TD3 - Modelization and Linear Programming

Indication of hardness: from (*) to (****).

1 *Theory on the Simplex algorithm*

Exercise 1 - Correctness of the Simplex algorithm. (**)
Consider the following LP in the standard form.

$$\max c^T x$$
$$\begin{pmatrix} I & A \end{pmatrix} \begin{pmatrix} x_B \\ x_N \end{pmatrix} = b$$
$$x \geq 0$$

where $b \geq 0$ and the vector x_B is a basic feasible solution.

(a) Show that if x is a solution of the system (resp. of the system without the non-negativity constraints) then x is still a solution after the new solution of the system after the pivot operation (resp. of the system without the non-negativity constraints).
i.e. x satisfies the constraint before the pivot iff x satisfies the constraint after the pivot.

(b) Prove the same for the objective function. Deduce that the value of the optimal solution is not modified by a pivoting rule.

(c) Show that after a pivot operation, the vector b is still non negative.

(d) Deduce that if the Simplex algorithm starts from a BFS, then the current solution is a BFS at any step.

Exercise 2 - Multiple optimal solutions. (**)

(a) Solve geometrically the following LP

$$\max 4x_1 + 14x_2$$
Subject to:
$$\begin{aligned} 2x_1 + 7x_2 &\leq 21 \\ 7x_1 + 2x_2 &\leq 21 \\ x_1, x_2 &\geq 0 \end{aligned}$$

What can you say about the set of optimal solutions?

(b) Solve it using the Simplex algorithm. What can you remark?

(c) Let c^* be the objective function in an optimal tableau. Prove that if a LP has several solutions then a non-basic variable MUST have coefficient 0 in the objection function c^*.

(d) Show that the above condition is sufficient if the LP is non degenerate.

(e) **(***)** Is it sufficient in general?

Exercise 3 - Non-necessarily optimal rule. (*)
Show on an example that choosing the entering variable having highest coefficient in z (in the current objective function) does not guarantee that the increase of the constant term of z is maximum amongst all possible pivots.

Exercise 4 - Infinite solutions

(a) Prove that if the ratio does not give any constraint, then the LP has an arbitrarily large optimal value.

(b)

$$\begin{aligned} \max z = 2x_1 + x_2 \\ \text{subject to} \\ x_1 - x_2 \leq 10 \\ 2x_1 - x_2 \leq 40 \\ x_1, x_2 \geq 0 \end{aligned}$$

Solve geometrically and with the Simplex algorithm this LP.

$$\begin{aligned} \max z = 3x_1 + 2x_2 \\ x_1 + x_2 \leq 10 \\ x_2 \leq 6 \\ 3x_1 + 2x_2 = 18 \\ x_1, x_2 \geq 0 \end{aligned}$$

Solve geometrically and with the Simplex algorithm this LP.

2 *Cycling and Running time*

Exercise 5 - Cycling and points of the polytope. (***)
Assume that we have a cycling $B_1, \ldots, B_\ell$ in an execution of the simplex algorithm. A variable is *busy* if it leaves the basis in the cycle (or similarly if it enters).

1. Assume that the Simplex algorithm do not stricly improve the objective value at step t. Then the two solutions correspond to the same point.
2. Deduce that if the simplex algorithm is cycling then all the busy variables equal 0.

Exercise 6 - Cycling example. (****)

$$\begin{aligned} \max z = 10x_1 - 57x_2 - 9x_3 - 24x_4 \\ \text{subject to} \\ 1/2x_1 - 11/2x_2 - 5/2x_3 + 9x_4 \leq 0 \\ 1/2x_1 - 3/2x_2 - 1/2x_3 + x_4 \leq 0 \\ x_1 + x_2 + x_3 + x_4 \leq 1 \\ x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

The entering variable will be the one with the most negative coefficient in the objective vector. The leaving variable will be the candidate for departing which has the smallest subscript. Prove that there is a cycle !

Exercise 7 - Exponential number of steps (***)
Consider the following LP:

$$\begin{aligned}
&\max x_d \\
&\text{subject to} \\
x_1 - r_1 &= \epsilon \\
x_1 + s_1 &= 1 \\
x_j - \epsilon x_{j-1} - r_j &= 0 \quad j = 2, \ldots, d \\
x_j + \epsilon x_{j-1} + s_j &= 1 \quad j = 2, \ldots, d \\
x_j, r_j, s_j &\geq 0 \quad j = 1, \ldots, d
\end{aligned}$$

(a) Prove that any basic feasible solution for the LP has the following properties:
- Each x_i is in the basis.
- Exactly one of $\{r_j, s_j\}$ is in the basis for each $j = 1, 2, \ldots, d$.

(b) So we can associate to each bfs, the subset S of indices j in $\{1, 2, \ldots, d\}$ such that r_j is in the basis. We denote by x^S the corresponding (uniquely) bfs. Prove that if $d \in S$ but $d \notin S'$, then for the corresponding basic feasible solutions' we have $x_d^S > x_d^{S'}$. Moreover, if $S = S' \cup \{d\}$, then $x_d^S = 1 - x_d^{S'}$.
Note that in particular each of these basis gives a different vertex of the polytope.

(c) Conclude that the Simplex algorithm might need an exponential number of steps.

Hint for (c): Apply induction.

3 *Determining properties of polyhedra using LPs*

Exercise 8 - Empty or bounded polyedra. (**)
Let P be a polyhedron (defined as an intersection of halfspaces you know). Explain how you can determine using Linear Programs:

(a) if P is empty.

(b) if P is bounded.

Exercise 9 - Empty polyhedra (II) (**)
Determine if the following polyhedra is empty. If not, exhibit a vertex.

$$\begin{aligned}
x_1 - 6x_2 + x_3 - x_4 &= 5 \\
-2x_2 + 2x_3 - 3x_4 &\geq 3 \\
3x_1 - 2x_3 + 4x_4 &= -1 \\
x_1, x_3, x_4 &\geq 0 \\
x_2 &\text{ free}
\end{aligned}
\qquad\qquad
\begin{aligned}
3x_1 - 2x_2 + 4x_3 &\geq 8 \\
-2x_2 + 2x_3 &\geq 3 \\
3x_1 - 2x_3 &\leq -1 \\
x_1 + x_2 - x_3 &= 1 \\
x_1, x_2 x_3 &\geq 0
\end{aligned}$$

Exercise 10 - Solutions of LPs. (**)
Find the necessary and sufficient conditions on the real values a, b, c such that the LP

$$\begin{aligned} \max x_1 + x_2 \\ \text{subject to} \\ ax_1 + bx_2 \leq c \\ x_1, x_2 \geq 0 \end{aligned}$$

(a) has a unique optimal solution.
(b) is infeasible.
(c) is unbounded.

Exercise 11 - Included or not? (**)
The goal of the exercise is to solve the following problem:
Given two polyhedra

$$P_1 = \{x | Ax \leq b\}, \qquad P_2 = \{x | Cx \leq d\} \tag{1}$$

Determine if $P_1 \subseteq P_2$, or find a point in P_1 that is not in P_2.

(a) Let P be a polyhedron and H be a **open** halfspace (*i.e.* a set satisfying $H = \{x / c^t x > b\}$ for some c, b). How can you test if $P \cap H$ is empty? How can you find a point of $P \cap H$ if it exists?
(b) Answer the initial question (if necessary, several calls to the Simplex Algorithm can me made).

4 Simplex algorithm

Exercise 12 - Phase I/II. (*)
Using Phase I/II method, solve the following LPs:

$$\begin{aligned} \max 2x_1 + 3x_2 + x_3 & \\ \text{subject to} & \\ x_1 + x_2 + x_3 &\leq 40 \\ 2x_1 + x_2 - x_3 &\geq 10 \\ -x_2 + x_3 &\geq 10 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

$$\begin{aligned} \max 1000x_1 + 1200x_2 & \\ \text{subject to} & \\ 10x_1 + 5x_2 &\leq 200 \\ 2x_1 + 3x_2 &= 60 \\ x_1 &\leq 12 \\ x_2 &\geq 6 \\ x_1, x_2 \geq 0 & \end{aligned}$$