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Covering dicycles of a given parity in a directed graph

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Introduction

This report describes the work done during the internship with Professor Ken-ichi Kawarabayashi from May 26 to August 10, 2016. It took place at the NII laboratory in Tokyo, Japan.

During this internship, we first studied the problem of packing and covering directed cycles in a directed graph. This two problems are quite important in graph theory. They belong to a category of problems that try to measure how far a given graph is from a simple graph. For instance, VertexCover measure how far a graph is from a stable graph. Covering the cycles of G consists in finding a set of vertices the removal of which makes G acyclic, and this problem measure the "distance" between our graph and an acyclic one. In the case of undirected graphs, the smallest set that cover all the odd cycles is also the smallest set which removal makes the graph bipartite. The covering vertices given by all this parameters can therefore be considered as noisy vertices, and their removal enlighten simple structures that are the core of the graph. That is why the odd cycle covering problem seemed also interesting in the case of directed graphs, as another parameter that gives a simple structure. Indeed, digraphs with no odd dicycle are the one in which all the connected components are bipartite.

The dual problem can sometimes be helpful. The dual of the covering problem in a graph is the packing problem. Packing a graph G with directed cycles consists in finding a set of disjoint directed cycles in G. If cp is the maximal number of cycles packing a graph, and cc the minimal number of vertices to cover all the cycles in a graph, Reed and Shepherd showed in [5] that when the directed graph is planar we have $cc = O(cp \log(cp) \log\log(cp))$. The main point in the proof of that result is that the fractional optimal of each problem is close to the integer optimal. The result we focused on is the one given by Seymour [7]: if cc^* is the fractional optimal of the cycle covering problem, we have: $cc = O(cc^* \log(cc^*) \log(\log(cc^*)))$. In this report, we show how we tried to extend these results to cycles of a given parity.

Finding directed cycles of a given parity in a directed graph can be really different from finding any directed cycle. Thought detecting an odd directed cycle is quite easy, mostly because an odd length circuit contains at least an odd length directed cycle, stating if a directed graph contains an even cycle is a much more difficult problem. In [8], Thomassen proved that digraph with large outdegrees can be even dicycles free. Later he provided, in [9], a general polynomial algorithm to find an even cycle in the case of a planar digraph with complexity $O(n^6)$ where n is the number of vertices.

The parity property modifies the study of cycle's problems. Therefore it seems interesting to try to adapt the covering and packing results for odd and even cycles. As shown in Section 1, we cannot find a polynomial function that links the optimal integer covering and the optimal integer packing. To obtain an approximation algorithm of the even and odd dicycle covering problems, we focuse on adapting the result given in [7]. This leads to Theorem 2 for the odd dicycle covering and to the approximation Theorem 11. First we define formally the covering and packing problems (1). Then we give the proof for the odd dicycle covering problem (2): iff occ and occ^* are respectively the integer and the fraction optimals for the odd dicycle covering problem then $occ \leqslant \mu(3(occ^*+1))$ where $\mu(x) = 4x\log(4x)\log\log(4x)$. This result leads to an $O(\mu(3(occ^*+1)))$-approximation for the odd cycle covering vertex set problem (3). We end with some attempts to cover even cycles in a digraph (4).

1 The problems of packing and covering dicycles in a digraph

In this part we introduce the problem of packing and covering dicycles of a given parity in a digraph.

1.1 Definitions and notations

All the graphs we consider are finite, directed and simple (we can have one and only one edge from a vertex to another one). If G is a digraph, we denote by $V(G)$ its set of vertices and $E(G)$ its set of edges. We call directed path P in G a finite sequence of vertices $(v_1, \dots, v_k)$ such that for all $1 \leqslant i < k$, (v_i, v_{i+1}) is in $E(G)$, we denote $|P|$ its length. If F is a subgraph of G we note $F \sqsubset G$. If G and G' are two isomorphic graphs we write $G \simeq G'$.

If $X \subset V(G)$ then $G - X$ is a digraph with vertices $V(G) \backslash X$ and all the edges of G that do not end nor begin in X. $G[X]$ is the subgraph of G built on the vertices X and with all the edges of G that start and end in X.

The Erdös-Pósa property. *Let $\mathcal{F}$ be a family of graphs.*

$\mathcal{F}$ has the Erdös-Pósa property if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for every graph G and every integer α, either there are α disjoint subgraphs of G isomorphic to elements of $\mathcal{F}$, or there is a vertex set X of size at most $f(\alpha)$ such that $G - X$ has no subgraph that is isomorphic to an element of $\mathcal{F}$.

Let's define the covering and the packing problem for a family of graphs $\mathcal{F}$.

The packing problem by a family of graphs $\mathcal{F}$ consists in finding the maximal number of disjoint subgraphs of G that are isomorphic to elements of $\mathcal{F}$. The fractional relaxation of this problem is given by the following maximization linear problem over w defined on all subgraphs of G isomorphic to an element of $\mathcal{F}$:

$$\max_{w} \sum_{\substack{F \sqsubset G \\ F \simeq F', F' \in \mathcal{F}}} w(F)$$

$$\text{subject to } \sum_{F \ni v} w(F) \leqslant 1, v \in V(G)$$

The covering problem by a family of graphs $\mathcal{F}$ consists in finding the smallest subset X of $V(G)$ such that no subgraph of $G - X$ is isomorphic to an element of $\mathcal{F}$. The fractional relaxation of this problem is given by this minimization linear problem over w defined on the vertices of G:

$$\min_w \sum_{v \in V} w(v)$$

$$\text{subject to } \sum_{v \in F} w(v) \geqslant 1, F \sqsubset G, F \simeq F' \in \mathcal{F}$$

These two problems are dual to each other and therefore have the same optimal solution.

1.2 Known results

A special case of the Erdös-Pósa property for directed graphs is the Gallai-Younger property ($\mathcal{F}$ being here the set of all cycles).

The Gallai-Younger property. *Let k be an integer.*

There is a function $g : \mathbb{N} \to \mathbb{N}$ such that, if G is a digraph, one of the two following propositions is always true:

- *G has k vertex-disjoint dicycles*
- *there is a set $X \subset V(G)$ of at most $g(k)$ vertices such that $G - X$ has no dicycle.*

It was proved in [4]. We focus on the proof of the planar digraphs case given in [5].

The proof uses the fractional relaxation defined in Section 1.1 (where $\mathcal{F}$ the set of all cycle). The main point is that the optimal integer solution of each problem is polynomially close to the optimal solution. We call cycle (resp. odd cycle, even cycle) covering problem the covering problem by the family of all cycles (resp. odd cycles, even cycles). We call cycles (resp. odd cycles, even cycles) packing problem the packing problem by the family of all cycles (resp. odd cycles, even cycles). The following theorem gives the precise result for the cycles covering problem:

Theorem 1 (Seymour [7])**.** *Let G be a digraph, cc^* (resp. cc) be the fractional (resp. integer) optimal solution of the cycles covering problem for G.*

Then $cc = O(cc^* \log(cc^*) \log(\log(cc^*)))$.

This covering theorem is the main subject of our research during the internship. From now on, we use the following notations:

- Let occ (resp. occ^*) be the integer (resp. fractional) optimal for the odd cycles covering problem, and let ocp (resp. ocp^*) be the integer (resp. fractional) optimal for the odd cycles packing problem.
- Let ecc (resp. ecc^*) be the integer (resp. fractional) optimal for the even cycles covering problem, and let ecp (resp. ecp^*) be the integer (resp. fractional) optimal for the even cycles packing problem.

Remark. The duality argument give: $ocp^* = occ^*$ and $ecp^* = ecc^*$. Therefore we have:

$$ocp \leqslant ocp^* = occ^* \leqslant occ$$

$$ecp \leqslant ecp^* = ecc^* \leqslant ecc$$

1.3 New questions

With the previous remark we have $ocp \leqslant occ$ and $ecp \leqslant ecc$. That makes us wonder if we can find two polynomial functions g_o and g_e such that $occ \leqslant g_o(ocp)$ and $ecc \leqslant g_e(ecp)$. As shown in Example 1 (resp. Example 2), the set of odd (resp. even) dicycles does not satisfy the Erdös-Pósa property, therefore such functions cannot exist.

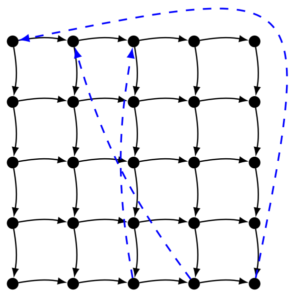

Figure 1: Grid in which covering odd cycles is more costly than packing.

Example 1. We consider the digraph given in Figure 1. Only one odd cycle can be packed. Indeed, all the odd cycles go through one of the dashed blue edges. And the cycles that go through dashed blue edges all have common vertices on the first or the last line. However here, to cover all the odd cycles, 3 vertices are needed. This example can be generalized with a grid of odd length n: the number of vertices needed to cover all the odd cycles tends to infinity with the size of the grid ($(n+1)/2$ vertices are needed), but there is always only one cycle packing the graph.

Example 2. We consider all the digraphs that have the same shape as the digraph of Figure 2, with $n = 2k+1$ concentric layers and $n+1$ lines going alternatively from and to the center. In this graph we can only pack two even dicycles: all even dicycles go through either one vertex below u or one vertex on the right of v. Indeed either they go through a, b or c and pass through u or v (otherwise they cannot either reach or cannot leave the edges a, b or c), or they go through the center vertex. Therefore only 2 cycles can be packed: one passing by a, b or c and one passing by the center. However the integer optimal covering number is $n+1$.

The covering and packing problems for a fixed parity can be quite different from the general case, since the Erdös-Pósa property is fulfilled when all cycles are taken. A natural idea is to try to find an equivalent of Theorem 1 for the odd and for even dicycle covering problems. In other word, can we find two functions f_o and f_e such that $occ \leqslant f_o(occ^*)$ and $ecc \leqslant f_e(ecc^*)$?

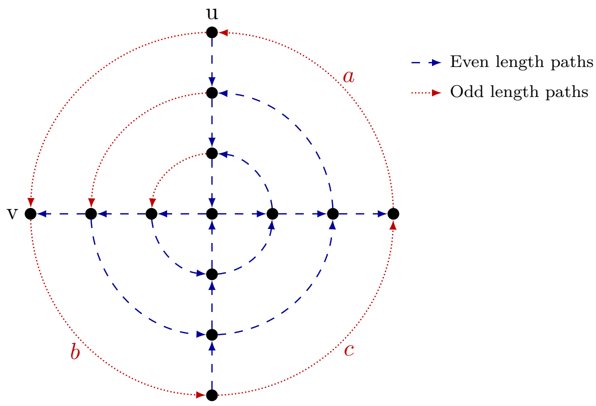

Figure 2: Graph in which covering even cycles is more costly than packing.

2 Adapting the problem to the covering of odd dicycles

In this section we show that the integer optimal covering is close to the fractional optimal in the case of odd dicycles.

The following proofs are inspired by [7]. The main result that we prove is Theorem 2:

Theorem 2. *Let G be a digraph. Then:*

$$occ \leqslant \mu(3(occ^* + 1)) \text{ , where } \mu(x) = 4x \log(4x) \log\log(4x)$$

2.1 Construction of a digraph with good properties

Let w be a valid distribution on $V(G)$ for the odd dicycle covering problem, and let k be the fractional weight associated with w. We are going to show that $occ \leqslant \mu(3(k+1))$.

The first step is to make a digraph H in which all the odd dicycles are big and each odd (resp. even) dicycle of H corresponds to an odd (resp. even) dicycle in G. We assume, by the continuity argument, that w is positive and that it is rational for every vertex v. Let n be a common denominator of all the $w(v)'s$, where $n \geqslant |V(G)|$. We construct a new digraph H that has big odd dicycles. We define $w'(v) = n \times w(v)$.

For each vertex v of G, we build the following vertices of H:

- If $w'(v)$ is odd we create $w'(v)$ vertices $(v, 1)$, $(v, 2)$, $\dots$, $(v, w'(v))$. We call $w'(v)$ the last rank of v.
- If $w'(v)$ is even we create $w'(v)+1$ vertices $(v, 1)$, $(v, 2)$, $\dots$, $(v, w'(v))$, $(v, w'(v)+1)$. We call $w'(v) + 1$ the last rank of v.

For all the vertices v, there is an edge in H from (v, i) to $(v, i + 1)$. If (u, v) is an edge of G then there is an edge between the (u, i) where i is the last rank of u and $(v, 1)$. Thus we always split the vertices of G into an odd number of vertices.

This construction is illustrated by Figure 3.

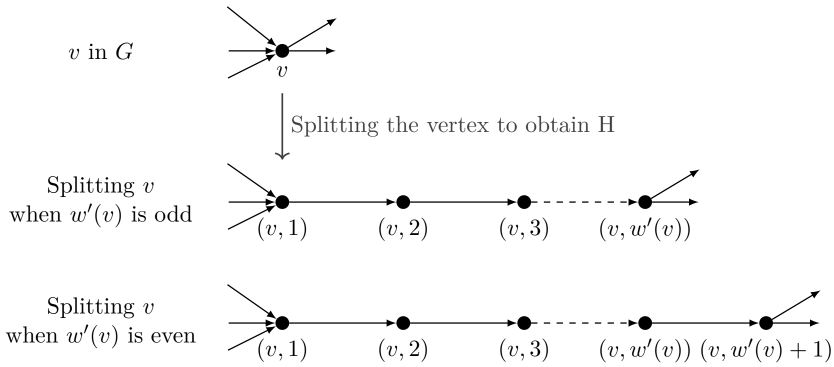

Figure 3: Transformation of a vertex in the new digraph.

Lemma 3. *To every dicycle C' in H corresponds a dicycle C in G that has the same parity.*

Proof. Let C' be a dicycle of H.
$((u,i),(v,j))$, where $u \neq v$, is an edge of H iff i is the last rank of u and $j=1$. Moreover $((v,i),(v,i+1))$ is an edge and no other edge goes out of (v,i) for $i+1$ smaller than the last rank. Therefore if a vertex (v,i), where v is in $V(G)$, is in C' then all the vertices $(v,1)$, $(v,2)$, …, (v,j) (where j last rank of v) are in the dicycle. By construction, this gives us a dicycle in G (formed by taking the first components v of the vertices (v,i) of C').

Now let's show that C and C' have the same parity. By construction the last rank of each vertex is always odd. Therefore C' is odd iff an odd number of vertices v of G intervene in the dicycle. This is equivalent to the fact that C is odd. Therefore C' and C have the same parity. □

Remark. This means that if we cover all the odd (resp. even) dicycles of H with α vertices, we can find a covering of all the odd (resp. even) dicycles of G with at most α vertices. That is why from now on we work on H instead of G.

Theorem 4. *For all odd dicycles C' of H we have $|V(C')| \geqslant \frac{|V(H)|}{k+1}$.*

Proof. Let C be the dicycle of G corresponding to the odd dicycle C' of H. We denote e (resp. f) the number of vertices v of G (resp. of the dicycle C in G) where $w'(v)$ odd. Then, because the sum of the $w(v)$'s is more than 1 on all odd dicycles we have:

$$|V(C')| = f + \sum_{v \in V(C)} nw(v) \geqslant f + n \geqslant n$$

Moreover $\sum_{v \in V(G)} w(v) = k$ so we have:

$$|V(H)| = \sum_{\substack{v \in V(G) \\ w'(v) \text{ odd}}} w'(v) + \sum_{\substack{v \in V(G) \\ w'(v) \text{ even}}} (w'(v)+1) = e + \sum_{v \in V(G)} nw(v) = e + nk$$

However $e \leqslant |V(G)| \leqslant n$ so $|V(H)| \leqslant (k+1)\,n \leqslant (k+1)|V(C')|$. Therefore

$$|V(C')| \geqslant \frac{|V(H)|}{k+1}$$

□

2.2 Construction of the layer

We want to find a partition of the digraph that leads to a good induction. To do so we want to partition the vertices into many sets. We call layers of distances of a vertex u the family of sets $X_i = \{v|\text{dist}(u,v) = i\}$ where $\text{dist}(u,v)$ is the size of the smallest path between u and v. We call number of layers of u the maximal integer i such that X_i is not empty.

In the article all the dicycles are of big length, so to make a good partition, it was enough to take a vertex randomly and make n layers. Here this method does not always work because sometimes the maximal number of layers is small. Example 3 illustrates this problem.

Example 3. In Figure 4 we have two representations of the same graph. The smallest odd cycle of this graph is of length 13.

The first representation shows the layers of distance of u. There are only 4 layers so taking randomly a vertex to build its layers of distances does not always work in our case.

However this problem can be solved by choosing the vertex for which we build the layers. The second representation of the graphs gives the layers of distances of v, the vertex that is at the opposite of the only odd cycle of the graph. This vertex has 11 layers of distance. This observation gives the intuition of the next theorem.

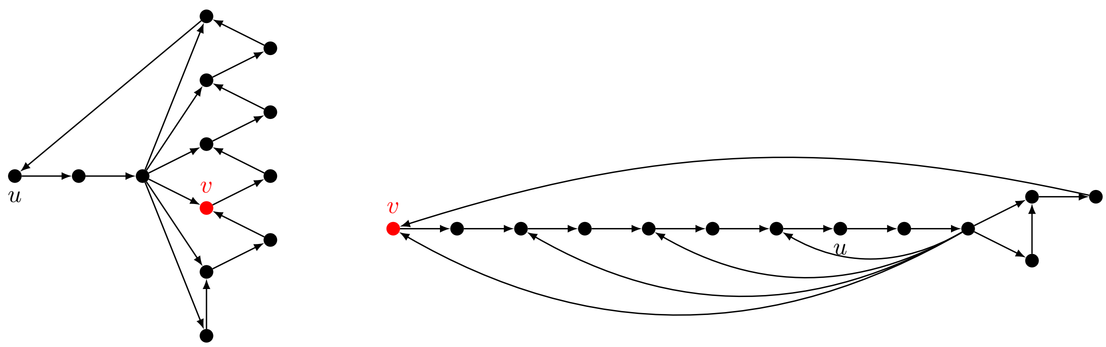

Figure 4: Finding a good vertex to make layers.

Theorem 5. *Let H be a strongly connected digraph with at least 2 vertices. We assume that all odd dicycles of H are of length at least n (n odd).*

Then there are two vertices u and v in $V(H)$ such that the shortest path between u and v is at least $(n-1)/2$.

Proof. Let's prove this theorem by contradiction: we assume that there is always a path of length inferior to $(n-1)/2$.

Let $C = (v_0, \ldots, v_l)$ be the smallest odd dicycle in H (if two or more dicycles have the same size, we just take one of them). Without loss of generality we assume that $l = n - 1$.

Let v_0 be a vertex of the dicycle and and $v_m = v_{(n-1)/2}$. Let P_0 (resp. P_1) be the path from v_0 to v_m (resp. from v_m to v_0) obtained with the dicycle. Then $|P_0| = (n-1)/2$ and $|P_1| = (n+1)/2$. We know that $|P_0|$ and $|P_1|$ do not have the same parity: we assume that P_0 is even and P_1 odd (if not exchange v_0 and v_m).

By hypothesis there is a path Q_0 (resp. Q_1) from v_0 to v_m (resp. from v_m to v_0) of length inferior to $(n-1)/2$ (because the digraph is strongly connected). Figure 5 illustrates the different paths we have.

Let's show that Q_0 cannot be of even length. If it is, then the path that is the concatenation of Q_0 and P_1 is of odd length, and it starts and ends in v_0. Therefore it contains an odd dicycle C'. However $|Q_0| + |P_1| < \frac{n-1}{2} + \frac{n+1}{2} = n$ if $|C'| < n$. This is absurd so $|Q_0|$ is odd.

With the same arguments we prove that Q_1 is even.

Therefore the concatenation of Q_0 and Q_1 is of odd length, and it starts and ends in v_0. Therefore it contains an odd dicycle. However $|Q_0| + |Q_1| < \frac{n-1}{2} + \frac{n-1}{2} < n$: contradiction.

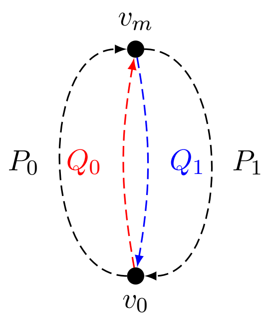

Figure 5: Using paths to make smaller odd dicycles.

This proves the theorem. □

2.3 Using the layers to divide the digraph

Lemma 6. *Let H be a strongly connected digraph with at least 2 vertices in which all odd dicycles are of length at least n (n odd). Define $m = \frac{n-1}{2}$.*

Then there is a partition a partition $X_1, \ldots, X_m$ of $V(G)$ (the sets being non empty) such that for $1 \leqslant i, j \leqslant m$, if $i + 1 < j$ there is no edge between X_i and X_j.

Proof. As shown in Theorem 5, there are two vertices u and v in $V(H)$ such that the shortest path between u and v is at least m. For $1 \leqslant i \leqslant m$ let X'_i be the set of elements that are at distance i of u, and $X'_{m+1} = V(H) \backslash (X'_1 \cup \dots \cup X'_m)$.

Let's show that all these sets are not empty. X_{m+1} contains u so it is not empty. X'_m is not empty because it contains v. This implies that all the other sets are not empty (there cannot be any element at distance m if there is no element at distance $i < m$).

Moreover, there is no edge between X'_i and X'_j where $1 \leqslant i, j \leqslant m$ and $i + 1 < j$. Otherwise an element in X'_j would also be at distance $i+1$ which is a contradiction. For the same reason, the only possible edges between X'_i and X_{m+1} if $i \leqslant m$ are from an element of X_i to u. And there is no edge between X'_i and X_{m+1} if $i \leqslant m$ is even because this would create an odd dicycle of length inferior to n.

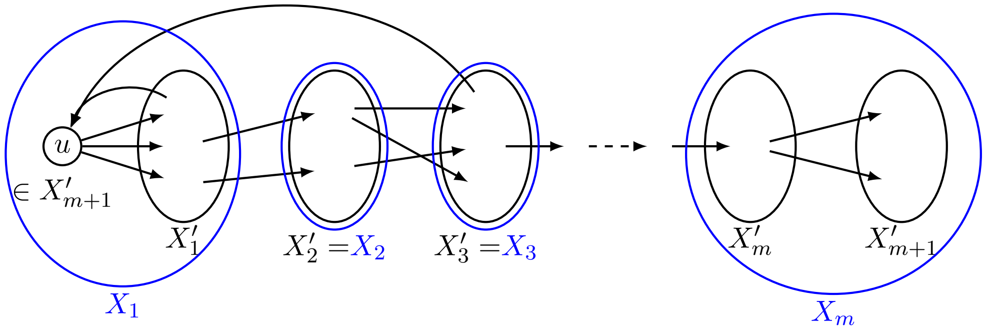

Figure 6: Building layers.

Define $X_1 = X'_1 \cup \{u\}$, for $1 < i < m$, $X_i = X'_i$ and $X_m = X'_m \cup X'_{m+1} \backslash \{u\}$. The previous remarks imply that these sets are not empty and that for $1 \leqslant i, j \leqslant m$, if $i + 1 < j$ there is no edge between X_i and X_j. $\square$

Definition 2.1. Let μ be a real-value function on $\mathbb{R}^+$. μ is suitable if:

1. $\mu \geqslant 0$
2. $\forall x \geqslant 1, \mu(x) \geqslant x$
3. $\forall x, y > 0, \mu(x + y) \geqslant \mu(x) + \mu(y)$
4. $\forall y \geqslant x \geqslant \frac{1}{4}$ if $x+y \geqslant 1$ then $\mu(x+y)-\mu(x)-\mu(y) \geqslant 4x \log\left(1 + \frac{y}{x}\right) \log\log(4(x+y))$

Theorem 7. *Let μ be a suitable function, $k > 0$ a real number. Let y be a continuous real-valued function on $[0, 1]$, we assume that $y(0) \geqslant 0$, $y(1) \leqslant 1$, there is a finite set I such that y is differentiable on $[0, 1] \backslash I$ and $\frac{dy}{dx} \geqslant \frac{1}{k}$.*

Then there is h such that: $k \frac{dy}{dx}(h) \leqslant \mu(k) - \mu(ky(h)) - \mu(k(1 - y(h)))$

Remark. This kind of functions suits well the problem because of the previous theorem. Moreover there is a quasi-linear function that is suitable: $\mu(x) = 0$ for $0 < x < 1$ and

$\mu(x) = 4x \log(4x) \log\log(4x)$ for $x \geqslant 1$. The proof of this fact and of the theorem can be found in [7].

Theorem 8. *Let μ be a suitable function, H a digraph with at least 2 vertices and $n \geqslant 3$ an odd integer. We assume that all odd dicycles in H have length at least n. Define $m = \frac{n-1}{2}$.*

Then there is a partition (A, B, C) of $V(H)$ where $A, B \neq V(H)$ and there is no edge from A to B such that:

$$|C| \leqslant \mu\left(\frac{|V(H)|}{m}\right) - \mu\left(\frac{|A|}{m}\right) - \mu\left(\frac{|B|}{m}\right)$$

Proof. (Adapted from [7])

If the digraph is not connected then there are A and B non empty with no edge from A to B. Therefore $C = \emptyset$ is a good choice.

Let's assume that the digraph is connected. By Lemma 6 there is a partition a partition $X_1, \dots, X_m$ of $V(H)$ such that for $1 \leqslant i, j \leqslant m$, if $i + 1 < j$ there is no edge between X_i and X_j.

We define y on $[0, 1]$ such that $y(0) = 0$ and for $0 < x \leqslant 1$, if $i = \lceil mx \rceil$:

$$y(x) = (|X_1| + \cdots + |X_{i-1}| + (mx - i + 1)|X_i|)\frac{1}{|V(H)|}$$

y is a continuous function. Define $I = \{0, \frac{1}{m}, \frac{2}{m}, \dots, 1\}$. Then y is differentiable on $[0, 1] - I$. We denote $l = \frac{|V(H)|}{m}$. For x in $[0, 1] - I$, if $i = \lceil mx \rceil$, then $\frac{dy}{dx}(x) = \frac{|X_i|}{l}$. Lemma 6 implies that all the X_i's are not empty. Therefore $\frac{dy}{dx}(x) \geqslant \frac{1}{l}$.

Thus by Theorem 7 y is differentiable in a point $h \in [0, 1]$ such that: $l\frac{dy}{dx}(h) \leqslant \mu(l) - \mu(ly(h)) - \mu(l(1 - y(h)))$.

If $i = \lceil mh \rceil$, $A = X_1 \cup \cdots \cup X_{i-1}$, $B = X_{i+1} \cup \cdots \cup X_m$, $C = X_i$, as in the article we have:

$$l = \frac{|V(H)|}{m} = |C|,\ \mu(ky(l)) \geqslant \mu\left(\frac{|A|}{m}\right),\ \mu(k(1 - y(l))) \geqslant \mu\left(\frac{|B|}{m}\right)$$

Therefore we obtain:

$$|C| \leqslant \mu\left(\frac{|V(H)|}{m}\right) - \mu\left(\frac{|A|}{m}\right) - \mu\left(\frac{|B|}{m}\right)$$

□

2.4 Covering the digraph

Theorem 9. *Let μ be a suitable function, and $k > 0$. Let H be a digraph such that all the odd dicycles have length at least $\frac{|V(H)|}{k+1}$. Let $m = \frac{n-1}{2}$ where n is the smallest odd number that is bigger than $\frac{|V(H)|}{k+1}$.*

Then we can find a set X covering all the odd dicycles in H and of cardinality at most $\mu\left(\frac{|V(H)|}{m}\right)$ if $m \neq 0$, and $\mu(3(k + 1))$ otherwise.

Proof. Let's prove this result by induction on $|V(H)|$.

If $|V(H)| = 1$ the empty set covers all odd dicycle.

Let's now assume that this is true for all the digraphs that have strictly less that $|V(H)|$ vertices. If there is no odd dicycle in the digraph then the empty set is a good choice.

If $n = 1$, then $m = 0$, and $k+1 \geqslant |V(H)|$. Therefore if $X = V(H)$, X covers all odd dicycles and $|V(H)| \leqslant k+1 \leqslant 3(k+1) \leqslant \mu(3(k+1))$ because μ is suitable.

Let's assume that $n \geqslant 3$. Then there is a partition (A, B, C) of $V(H)$ where $A, B \neq V(H)$ and there is no edge from A to B such that:

$$|C| \leqslant \mu\left(\frac{|V(H)|}{m}\right) - \mu\left(\frac{|A|}{m}\right) - \mu\left(\frac{|B|}{m}\right)$$

Then all the odd dicycles of A (resp. B) have a length that is bigger than $\frac{|V(H)|}{k+1}$ so we have the same integers $n > 1$ and m, so there is Y (resp. Z) of size at most $\mu\left(\frac{|A|}{m}\right)$ (resp. $\mu\left(\frac{|B|}{m}\right)$) that covers all the odd dicycles of A (resp. B).

Define $X = C \cup Y \cup Z$: X covers all the dicycles of H because the is no edge from A to B. Moreover:

$$\begin{aligned}
|X| &\leqslant |C| + |Y| + |Z| \\
&\leqslant \mu\left(\frac{|V(H)|}{m}\right) - \mu\left(\frac{|A|}{m}\right) - \mu\left(\frac{|B|}{m}\right) + \mu\left(\frac{|A|}{m}\right) + \mu\left(\frac{|B|}{m}\right) \\
&\leqslant \mu\left(\frac{|V(H)|}{m}\right)
\end{aligned}$$

Therefore the property is true for H. This proves the theorem. $\square$

Remark. In the case where $n > 1$, we have $\frac{n-1}{2} \geqslant \frac{n}{3}$ so:

$$\frac{|V(H)|}{m} \leqslant 3\frac{|V(H)|}{n} = 3(k+1)$$

Therefore in all the cases we have covered the odd dicycles of H with a set of cardinality at most $\mu(3(k+1))$. So we proved 2 when w is the optimal solution.

3 The algorithmic point of view: the minimum odd cycle covering set

As in the article [1], we will use the odd dicycle covering result found in Section 2 to approximate minimum odd cycle covering set problem in the case of odd cycle covering.

We formally define the OCC problem as:

Input: A digraph G.

Question: What is the smallest set of vertices that cover all the odd dicycles in the graph?

Theorem 10. *The OCC problem is NP-complete.*

Theorem 11. *There is a polynomial time $O(\mu(3(occ^* + 1)))$-approximation for the OCC problem.*

The following algorithm gives the approximation.

Computing the fractional optimal

The first thing we need is to find the fractional odd cycle covering optimal k. We can obtain it by solving the linear program, but we still have to be careful because there can be an exponential number of constraints. To show that the linear program is solvable in polynomial time we can use a separability argument and then apply the ellipsoid algorithm.

The system is separable if we can state that a given distribution does not satisfy the constraints in polynomial time. Let w a function on the vertices. We want to show that this function satisfies all the covering problem constraints. This is equivalent to the fact that the smallest odd cycle going through a vertex has weight at least 1. Therefore proving separability is equivalent to finding an algorithm that gives the smallest odd cycle going through a vertex.

To do so we use Dijkstra algorithm on a graph reduced from our graph. In the new graph G' we double the vertice of $V(G)$ into two set V_1 and V_2. Then the edges are:

- the (u_1, v_2) with $u_1 \in V_1$ (resp. $v_2 \in V_2$) corresponding to $v \in V(G)$ and $(u, v) \in E(G)$
- the (v_2, u_1) with $u_1 \in V_1$ (resp. $v_2 \in V_2$) corresponding to $v \in V(G)$ and $(v, u) \in E(G)$

The weight of an edge (u, v) is $w(v)$. Then we compute for u in $V(G)$ the path of smallest weight between u_1 and u_2 which gives the odd cycle of smallest weight. This proofs the separability.

Therefore we can find the fractional optimal in polynomial time.

We call w the function that satisfies this optimal. Let's describe an algorithm that gives an approximation to OCC. Here we denote $V = V(G)$

The algorithms that follow are adapted from [1].

Building H

First we build the graph H (Section 2.1) in polynomial time: to do so we should have a common denominator n that is polynomial in $|V|$ to ensure that we add a polynomial number of vertices. To do so we create a new feasible function φ for the OCC problem:

$$\varphi(v) = \begin{cases} \frac{1}{2|V|} & \text{if } w(v) < {}^{1}\!/_{2|V|} \\ \frac{\lceil 4nw(v) \rceil}{2|V|} & otherwise \end{cases} \pmod 2.$$

This new function gives a valid covering with weight at most $k = 4occ^*$. Therefore we can build a graph H that has a polynomial number of vertices in $|V|$ (H has $O(|V|^2)$ vertices and therefore can be build in $O(|V|^2)$)

Find the smallest odd cycle

To make the layers, we need to find the smallest odd dicycle. To do so we use the following algorithm:

Input : the graph H.
Output: the smallest odd dicycle of H.

pick a vertex v randomly
with a BFS algorithm find the smallest odd dicycle C going through v
C' =SmallestOddCycle($H - v$)
if $|C| \leqslant |C'|$ **then**
  return C
else
  return C'

Algorithm 1: SmallestOddCycle

This algorithm runs $|V(H)|$ times the BFS and therefore is in $O(|V(H)|(|E(H)| + |V(H)|))$ which is polynomial in $|V|$. More precisely, $|E(H)| = O(|V(H)| + |E(G)|) = O(|V|^2)$ so this algorithm is in $O(|V|^4)$.

Building the layers

To build the layers we need to try one vertex in the cycle and if it has less than $(n-1)/2$ layers, build the layers for the vertex at the other extremity of the cycle:

Input : the graph H.
Output: A partition of $V(H)$ into $m = (n-1)/2$ layers

C = SmallestOddCycle(H)
Pick v in $V(C)$.
$i = 0$, $X_0 = \{v\}$ **while** $X_i' \neq \emptyset$ **do**
  $i = i + 1$, $X_i' = \{v'$ at distance i from v in $H\}$
if $i > m$ **then**
  $X_{m+1}' = V(H) \backslash (X_1' \cup \cdots \cup X_m')$
  $X_1 = X_1' \cup \{v\}$, for $1 < i < m$, $X_i = X_i'$, $X_m = X_m' \cup X_{m+1}' \backslash \{v\}$
else
  Pick u at distance m in the cycle from v
  for $1 \leqslant i \leqslant m$ **do**
    $X_i' = \{v'$ at distance i from u in $H\}$
  $X_{m+1}' = V(H) \backslash (X_1' \cup \cdots \cup X_m')$
  $X_1 = X_1' \cup \{u\}$, for $1 < i < m$, $X_i = X_i'$, $X_m = X_m' \cup X_{m+1}' \backslash \{u\}$

Algorithm 2: BuildLayers

This algorithm is in $O(|V(H)|) = O(|V|^2)$.

General algorithm

We obtain the following general algorithm:

Input : the graph G.
Output: $X \subset V$ that covers odd cycles.

1 Build H
2 $i = 0$ $A = \emptyset$, $B = V(H)$
3 **for** $1 \leqslant j \leqslant m$ **do**
4     $i = i + 1$, $A = A \cup X_i$, $B = B - X_i$
5   **if** $|X_i| \leqslant \mu\left(\frac{|V(H)|}{m}\right) - \mu\left(\frac{|A|}{m}\right) - \mu\left(\frac{|B|}{m}\right)$ **then**
6   **goto** 7
7 Y =CoverOddCycles($H[A]$)
8 Z =CoverOddCycles($H[B]$)
9 $X' = Y \cup Z \cup X_i$
10 Obtain the cover X of G by taking the first component of the vertices covering X' of H
11 **return** X

Algorithm 3: CoverOddCycles

This covering is a $O(\mu(3(k+1)))$-approximation and therefore a $O(\mu(3(occ^* + 1)))$-approximation.

Remark. This algorithm can be implemented in the case of weighted vertices. The only thing that might be harder is that the fractional optimal may be huge compared to the number of vertices. However this problem is solved in [1].

4 Attempts on the covering of even dicycles

In this section we give some results about the even cycle problem and our attemps on the covering of even cycles in a graph. More formally it gives the different strategy used to find a polynomial function f_e such that $ecc \leqslant f_e(ecc^*)$.

4.1 Difficulties of the problem

Finding an even dicycle is much more difficult than finding an odd dicycle. As said in [6], the reason for that is that once we find a odd circuit in a digraph, we know that at least one of the cycles that decompose this circuit is odd. However an even circuit can be composed of an even number of odd dicycles.

If we look at the proof of Section 2 for the odd dicycle problem, we note that most of the proof can be adapted for the even cycle problem. Indeed, the graph made in Section 2.1 is still valid in the case of the even dicycle problem and has the same properties transposed to even dicycles. And if we manage to make layers of distances, we can divide the digraph and then make a recursion. The hard part here is to make layers of distances.

Indeed, because of the problem of the division of circuits, we cannot apply the method used in the proof of Theorem 5. If we take the smallest even dicycle, Figure 7 shows that a point in this dicycle and its opposite can both have little layers. Here the smallest even cycle is of length 12, u has only 3 layers, and v has 5 layers which is less than $12/2 = 6$. And we can generalize this graph by adding vertices in the third and fourth layer: we obtain a cycle of length $4 \cdot l$ with still 3 layers for u and 5 for v.

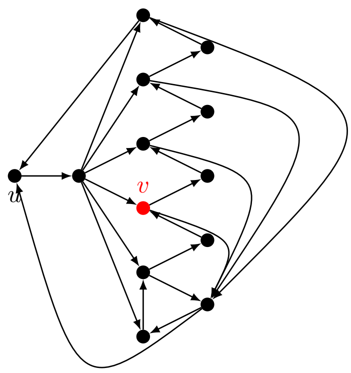

Figure 7: A graph in which most vertices do not have many layers.

However we remark that some vertices still have many layers. If we try to obtain few layers for all the vertices of this graph we get the graph in Figure 8. One observation we made is that this graph, on the contrary of all the others found up until now, is not planar. Therefore we made Conjecture 1.

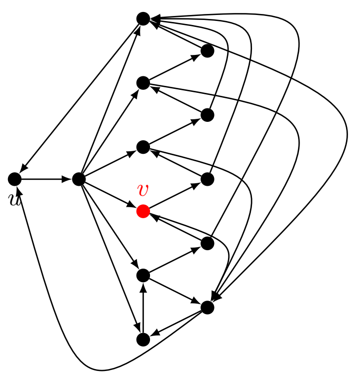

Figure 8: A graph with few layers for all vertices.

Conjecture 1. *Let H be a strongly connected digraph with at least 2 vertices. We assume that H is a planar graph and that all even dicycles of H are of length at least n (n even).*

Then there are two vertices u and v in $V(H)$ such that the shortest path between u and v is at least $n/2$.

With the remarks made before in the section, if this conjecture is proved, then we have a difference between the integer covering problem and the fractional covering problem that is of the same order as in the odd dicycle case. However we did not prove this conjecture yet. We tried a few ideas such as adapting the odd cycle paths proof, or using tree decomposition.

4.2 Some interesting approachs

In this section we discuss a few other non working ideas to solve the problem in the even cycle case.

4.2.1 Using connectivity

The first method tried was to divide the digraph into two parts A and B, where one of the two parts, say A, does not contain even dicycles. The main problem when we cut into two parts is that there can be even dicycles that are shared by A and B but not covered by the covering sets of A and B. The idea we had was to use the following theorem proved by McCuaig:

Theorem 12 (McCuaig [3]). *D_7 (see Figure 9) is the only strongly 2-connected digraph with no even cycle*

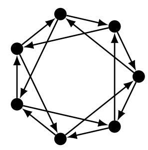

Figure 9: D_7 graph.

We can decompose A into 2-connected components, that form a directed acyclic graph. Theorem 12 implies that the 2-connected components composing the node of the DAG are either single node or D_7 digraphs because A does not contain even cycle (see Figure 10). We tried to use that shape to erase all the shared even cycles but it is hard to control because we cannot quantify the number of vertices going from B to the DAG, and the converse.

4.2.2 Planar separator

We assume that we have a graph G with a fractional even dicycle covering of weight k. As for the odd dicycle case, we can make a graph H such that all the even cycles of H are bigger that $\frac{|V(H)|}{k+1} = n$.

Following the idea of the article's proof in the general case of all dicycles, we want to divide $V(H)$ into three sets A, B and C such that:

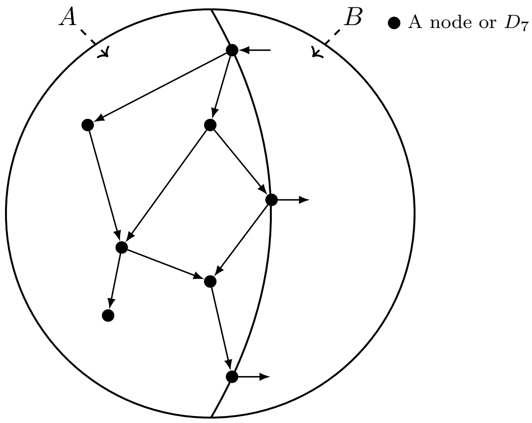

Figure 10: Graph divided with a part without even dicycle.

- C is small compared to A and B,
- A and B are well balanced,
- removing C cuts all the cycles shared by A and B.

This seemed close to the following planar separator theorem:

Theorem 13 (Lipton, Tarjan [2])**.** *Let G be a planar graph (not directed).*

Then $V(G)$ can be separated into 3 sets A, B and C such that there is no edge between A and B, A and B contains less that $\frac{2}{3}|V(G)|$ vertices and C contains $O(\sqrt{|V(G)|})$ vertices.

We wanted to apply this theorem to H but the main problem we had was that we do not know how small k is compared to $|V(H)|$. Therefore the set C can be much bigger than expected.

Conclusion

We tried to study the covering of odd and even dicycles in a digraph.

Adapting the covering result for odd cycle was not straightforward, but still possible. The problem is that not all cycles were large, and therefore all the vertices could be at a close distance to each other. But we managed to show that this cannot happen. This led us to an approximation algorithm for the odd cycle covering set problem with same ratio as for the general cycle covering vertex set problem. We can wonder if there is better approximation for this problem, since for the general case better approximations exist when the graph is unweighted.

However in the case of even dicycle, it appears that all the vertices can actually be close. The reason for that is the one that makes the even cycle problem difficult: an even circuit can be the sum of only odd dicycles. Therefore all vertices can be close as long as they just form small odd cycles. That is why we tried to solve it in the case of planar graphs. We are trying many different tools to tackle the problem (planar separator, connectivity, tree decomposition) but none of them gave us yet a good link between fractional and integer covering.

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