Least Squares Polynomial Approximations

We are looking for a degree-n polynomial

$$p^*(x) = p_n^* x^n + p_{n-1}^* x^{n-1} + \ldots + p_1^* x + p_0^*$$

that satisfies:

$$\int_a^b w(x)\left(f(x) - p^*(x)\right)^2 dx = \min_{p \in \mathcal{P}_n} \int_a^b w(x)\left(f(x) - p(x)\right)^2 dx. \quad (1)$$

Define $\langle f, g\rangle$ as

$$\langle f, g\rangle = \int_a^b w(x) f(x) g(x) dx.$$

The approximation p^* can be computed as follows

- Build a sequence (T_m), $(m \leq n)$ of *orthogonal* polynomials such that (T_m) is of degree m, and such that $\langle T_i, T_j\rangle = 0$ for $i \neq j$;
- compute the coefficients:

$$a_i = \frac{\langle f, T_i\rangle}{\langle T_i, T_i\rangle}; \qquad (2)$$

- compute

$$p^* = \sum_{i=0}^{n} a_i T_i.$$

Legendre polynomials

- $w(x) = 1; [a, b] = [-1, 1]$;
- definition:

$$\begin{cases} T_0(x) & = & 1 \\ T_1(x) & = & x \\ T_n(x) & = & \dfrac{2n-1}{n} x T_{n-1}(x) - \dfrac{n-1}{n} T_{n-2}(x); \end{cases}$$

- values of the scalar products:

$$\langle T_i, T_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ \dfrac{2}{2i+1} & \text{otherwise.} \end{cases}$$

Chebyshev polynomials

- $w(x) = 1/\sqrt{1-x^2}$, $[a,b] = [-1,1]$;
- definition:

$$\begin{cases} T_0(x) & = & 1 \\ T_1(x) & = & x \\ T_n(x) & = & 2xT_{n-1}(x) - T_{n-2}(x) = \cos\left(n \cos^{-1} x\right); \end{cases}$$

- values of the scalar products:

$$\langle T_i, T_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ \pi & \text{if } i = j = 0 \\ \pi/2 & \text{otherwise.} \end{cases}$$

Jacobi polynomials

- $w(x) = (1-x)^\alpha (1+x)^\beta$ $(\alpha, \beta > 1)$, $[a,b] = [-1,1]$;
- definition:

$$T_n(x) = \frac{1}{2^n} \sum_{m=0}^{n} \binom{n+\alpha}{m} \binom{n+\beta}{n-m} (x-1)^{n-m} (x+1)^m.$$

Least maximum (minimax) approximations

Theorem 1 (Weierstrass) *Let f be a continuous function. For any $\epsilon > 0$ there exists a polynomial p such that $||p - f||_\infty \leq \epsilon$.*

Theorem 2 (Chebyshev) *p^* is the minimax degree-n approximation to f on $[a, b]$ if and only if there exist at least $n + 2$ values*

$$a \leq x_0 < x_1 < x_2 < \ldots < x_{n+1} \leq b$$

such that:

$$p^*(x_i) - f(x_i) = (-1)^i \left[p^*(x_0) - f(x_0)\right] = \pm||f - p^*||_\infty.$$

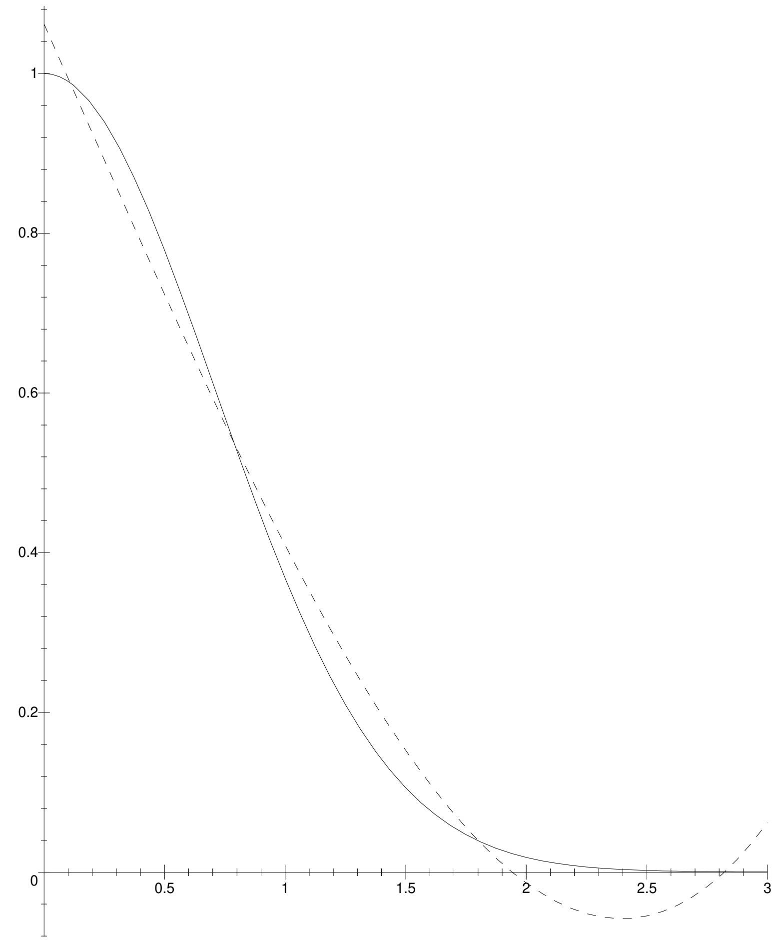

FIG. 1 - The $\exp(-x^2)$ function and its degree-3 minimax approximation on the interval $[0, 3]$ (dashed line).

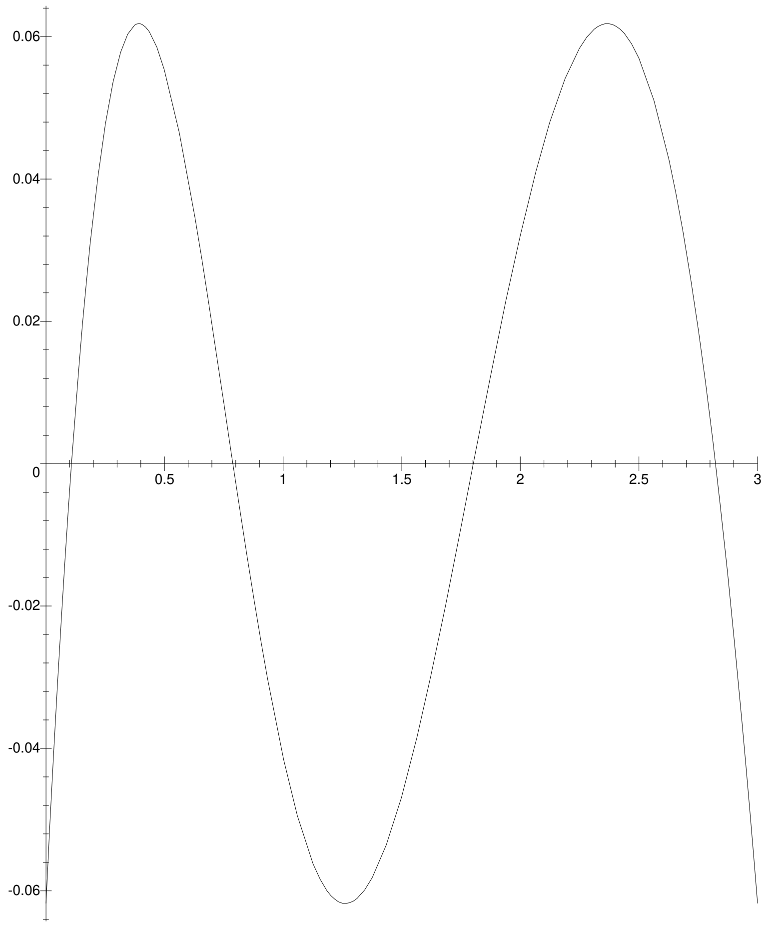

FIG. 2 - The difference between the $\exp(-x^2)$ function and its degree-3 minimax approximation on the interval $[0,3]$.

Example: approximations to $\exp(x)$ on [-1,1]

Least squares approximation using Legendre polynomials

$$\frac{15}{4}\left(e-\frac{7}{e}\right)x^2+\frac{3}{e}x+\frac{33}{4e}-\frac{3e}{4}\simeq 0.5367215x^2+1.103683x+0.9962940.$$

Least squares approximation using Chebyshev polynomials

$$0.5429906776x^2+1.130318208x+0.9945705392.$$

Minimax approximation

$$0.98903973+1.13018381x+0.55404091x^2$$

	Taylor	Legendre	Chebyshev	Minimax
Max. Error	0.218	0.081	0.050	0.045

TAB. 1 - Maximum absolute errors for various degree-2 polynomial approximations to e^x on $[-1, 1]$.

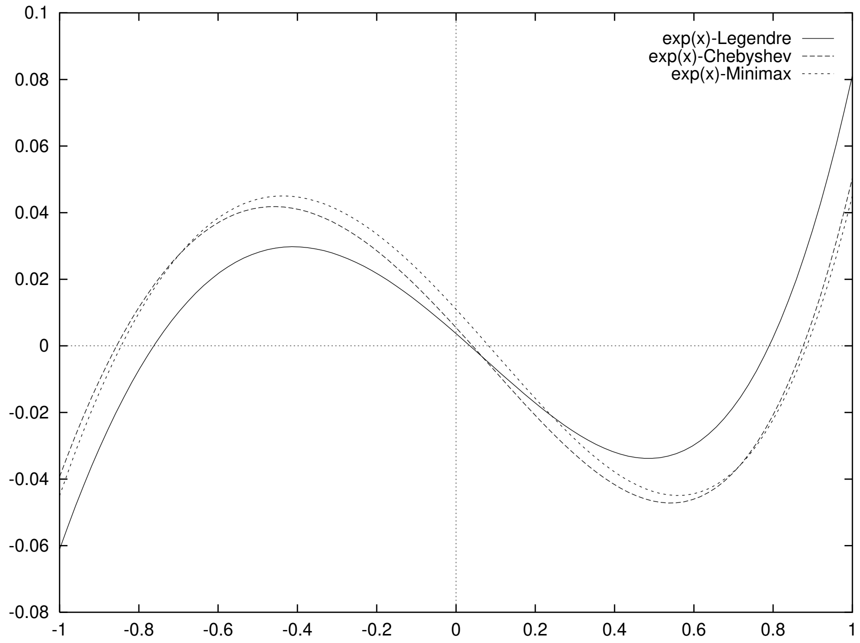

FIG. 3 - Errors of the various approximations to e^x. Legendre approximation is better on average, and Chebyshev approximation is close to the minimax approximation.

Example: the Cyrix FastMath Processor

– to compute $2^x - 1$ on $[-1, 1]$, a rational approximation of $2^{x/2} - 1$ is first computed, and then the identity

$$2^x - 1 = \left(2^{x/2} - 1\right) \left[\left(2^{x/2} - 1\right) + 2\right]$$

is used;

– to compute $\sin(x)$ on $[-\pi/4, +\pi/4]$, an odd polynomial approximation of the form $xP(x^2)$ is used. The cosine is obtained as

$$\cos(x) = \pm\sqrt{1 - \sin^2(x)}.$$

On the Cyrix FastMath processor, the square root is almost as fast as division, so there is little penalty from using it;

– to compute $\log_2(x+1)$ on $[1/\sqrt{2}-1, \sqrt{2}-1]$, the number

$$g = \frac{x}{x+2}$$

is computed, so that

$$\log_2(1+x) = \log_2\left(\frac{1+g}{1-g}\right)$$

is an *odd* function of g. Then an odd rational approximation of the form $g \times Q(g^2)$ is used. Using an odd approximation reduces the number of multiplications required to evaluate it;

– to compute $\arctan(x)$ on $[-\pi/32, +\pi/32]$, a five-segment odd rational approximation of the form $xP(x^2)/Q(x^2)$ is used.

All approximations have the general form $xR(x)$, where $R(x)$ is a rational function or a polynomial. In each case, the graph of R is relatively flat, so R can be accurately evaluated in fixed-point.