TD1 - Modelization and Linear Programming

Indication of hardness: from (*) to (****).

1 Modelization (again)

Exercise 1 - Computers production (*)
Pear© produces notebook computers and desktop computers. Pear© would like to know how many of each product to produce in order to maximize profit for the quarter. The major constraints are as follows:

1. Each computer (either notebook or desktop) requires a Processing Chip. Due to tightness in the market, the supplier has allocated 10,000 such chips to the company.
2. Each computer requires memory. Memory comes in 8GB chip sets. A notebook computer has 8GB memory installed (so needs 1 chip set) while a desktop computer has 16MB (so requires 2 chip sets). Pear© received a great deal on chip sets, so have a stock of 15,000 chip sets to use over the next quarter.
3. Each computer requires assembly time. Due to tight tolerances, a notebook computer takes more time to assemble: 4 minutes versus 3 minutes for a desktop. There are 38,000 minutes of assembly time available in the next quarter.

Given current market conditions, material cost, and our production system, each notebook computer produced generates $750 profit, and each desktop produces $1000 profit.

(a) Formulate the problem as a Linear Program.
(b) Solve it geometrically and with the Simplex Algorithm.

Exercise 2 - The world is (still) linear... (***)
(a) We want to maximize the following fraction $\frac{3+2x_1+3x_2+x_3}{1+3x_1+x_2+4x_3}$ subject to the constraints $5x_1 + x_2 + 6x_3 \leq 10$ and $x_1 + 2x_2 + x_3 \leq 2$ and non negative x_i for every i. Show that this problem can be modeled with the following linear program:

$$\begin{aligned} \max 3t + 2y_1 + 3y_2 + y_3 \\ \text{subject to} \\ t = 1 - 3y_1 - y_2 - 4y_3 \\ 5y_1 + y_2 + 6y_3 - 10t \leq 0 \\ y_1 + 2y_2 + y_3 \leq 0 \\ t, y_1, y_2, y_3 \geq 0 \end{aligned}$$

Hint: $t = \frac{1}{1+3x_1+x_2+4x_3}$.

(b) Explain how you can generalize it to any fraction.

2 Convex sets

Exercise 3 - Alternative definition of convext sets. (*)
Prove that: a set X is convex if and only if any convex combination of a finite number of points of X is in X.

Exercise 4 - Applications of the definition (*)
Show that:

- A hyperplane is a convex set.
- A polyhedron is a convex set.
- A cone is a convex set.

Exercise 5 - Convex hull is convex. (*)
Let X be a set of points of $\mathbb{R}^n$. Show that the set $Conv(X)$ is convex.

Exercise 6 - Convex hull of the extreme points (**)
Prove that if P is a polytope, then $P = Conv(V(P))$.

Exercise 7 - Operations on convex sets. (*)

(a) Show that the intersection of any collection (not necessarily finite) of convex sets is convex. What about the union of convex sets?

(b) Show that for any $X \subseteq \mathbb{R}^n$, the set $Conv(X)$ is the intersection of all convex sets that contain X.

Hint: If x, y are in the intersection, show that the "segment" $[x, y]$ also is. 2) Why it is included in $Conv(X)$? Since the convex set contains X, why does it contain $Conv(X)$?

Exercise 8 - Convex functions. (*)

(a) Show that the sum of convex functions is convex.

(b) Is it also true for the product of convex functions? For their multiplication by a scalar?

(c) Let $c \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Show that $f(x) = c^T x + \alpha$ is convex and concave.

3 Polytopes

Exercise 9 - Optimal solution and faces. (*)
Prove that any optimal solution of a LP on a polyhedron P has either an infinite optimal value or all every optimal solution is on a face F of P (strictly contained in P).

Exercise 10 - Alternative definition of polytope. (**)

(a) Prove that any minimal face of a polytope P is reduced to a single point.
(b) Consider the linear programming problem of minimizing $c^t x$ over a non-empty, polytope P. Prove that there exists an optimal solution which is an extreme point of P .

Exercise 11 - An example (*)
Give an example of convex set which is not the convex hull of its extreme points.

Hint: It is not a polytope (why?), thus...

Exercise 12 - Lines and Extreme Points. (**)
We say that a polyhedron P contains a line if there exists a point $x \in P$ and a nonzero vector $d \in \mathbb{R}^n$ such that $x + \lambda d \in P$ for all scalars λ.

(a) Prove the following: Let P be a non-empty polyhedron. Show that if P does not contain a line, then P contains an extreme point.
(b) Is the converse also true? Proof or counter-example.

4 Harder exercises on polytopes

Exercise 13 - Extremal points of the unit disk (***)
We want to prove formally that the set of extremal points of the unit disk

$$C = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ such that } x^2 + y^2 \leq 1 \right\}$$

is the set of points satisfying $x^2 + y^2 = 1$.

(a) Let $\binom{x_0}{y_0}$ be a point satisfying $x_0^2 + y_0^2 = 1$. Give an equation of the line tangent to the sphere at point $\binom{x_0}{y_0}$ as $\alpha x + \beta y = c$ (explicit α, β and c).
(b) Maximize $\alpha x + \beta y$ on C. What is the optimal value? Show that $\binom{x_0}{y_0}$ is the unique point reaching the optimal value.
(c) Let X be a subset of C such that $\binom{x_0}{y_0} \notin X$. What can you say about the optimal value of the objective function in $Conv(X)$? Conclude.
(d) Recall why it implies that C is not a polytope.
(e) How can you adapt this proof for higher dimensional spaces?

Hint for (b): We denote by $(u|v) = \sum_i u_i v_i$ the scalar product of two vectors u and v. Recall that (Cauchy-Schwarz theorem)

$$|(u|v)| \leq ||u|| \cdot ||v||$$

And the equality case happens if and only if $u = \lambda v$ for $\lambda \geq 0$.

Exercise 14 - Extended formulations - The cross polytope (***)
Consider the following polytope:

$$C_d = \{x \text{ such that } \|x\|_1 = 1\} = \{x \in \mathbb{R}^d \text{ such that } \pm x_1 \pm x_2 \ldots \pm x_d \leq 1\}$$

(a) Represent C_d when $d = 2$ and $d = 3$.

(b) Show that no constraint of the following polyhedron is useless. In other words, prove that none of the constraints $\pm x_1 \pm x_2 \ldots \pm x_d \leq 1$ can be deleted without modifying the polytope.
What is the number of facets of this polytope?

(c) Find an extended formulation C_d with a linear number of constraints.

Hint for (c): How did we transform absolute values into variables in TD1?

5 Simplex algorithm

Exercise 15 - Solve the following LP (**) Solve the following linear programs using the simplex algorithm:

$$\begin{aligned} \max 6x_1 + 8x_2 + 5x_3 + 9x_4 \quad & \text{subject to:} \\ 2x_1 + x_2 + x_3 + 3x_4 &\leq 5 \\ x_1 + 3x_2 + x_3 + 2x_4 &\leq 3 \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

Exercise 16 - Solve the following LP (**)

$$\begin{aligned} \max 3x_1 + 3x_2 + 4x_3 \quad & \text{subject to:} \\ x_1 + x_2 + 2x_3 &\leq 4 \\ 2x_1 + 3x_3 &\leq 5 \\ 2x_1 + x_2 + 3x_3 &\leq 7 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

Exercise 17 - Solve the following LP (**)

$$\begin{aligned} \max 2x_1 + x_2 \quad & \text{subject to:} \\ 2x_1 + x_2 &\leq 3 \\ 2x_1 + x_2 &\leq 1 \\ 2x_1 + x_2 &\leq 4 \\ 2x_1 + x_2 &\leq 5 \\ x_1, x_2 &\geq 0 \end{aligned}$$