$. It is an "ideal" system, with no overflows or underflows. We will show results in $\mathbb{M}_n$. These results will remain true in actual systems that implement the IEEE-754 standard [6, 1], provided that no overflows or underflows do occur. The **mantissa** or **significand** of a nonzero element $M \times 2^E$ of $\mathbb{M}_n$ is the number $m(x) = M/2^{n-1}$, its **integral significand**, noted $M_x$ is $M$ and its corresponding **exponent**, noted $e_x$ is $E$. We assume that the reader is familiar with the basic notions of floating-point arithmetic: rounding modes, ulps, ... See [10] for definitions. In the following $\circ(t)$ means $t$ rounded to the nearest even.

## 3 Previous results and preliminary properties

We will use the 2sum and Fast2Sum algorithm, presented below. They do not require the availability of a fused-mac, and make it possible to compute the error of a



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floating-point addition exactly, represented by an FP number. The first one [14, 18] only assumes $a$ and $b$ are normalized FP numbers (i.e., elements of $\mathbb{M}_n$).

**Property 1 (2Sum Algorithm)** *Let $a$, $b \in \mathbb{M}_n$. Define $x$ and $y$ as*

$$\begin{array}{rcl} x & = & \circ(a+b) \\ b' & = & \circ(x-a) \\ a' & = & \circ(x-b') \\ \epsilon_b & = & \circ(b-b') \\ \epsilon_a & = & \circ(a-a') \\ y & = & \circ(\epsilon_a+\epsilon_b) \end{array}$$

*We have:*

- $x + y = a + b$ *exactly;*
- $|y| \leq \frac{1}{2}\mathrm{ulp}(x)$. □

If we know in advance that $|a| \geq |b|$ (as a matter of fact, it suffices to have $e_a \geq e_b$), a much faster algorithm can be used [9, 14]:

**Property 2 (Fast2Sum Algorithm)** *Let $a$, $b \in \mathbb{M}_n$, with $|a| \geq |b|$. Define $x$ and $y$ as*

$$\begin{array}{rcl} x & = & \circ(a+b) \\ b' & = & \circ(x-a) \\ y & = & \circ(b-b') \end{array}$$

*We have:*

- $x + y = a + b$ *exactly;*
- $|y| \leq \frac{1}{2}\mathrm{ulp}(x)$. □

Although we have presented these properties assuming a radix-2 number system, it is worth being noticed that the 2Sum algorithm (property 1) works in any radix $\geq 2$, and that the Fast2Sum algorithm (property 2) works in radices 2 and 3. And yet, rounding to nearest is mandatory: with "directed" roundings it is possible [14] to exhibit cases where the difference between the computed value of $a + b$ and the exact value cannot be exactly expressed as an FP number.

The 2Sum algorithm satisfies the following property, that will be needed in Section 5.

**Property 3** *If $(x, y) = 2\mathrm{Sum}(a, b)$ then $|y| \leq |b|$.* □

**Proof.** $x$ is the FP number that is closest to $(a + b)$. This implies that $x$ is closer to $(a + b)$ than $a$. Hence, $|(a + b) - x| = |y|$ is smaller than $|(a + b) - a| = |b|$. □

A well known and useful property of the fused-mac instruction, noticed by Karp and Markstein [13], is that it allows to very quickly compute the product of two FP numbers $x$ and $y$ exactly, expressed as the sum of two FP numbers $u$ and $v$. More precisely,

**Property 4 (Fast2Mult Algorithm)** *Let $a$, $b \in \mathbb{M}_n$. Define $x$ and $y$ as*

$$\begin{array}{rcl} x & = & \circ(ab) \\ y & = & \circ(ab-x) \end{array}$$

*we have:*

- $x + y = ab$ *exactly;*
- $|y| \leq \frac{1}{2}\mathrm{ulp}(x)$. □

Without a fused-mac, computing $x$ and $y$ is possible, but requires much more computation [9] (the significands of $x$ and $y$ are splitted, then partial products are computed and summed up).

# 4 Basic functions computable with a fused-mac

## 4.1 Checking if the last bit of the significand of some number is a zero

Brisebarre, Muller and Raina [4] have suggested an algorithm for division by a constant that works when the last bit of the divisor significand is a zero. Checking that condition is easily done with a fused-mac.

**Property 5 (Algorithm IsEven)** *The following algorithm on $x$ checks if the last significand bit of $x$ is a zero.*

$$\begin{array}{rcl} \alpha & = & \circ(3x) \\ \beta & = & \circ(\alpha-2x) \\ \mathrm{IsEven} & = & (\beta = x) \end{array}$$

□

One may notice that the same algorithm also works with the usual (addition and multiplication) floating-point instructions. The availability of a fused-mac, here, only saves one operation.



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### 4.2 Checking if a number is a power of 2.

The following algorithm requires storage of the constant

$$C = 2^n - 1.$$

Of course, $C \in \mathbb{M}_n$: it is exactly representable as a floating-point number.

**Property 6 (Algorithm IsAPowerOf2)** *The following algorithm on $x$ returns "true" if $x$ is a power of 2.*

$$\begin{array}{lcl} y_h & = & \circ(xC) \\ y_\ell & = & \circ(xC - y_h) \\ \mathrm{IsAPowerOf2} & = & (y_\ell = 0). \end{array}$$

□

**Proof** if $x$ is not a power of 2 then $M_x$ has at least a prime factor different from 2, thus $M_x C$ is of the form $P2^\alpha$, where $P$ is odd and larger than $2^n$. Hence $P$ cannot be exactly representable with $n$ bits, hence $y_h \neq xC$, hence $y_\ell \neq 0$. □

**Important remark** The above given algorithm works in the "ideal" set $\mathbb{M}_n$, which means that with "real world" floating-point arithmetic it will work provided that no overflow or underflow occur. To minimize the risk of overflow/underflow, one should choose

$$C = (2^n - 1)/(2^n),$$

instead of the previously given constant. The proof will be the same, overflow will never occur, and underflow will occur only where $x$ is a subnormal FP number.

### 4.3 Floating-point successors

There are several notions of "floating-point successor" that can be defined. The IEEE-754 standard for FP arithmetic[1] [1] *recommends* (but does not *require*) the availability of function `Nextafter`. $\mathrm{Nextafter}(x, y)$ returns the next representable neighbor of $x$ in the direction toward $y$. If $x = y$, then the result is $x$ without any exception being signaled. If either $x$ or $y$ is a NaN, then the result is a NaN. Overflow is signaled when $x$ is finite but $\mathrm{Nextafter}(x, y)$ is infinite; underflow is signaled when the result is subnormal or zero. Cody and Coonen [5] provide a portable C version of that function.

Let us show how such a function can be implemented using fused-mac instructions. First, define the following four functions.

[1] See `http://754r.ucbtest.org/standards/754.txt`

**Definition 1** *The* successor *of an FP number $x$, denoted $x^+$ is the smallest FP number larger than $x$. The* predecessor *$x^-$ of $x$ is the largest FP number less than $x$. The* symmetrical successor *of $x$, denoted $\mathrm{succ}(x)$ is $x^-$ if $x < 0$, and $x^+$ if $x > 0$. The* symmetrical predecessor *$\mathrm{pred}(x)$ of $x$ is $x^+$ if $x < 0$ and $x^-$ if $x > 0$.*

The following algorithm will use the constant

$$s = 2^{-n} + 2^{-2n+1}.$$

Notice that $s \in \mathbb{M}_n$. Even on "real life" floating-point systems, $s$ will be representable: on all floating-point systems of current use, the number of significand bits is less than the absolute value of the smallest exponent. This is required by the IEEE-854 Standard for Floating-Point arithmetic [12], that says that $(E_{max} - E_{min})/n$ shall exceed 5 and should exceed 10, and that $b^{E_{max}+E_{min}+1}$ should be the smallest integral power of $b$, that is greater than or equal to 4, where $b$ is the radix.

**Property 7** *Computation of $\mathrm{succ}(x)$ If $n \geq 2$ and $x \neq 0$, then*

$$\mathrm{succ}(x) = \circ(x + sx)$$

□

**Proof** Assume $x > 0$ and $2^e \leq x < 2^{e+1}$ (i.e., the exponent of $x$ is $e$). Since, in that case, $\mathrm{succ}(x) = x + 2^{e-n+1}$ and $\mathrm{ulp}(x) = 2^{e-n+1}$, to show that $\circ(x + sx)$ is equal to $\mathrm{succ}(x)$ it suffices to show that

$$x + 2^{e-n} < x + sx < x + 3 \times 2^{e-n}$$

(i.e., that $x + sx$ is within $1/2\mathrm{ulp}$ from $\mathrm{succ}(x)$).

Thus, it suffices to show that

$$2^{e-n} < sx < 3 \times 2^{e-n}. \tag{1}$$

Since $x \geq 2^e$, $sx > 2^{e-n}$. Since $x < 2^{e+1}$, $sx < 2(1 + 2^{-n+1})2^{e-n}$, which is less than $3 \times 2^{e-n}$ as soon as $n \geq 2$. □

Property 7 shows that $\mathrm{succ}(x)$ can be computed with one fused-mac only.

Function $\mathrm{pred}(x)$ is also computable with one fused-mac only. The proof is very similar to that of Property 7.

**Property 8** *Computation of $\mathrm{pred}(x)$ If $n \geq 2$ and $x \neq 0$, then*

$$\mathrm{pred}(x) = \circ(x - sx)$$

□



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Now, from functions succ and pred, one can very easily compute functions `Nextafter`, $x^+$ and $x^-$:

**Property 9** *if $x \neq 0$ then*

$$\begin{array}{lcl} x^+ & = & \circ(x + s|x|) \\ x^- & = & \circ(x - s|x|) \\ \mathrm{Nextafter}(x,y) & = & \left\{ \begin{array}{lll} x^+ & \text{if} & y > x \\ x & \text{if} & y = x \\ x^- & \text{if} & y < x \end{array} \right. \end{array}$$

□

**Important remark:** although we have proven these algorithms assuming an ideal FP arithmetic with unbounded exponents, they work well with "real life" arithmetic. From the definition of $\mathrm{succ}(x)$, underflow is impossible. Also, if $|x|$ is equal to the largest representable FP number, then on a machine compliant with the IEEE 754 standard, $\pm\infty$ (depending on the sign of $x$) will be returned[2], which is the right answer. If $x$ is a NaN, then the fused-mac operation will return a NaN. Hence, our algorithm for $\mathrm{succ}(x)$ is always correct, unless $x$ is a subnormal number. Function $\mathrm{pred}(x)$ cannot generate an overflow, correctly propagates NaNs, and correctly signal underflows, however, it does not work correctly if $x$ is a subnormal number: that (rare) case should be handled separately.

If we use rounding to nearest, then a fused-mac instruction looks mandatory for designing such algorithms (at least, they cannot be implemented using one addition or multiplication). For example:

**Property 10** *Apart from the "toy case" $n \leq 2$, there is no constant $C \in \mathbb{M}_n$ such that $\circ(xC)$ always equals* $\mathrm{succ}(x)$. □

**Proof:** Suppose that there exists $C \in \mathbb{M}_n$ such that $\circ(xC)$ always equals $\mathrm{succ}(x)$. Assume $1 \leq x < 2$ (the other cases are easily deduced from this one). This implies

$$x + 2^{-n} \leq Cx \leq x + 3 \times 2^{-n}.$$

Hence,

$$2^{-n} \leq (C-1)x \leq 3 \times 2^{-n}$$

for any $x \in \mathbb{M}_n$, $1 \leq x < 2$. For $x = 1$, this implies $C \geq 1 + 2^{-n}$. Since the smallest element of $\mathbb{M}_n$ larger than or equal to $1 + 2^{-n}$ is $1 + 2^{-n+1}$, we then have $C \geq 1 + 2^{-n+1}$. And yet, for $x$ equal to the largest element of $\mathbb{M}_n$ less than 2 (i.e., $2 - 2^{-n+1}$), $C \geq 1 + 2^{-n+1}$ implies $(C-1)x \geq 2^{-n+1}(2 - 2^{-n+1}) = 4 \times 2^{-n} - 2^{-2n+2}$. Therefore, in that case, $(C-1)x > 3 \times 2^{-n}$, unless $n \leq 2$. □

This may be different with other rounding modes. For instance, if rounding towards zero $\mathcal{Z}(x)$ is used, then $\mathcal{Z}(x\sigma)$ returns $\mathrm{pred}(x)$ for any nonzero $x \in \mathbb{M}_n$, with $\sigma = 1 - 2^{-n}$. One can also implement $x^+$ as $(x + \epsilon)$ rounded towards $+\infty$, where $\epsilon$ is the smallest positive nonzero subnormal number. And yet, in practice, changing the rounding mode may be quite time consuming: this is why an algorithm that works in the default mode (i.e., round-to-nearest) is preferable.

### 4.4 Function $\mathrm{ulp}(x)$

Function ulp (unit in the last place) is very frequently used for expressing the accuracy of a floating-point result. Several definitions have been given(see [11] for a discussion on that topic), they differ near the powers of 2. If we use as a definition, when $x$ is an FP number:

$$\mathrm{ulp}(x) = |x|^+ - |x|$$

then (if $x \in \mathbb{M}_n$ is nonzero) one can compute function ulp through the following sequence

$$\begin{array}{lcl} y & = & \circ(x + sx) \\ \mathrm{ulp} & = & |y - x| \end{array}$$

where $s$ is the same constant as in Section 4.3. If we define $\mathrm{ulp}(x)$ as

$$\mathrm{ulp}(x) = |x| - |x|^-$$

then function ulp is computed through

$$\begin{array}{lcl} y & = & \circ(x - sx) \\ \mathrm{ulp} & = & |y - x| \end{array}$$

The two functions differ only when $x$ is a power of 2. The first one is compatible with Goldberg's definition [10] (which is given for *real* numbers, not only for floating-point ones), the second is compatible with Kahan's one[3] and Harrison's one [11] (they differ for real numbers but coincide on FP numbers).

## 5 Computing the error term of a fused-mac

We require here that $n \geq 3$. The correcting term cannot be a single FP number, even in rounding to the nearest. We will therefore compute two FP numbers such that their sum is the exact correcting term of the fused-mac.

[2] This is due to the definition of rounding to the nearest: the standard specifies that *An infinitely precise result with magnitude at least $2^{Emax}(2 - 2^{-n})$ shall round to $\infty$ with no change in sign.*

[3] Kahan's definition is: $\mathrm{ulp}(x)$ is the gap between the two finite floating-point numbers nearest $x$ , even if $x$ is one of them (But ulp(NaN) is NaN .)



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### 5.1 The algorithm ErrFmac

**Property 11 (Algorithm ErrFmac)** *Let $a$, $x$, $y \in \mathbb{M}_n$. Define $r_1$, $r_2$ and $r_3$ as*

$$\begin{array}{lcl} r_1 & = & \circ(ax+y) \\ (u_1,u_2) & = & \mathrm{Fast2Mult}(a,x) \\ (\alpha_1,\alpha_2) & = & 2\mathrm{Sum}(y,u_2) \\ (\beta_1,\beta_2) & = & 2\mathrm{Sum}(u_1,\alpha_1) \\ \gamma & = & \circ(\circ(\beta_1-r_1)+\beta_2) \\ (r_2,r_3) & = & \mathrm{Fast2Sum}(\gamma,\alpha_2) \end{array}$$

*we have:*

- $ax+y = r_1+r_2+r_3$ *exactly;*
- $|r_2+r_3| \le \frac{1}{2}\mathrm{ulp}(r_1)$*;*
- $|r_3| \le \frac{1}{2}\mathrm{ulp}(r_2)$. □

Figure 1 gives the idea behind the algorithm: we want to exactly add the 3 FP numbers $y$, $u_1$ and $u_2$. This is usually difficult, but as we know the correct answer ($r_1$) thanks to the fused-mac computation, we just have to get the two error terms. We first compute the "small" error, namely $\alpha_2$. Then the other terms $u_1$ and $\alpha_1$ are bigger than this value and can be combined with $r_1$ into a single value $\gamma$.

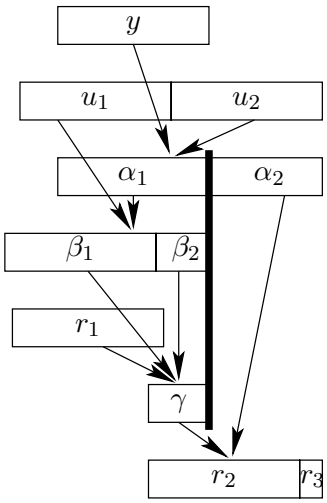


**Figure 1. Intermediate values of the ErrFmac algorithm.**

### 5.2 Proof of the correctness of the ErrFmac algorithm

If $\gamma = \circ(\circ(\beta_1 - r_1) + \beta_2)$ is equal to $(\beta_1 - r_1) + \beta_2$, then $r_1+r_2+r_3 = r_1+\gamma+\alpha_2 = r_1+\beta_1-r_1+\beta_2+\alpha_2 = u_1+\alpha_1+\alpha_2 = u_1+u_2+y = y+ax$. If this equality holds, we easily also have that $|r_2+r_3| \le \frac{1}{2}\mathrm{ulp}(r_1)$ and $|r_3| \le \frac{1}{2}\mathrm{ulp}(r_2)$, from previous properties.

There is left to prove that $\beta_1 - r_1$ and $(\beta_1 - r_1) + \beta_2$ are in $\mathbb{M}_n$. If they are, then they are exactly computed and the algorithm is correct. To guarantee that a value $v$ is in $\mathbb{M}_n$, we just have to find an exponent $e$ such that $v2^{-e}$ is an integer and $|v2^{-e}| < 2^n$. There may exist more than one suitable $e$, but the existence of one is enough. We split the proof into two subcases.

**If we have $\beta_2 = 0$,**

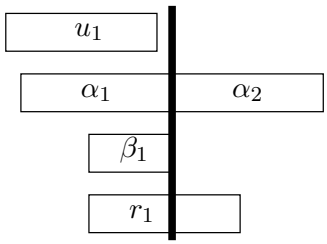


**Figure 2. Intermediate values when $\beta_2 = 0$.**

Figure 2 reminds the compared positions of the FP numbers involved. As $\beta_2 = 0$, we have left to prove that $\beta_1 - r_1$ is in $\mathbb{M}_n$. If $\beta_1 = 0$, then this is correct. Let us assume that $\beta_1 \neq 0$. We then know that $r_1 = \circ(\beta_1 + \alpha_2)$ as $\beta_2 = 0$.

But we also have that $|\alpha_2| \le \frac{1}{2}\mathrm{ulp}(\alpha_1)$ from Property 1 and that $|\alpha_2| \le |u_2| \le \frac{1}{2}\mathrm{ulp}(u_1)$ from Property 3 and by definition $\beta_1 = \circ(u_1 + \alpha_1)$. This means that $|\alpha_2| \ll |\beta_1|$. More precisely, we either have:

- the general case: $|\beta_1| \ge 4\,|\alpha_2|$;
- the special case where $\beta_1$ is a result of a near-total cancellation: $\beta_1 = 2^{\min(e_{u_1}, e_{\alpha_1})}$ and $|\beta_1| \ge 2\,|\alpha_2|$.

In the general case, we are in the conditions of Sterbenz's theorem [19]: $r_1$ and $\beta_1$ share the same sign and

$$\begin{array}{rcl} |r_1| & \le & \dfrac{|\beta_1+\alpha_2|}{1-2^{-n}} \le \dfrac{5}{4}\dfrac{1}{1-2^{-n}}|\beta_1| \le 2\,|\beta_1| \\ |r_1| & \ge & \dfrac{|\beta_1+\alpha_2|}{1+2^{-n}} \ge \dfrac{3}{4}\dfrac{1}{1+2^{-n}}|\beta_1| \ge \dfrac{1}{2}|\beta_1| \end{array}$$

In the special case, we have $4\,|\alpha_2| > |\beta_1| \ge 2\,|\alpha_2|$. As $\beta_1$ is a power of 2, we know that $e_{\beta_1} - 1 \le e_{r_1} \le e_{\beta_1}$, so $e_{r_1}$ is a suitable exponent for $\beta_1 - r_1$ and

$$\begin{array}{rcl} |\beta_1 - r_1|2^{-e_{r_1}} & = & |\beta_1 - \circ(\beta_1+\alpha_2)|2^{-e_{r_1}} \\ & \le & \left(\dfrac{1}{2}\mathrm{ulp}(r_1) + |\alpha_2|\right)2^{-e_{r_1}} \\ & \le & \dfrac{1}{2} + |\beta_1|2^{-e_{r_1}-1} \\ & \le & \dfrac{1}{2} + (2^n-1)2^{e_{r_1}+1-e_{r_1}-1} < 2^n. \end{array}$$



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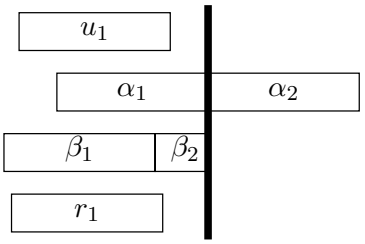


**Figure 3. Intermediate values when $\beta_2 \neq 0$.**

**If we have $\beta_2 \neq 0$,**

Figure 3 reminds the compared positions of the FP numbers involved. In the general case, we have here that $\beta_1 = r_1$, then of course $\beta_1 - r_1 = 0$ and $(\beta_1 - r_1) + \beta_2 = \beta_2$ are in $\mathbb{M}_n$. If not, as $\beta_2 \neq 0$, the only possibility for $\beta_1 = \circ(\beta_1 + \beta_2)$ not to be equal to $\circ(\beta_1 + \beta_2 + \alpha_2) = r_1$ is that either $|\beta_2| = \frac{1}{2}\mathrm{ulp}(\beta_1)$ or $\beta_2 = -\frac{1}{4}\mathrm{ulp}(\beta_1)$ if $\beta_1$ is a power of 2.

We also deduce that the exponent of $r_1$ and of $\beta_1$ differ from at most 1. Lastly, we know that $|\alpha_2| \leq |\beta_2| \leq 2^{e_{\beta_1}-1}$. The value $\min(e_{r_1}, e_{\beta_1})$ is a suitable exponent for $\beta_1 - r_1$ and

$$\begin{aligned} & |\beta_1 - r_1| 2^{-\min(e_{r_1}, e_{\beta_1})} \\ = \; & |\beta_1 - \circ(\beta_1 + \beta_2 + \alpha_2)| 2^{-\min(e_{r_1}, e_{\beta_1})} \\ \leq \; & \left(\frac{1}{2}\mathrm{ulp}(r_1) + |\beta_2| + |\alpha_2|\right) 2^{-\min(e_{r_1}, e_{\beta_1})} \\ \leq \; & \left(2^{e_{r_1}-1} + 2^{e_{\beta_1}-1} + 2^{e_{\beta_1}-1}\right) 2^{-\min(e_{r_1}, e_{\beta_1})} \leq 4 \end{aligned}$$

So $\beta_1 - r_1 \in \mathbb{M}_n$ as $n \geq 3$. There is left to prove that $(\beta_1 - r_1) + \beta_2 = u_1 + \alpha_1 - r_1$ is in $\mathbb{M}_n$. We know that $e_{\beta_1} + 1 \geq e_{r_1} \geq e_{\beta_1} - 1$ and that $\beta_2$ is either $2^{e_{\beta_1}-1}$ or $2^{e_{\beta_1}-2}$, so $e_{\beta_1} - 2$ is a suitable exponent for $(\beta_1 - r_1) + \beta_2$ and

$$\begin{aligned} & |(\beta_1 - r_1) + \beta_2| 2^{-e_{\beta_1}+2} \\ = \; & |u_1 + \alpha_1 - \circ(u_1 + \alpha_1 + \alpha_2)| 2^{-e_{\beta_1}+2} \\ \leq \; & \left(\frac{1}{2}\mathrm{ulp}(r_1) + |\alpha_2|\right) 2^{-e_{\beta_1}+2} \\ \leq \; & \left(2^{e_{r_1}-1} + 2^{e_{\beta_1}-1}\right) 2^{-e_{\beta_1}+2} \leq 6 \end{aligned}$$

So $(\beta_1 - r_1) + \beta_2 \in \mathbb{M}_n$ as $n \geq 3$. □

### 5.3 With other rounding modes

Such correcting terms for the fused-mac are only representable when the rounding is to the nearest. For example, when rounding up, if $a = x = 2^n - 1$ and $y = 2^{4n}$ then $ax + y = 2^{4n} + 2^{2n} - 2^{n+1} + 1$ and therefore $r_1$ must be strictly greater than $2^{4n}$ so $r_1 = \triangle(ax+y) = 2^{4n} + 2^{3n+1}$. So $r_2 + r_3$ should be exactly equal to $-2^{3n+1} + 2^{2n} - 2^{n+1} + 1$ that cannot be represented as the sum of two FP numbers in $\mathbb{M}_n$.

### 5.4 Cost of the algorithm

The basic cost of the algorithm is 20 fused-mac delays (FMD), but this can be tremendously reduced.

The first enhancement is when we know that $|y| \geq |ax|$ or that $|y| \geq |u_1|$. Then, the first $2\mathrm{Sum}$ is useless as $\alpha_1 = y$ and $\alpha_2 = u_2$. This is typically the case in range reduction [8, 15].

The second enhancement is to get rid of the final $\mathrm{Fast2Sum}$: this means that the result will not be compressed. It means that we only have:

- $ax + y = r_1 + r_2 + r_3$ exactly;
- $|r_2 + r_3| \leq \frac{1}{2}\mathrm{ulp}(r_1)$;
- $r_2 = 0$ or $|r_2| > |r_3|$.

The last enhancement is if the processor can use several floating-point units (FPUs) in parallel. There are indeed several computations that can be done either at the same time or at consecutive steps in a pipe-line, as there is no dependence between them. For example, the computations of $a'$ and $\epsilon_b$ in the $2\mathrm{Sum}$ algorithm (Property 1) can be performed in parallel.

If 3 FPUs are available, the algorithm only costs 12 FMDs. The tasks given to each processor are given in Figure 4. More FPUs are useless to speed up the algorithm.

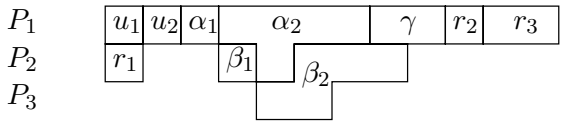


**Figure 4. Task repartition when 3 FPUs are available.**

If only 2 FPUs are available, the algorithm costs 14 FMDs. The tasks given to each processor are shown in Figure 5.

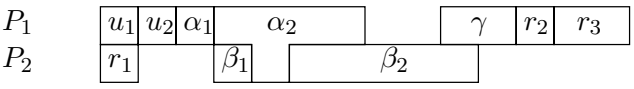


**Figure 5. Task repartition when 2 FPUs are available.**

The following table gives the cost of the $\mathrm{ErrFmac}$ algorithm depending on the conditions (number of FPUs, final



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compression and knowledge that the inequality $|y| \geq |ax|$ holds):

| Cost (in FMDs) | 1 FPU | 2 FPUs | 3 FPUs |
|---|---|---|---|
| Given algorithm | 20 | 14 | 12 |
| Without the final compression | 17 | 11 | 9 |
| When $\|y\| \geq \|ax\|$ | 14 | 10 | 10 |
| When $\|y\| \geq \|ax\|$ and without the final compression | 11 | 7 | 7 |

## 6 Conclusion

We have shown that the fused-mac instruction makes it possible to implement efficiently and in a portable way many functions that are useful for expert floating-point programming. We also have shown that, assuming rounding to nearest, the error of a fused-mac operation in a given format is exactly representable as a sum of two floating-point numbers of the same format. We have given a fast and portable algorithm that returns that error. We can take advantage of this algorithm for implementing a very accurate range reduction.

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